Optimal back-to-front airplane boarding

Research Thesis

Submitted in Partial Fulfillment of the Requirements
For the Degree of Master of Science
in Computer Science

Vassilii Khachaturov

Ben-Gurion University of the Negev

Kislev 5769 BEER-SHEVA DECEMBER 2008

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The research thesis was done under the supervision of Dr. Eitan Bachmat

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December 18, 2008

Abstract

The problem of finding an optimal back-to-front airplane boarding policy is explored, using the mathematical model introduced by Bachmat *et al.* A combined analytical and numerical solution is presented for the number of passenger groups $m = 2$. For a larger m, a similarly constructed recursive computation yielding an optimal policy is described. Applying it numerically, optimal policies for $m = 3, 4$ are found; modern transport aircraft design renders further increase of m impractical.

Acknowledgment

The author would like to thank his advisor, Dr. Eitan Bachmat, for his helpful and inspiring guidance throughout the research reported in this work, as well as in other projects. Thanks to the Ben-Gurion University of the Negev for its exceptional working atmosphere, where studies and research is the most popular topic even outside of the classrooms.

The generous financial help of the Richard A. and Edythe Kane Scholarship Endowment Fund is gratefully acknowledged.

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1 Introduction

Airlines and their passengers alike have a mutual interest in minimizing the time spent at the gate while the passengers are boarding the plane. For the airlines and airport infrastructure, reducing the boarding time means decreased operational costs and increased passenger throughput capacity. The passengers, in their turn, benefit from reducing the boarding time, because waiting, either in the line at the gate or aboard the plane while the remaining fellow travellers are boarding, is a contributing factor to the overall fatigue and dissatisfaction from the trip.

A common airline strategy aimed at decreasing the boarding time is to employ an *announcement policy*. The boarding is performed in stages, each stage announced through the public address system at the gate by the means of calls like "Flying Carpet Airlines flight number 123 to Kitezh, passengers with their row number 40 and above are now welcome to board the plane; all the other passengers, please remain waiting by the gate". Bachmat *et al.* in [3] study such policies based on the mathematical model introduced in Bachmat *et al.* [2]. They call each set of rows boarding at the same stage a *group*, and concentrate mostly on *uniform* policies, i.e., policies where all the groups have the same number of rows.

When considering deployment of a boarding policy, the first obvious thing to verify is that the policy shortens the boarding time, when compared against *random boarding*, i.e., having no policy at all. Having assured that, one has to consider the trade-off between the potential savings due to the boarding time decrease and the price of implementing the new policy. The price includes the expenses incurred to train the personnel, and the risk of the customer satisfaction reduction when an overly complicated boarding policy becomes used by the airline.

In the remainder of this work, we concentrate on the initial task of boarding policy evaluation, i.e., gauging it against random boarding in terms of the relative time savings.

Researchers have mostly been studying airline boarding through discrete simulation (e.g., [7], [10], [6]). Bachmat *et al.* in [2] developed a mathematical model of the process, which they further show in [3, Sec. 6] to be in nearly complete agreement with the discrete simulations. A key parameter of the model is the *congestion factor* k, the formal definition of which will be given later in Section 3. Intuitively, $1/k$ measures the fraction of passengers that can stand along the aisle, while preparing to be seated during the boarding. This parameter depends on the aircraft design, namely, on the inter-row distance and the number of passengers per seat. As argued in [3], modern airplanes have a congestion factor around $k = 4$.

Exploring this mathematical model, Bachmat and Elkin [4] established an upper bound on the performance of any *back-to-front* boarding policy, i.e., a policy in which each group is allowed to board only when all the groups with higher row numbers have already boarded. They showed that for realistic airplanes, i.e., those with a congestion factor $k \approx 4$, the upper bound on possible time savings is about 20%, no matter into how many groups the passengers are split.

For the remainder of this work, we shall employ the model to reflect the airplane boarding, and will assume that the real process behaves likewise.

As Bachmat and Elkin do not consider the optimal non-uniform back-to-front policies directly, but rather place an upper bound on the achieved savings, two questions remain. First, what is the exact saving achieved by an optimal m-group boarding policy over random boarding? Second, what is the maximum number of groups that can be justified by non-negligible additional savings over a boarding policy with less groups? Bachmat [1] conjectured that, for $k = 4$, no matter how large the number of groups m, the savings will not exceed 5%.

In the course of the present work, the following results on back-to-front policies were established:

- A combined analytical and numerical solution, describing exactly the optimal policy for $m = 2$, was found for any $k > 0$.

- Even for $m = 2$, the optimal policy can achieve approximately 8% savings under congestion $k = 4$, with further improvement possible through increasing the number of the passengers groups, thus disproving the conjecture. Therefore, while the passengers may not complain too much about the additional discipline imposed on them by the airlines, the airlines' usage of boarding policies is perhaps justified.

- The exact optimal partitioning of a passenger load into groups up to $m = 4$ for back-to-front boarding was found for various congestion factors, employing semi-numerical methods for $m = 2$ and numerical ones for $m > 2$.

- Based on the calculated savings, as achieved by these optimal policies, any further partitioning beyond 4 groups was found impractical until a 10^{-4}-th of the passenger load means at least one non-dismembered passenger. Taking into account the small increase in savings between 3 and 4 groups, 4-group boarding policies are probably meaningless as well.

The remainder of this work is organised as follows. In Section 2 we describe the preceding research in the field, mostly following [3, Sec. 1]. In Section 3 we describe the mathematical model of airplane boarding introduced by Bachmat *et al.* Starting with that section, we only consider back-to-front policies. We then proceed to calculations of an optimal two-group non-uniform back-to-front boarding policy in Section 4, further generalising this approach to m groups in Section 5. The m-group case involves a recursive computation which uses the uniform case from [3] as the base case, and a generalisation of the same techniques as used in the two-group case for the transition. Further on in Section 6 we present the numerical portion of the obtained results, including the actual optimal policies found and their ranking relatively to random boarding. Finally, in Section 7, we map out further work to be done exploring back-to-front boarding policies.

2 Related work

We shall now closely follow the survey from [3, Sec. 1] in describing the preceding research in the field.

More elaborate discussions of the importance of gate delay reductions are found in Marelli *et al.* [7], van Landeghem and Beuselinck [10], and van den Briel *et al.* [9].

Airplane boarding has been previously studied through discrete event simulations in [7], [10], and by Ferrari and Nagel [6]. In addition, van den Briel *et al.* [8] formulated a non-linear integer programming problem, related to airplane boarding time, to which they applied various heuristics in order to find efficient boarding policies. The policies were then tested using a discrete event simulation.

These studies have found that back-to-front policies are not necessarily effective, and might even be detrimental when compared with random boarding. Ferrari and Nagel show [6] that, in some cases, disturbances of the back-to-front scenario caused by passengers boarding outside the time slot assigned for their group will actually *shorten* the boarding time. Van Landeghem and Beuselinck argue [10] that back-to-front policies are ineffective because they cause local congestion in the airplane, but no explanation is given for the mechanism by which congestion affects boarding time.

These studies also show that *outside-in* boarding policies, in which window seat passengers board first, followed by middle and then aisle seat passengers, can improve boarding time. In such policies, and others suggested by the studies, passengers are divided into multiple *boarding groups* which are boarded in sequence.

The study by Marelli *et al.* [7] of Boeing Corp. emphasizes the effect of airplane interior design on boarding time, again using discrete event simulation methods. The paper describes a commercial simulation product, and simulation results are only sketched, making it difficult to analyse or validate the results of this work.

The results and observations of van Landeghem and Beuselinck, van den Briel *et al.*, and Ferrari and Nagel are of considerable value and interest. However, they do not address the need for a unified, analytic approach which can lead to a deeper understanding of the boarding process. As an example of the limitations of previous methods, Bachmat *et al.* [3] note that the simulations in all studies were carried out with particular airplanes in mind. It is therefore important to know how the success of airline boarding policies is related, if at all, to airplane design parameters, such as distance between rows, if the results are to be extended to other airplanes. The model of van den Briel *et al.* [8] does not take such design parameters into account.

Since the simulation-based models offer a black-box approach, every new potential strategy has to be evaluated from scratch as a dedicated simulation scenario, without any advance performance estimates, nor any clue as to how much a given policy might be further improved.

An analytical approach, not suffering from these inherent drawbacks of discrete simulations, was taken by Bachmat *et al.* [2]. The authors express the expected boarding time, as the number of passengers becomes sufficiently large, in terms of the solution to a variational problem. Their mathematical model turns out to be quite general, having both a geometric interpretation in terms of spacetime (Lorentzian) geometry, and also describing the behaviour of another operational research subject, the disk scheduling problem.

Deeper connections of this model with physics, random matrix theory, and other applications of the model are further mapped out by Bachmat [1], following the use of the model by Deift [5] in an even broader physical and mathematical context.

3 The mathematical model of airplane boarding

We shall introduce the mathematical model of Bachmat *et al.*, closely following the definitions and the description of [3, Sec. 2–5] and [4, Sec. 2], using the same notation. We shall only summarize the parts that pertain to back-to-front boarding. Since only such back-to-front boarding policies are the subject of this work, we shall omit the words "back-to-front" henceforth.

We consider only *row policies*, i.e., policies which segregate passengers into boarding groups based on the assigned row numbers. (See [3, Sec. 4.2] for more general boarding strategies, *multiclass policies*.)

Let

n be the number of passengers,

R — the number of rows in the airplane,

m — the number of boarding groups,

k — the congestion factor. The parameter k is the number of passengers per row, times the average aisle length occupied by a passenger when boarding, divided by the distance between successive rows in the airplane.

The passengers will be represented as points (q, r) in the unit square $[0,1]^2$. The q coordinate will represent the position of a given passenger in the boarding queue, divided by n, while the r coordinate will represent the assigned row number, divided by R. An m-group boarding policy F will be represented by a partition of the unit interval $1 = \rho_0 > \rho_1 > \ldots > \rho_m = 0$. For more compact computations, it will be more convenient to consider the equivalent partition in the queue coordinate. We shall denote this as

$F = (x_0, \dots, x_m)$, where $x_i = 1 - \rho_i$. Passengers in the i-th boarding group, $1 \le i \le m$, will have their normalised row numbers satisfy $\rho_{i-1} \ge r \ge \rho_i$. To implement such a policy, an airline would first announce the boarding of the passengers with the row numbers $R \cdot \rho_1$ and above, then — $R \cdot \rho_2$ to $R \cdot \rho_1$, etc.

Let $p(q, r)$ be a joint probability density function on the unit square, representing the passengers boarding the aircraft under such a policy. It is defined (see [4, eqs. (2,3)]) by

$$p(q,r) = \begin{cases} \frac{1}{\rho_{i-1} - \rho_i}, & \rho_{i-1} \ge r \ge \rho_i,\ 1 - \rho_{i-1} \le q \le 1 - \rho_i,\ i = 1, \dots, m; \\ 0, & \text{otherwise.} \end{cases}$$

When conditioned on a given boarding group, it induces a uniform distribution, since the ordering of passengers within a group is uniformly random. See Figure 1 for an example with two boarding groups, illustrating the effect of a group size on the density within the group.

Denote

$$\alpha(q, r) = \int_r^1 p(q, z)\, dz$$

as in [3, eq. (2)]. Let Ψ be the set of all piecewise differentiable functions $\varphi(q)$ defined on an interval $[q_0, q_1] \subset [0, 1]$ and with values in $[0, 1]$, such that

$$\varphi'(q) + k \cdot \alpha(q, \varphi(q)) \ge 0. \tag{1}$$

We shall now introduce several concepts related to the mathematical model we use, consistent with other works in the field. Giving full intuition behind the naming and origins of these concepts would require introduction of significant volume of material in differential geometry in general and Lorenzian geometry in particular, which we shall omit, since it is not needed to understand the present work.

When F is a boarding policy, and $p(q, r)$ is the corresponding probability density function, the *boarding time* under this policy is

$$T(F, k) = \max_{\varphi \in \Psi} L(\varphi), \tag{2}$$

where

$$L(\varphi) = \int_{q_0}^{q_1} \sqrt{p(q, \varphi(q)) \cdot (\varphi'(q) + k \cdot \alpha(q, \varphi(q)))}\, dq.$$

Note that $L(\varphi)$ is only well-defined for a φ satisfying (1); such curves φ are *legitimate*. For a legitimate curve φ, the non-negative value $L(\varphi)$ is the *length* of the curve. In the context of Lorentzian geometry, the functional L is known as *proper time*, and legitimate curves are *timelike*. The border case legitimate curves, namely, the ones for which $L(\varphi) = 0$, are *lightlike*. Curves maximising $T(F, k)$ are *geodesics*.

Denote by F_m the uniform m-group policy, i.e., one with R/m rows per group. For the random policy F_1, the length functional takes [3, eq. (13)] the form

$$L(\varphi) = \int_0^1 \sqrt{\varphi' + k(1 - \varphi)}\, dq. \tag{3}$$

According to [3, eqs. (16,17)], the general solution achieving (2) is

$$\varphi(q) = c_1 e^{2kq} + c_2 e^{kq} + 1, \tag{4}$$

and, as long as this curve stays within the unit square, for $q \in [\alpha, \beta]$

$$L(\varphi) = \left(e^{k\beta} - e^{k\alpha}\right)\sqrt{\frac{c_1}{k}}. \tag{5}$$

Notation 1. We denote by F_m^* an optimal m-group non-uniform policy. The corresponding maximal curve length, i.e., expected boarding time under this policy, will be denoted by T_m.

By [3, eqs. (19,20)],

$$T(F_1, k) = \begin{cases} \sqrt{\frac{e^k - 1}{k}}, & 0 < k \le \ln 2; \\ \sqrt{k} + \frac{1 - \ln 2}{\sqrt{k}}, & k \ge \ln 2. \end{cases} \tag{6}$$

Also, by [3, eqs. (27,28)],

$$T(F_2, k) = \begin{cases} \sqrt{\frac{1}{2k}} \cdot \left(k + \frac{e^k - 1}{4}\right), & 1 \le k \le 2\ln 2; \\ \sqrt{2k} + \frac{3/4 - 2\ln 2}{\sqrt{2k}}, & k \ge 2\ln 2. \end{cases} \tag{7}$$

Finally, according to [3, eq. (33)], for $m \ge 2$ and $k \ge 3/4 + \ln 2$

$$T(F_m, k) = \sqrt{mk} - \frac{m-2}{\sqrt{mk}} \cdot (\ln 2 + 1/4) - \frac{2\ln 2 - 3/4}{\sqrt{mk}}. \tag{8}$$

According to the upper bound established in [4], for an airplane model with the congestion factor $k \ge 1$, the savings of any policy F over the random policy F_1 is at most

$$1 - \frac{T(F,k)}{T(F_1,k)} \le 1 - \frac{\sqrt{k-1}}{\sqrt{k} + \frac{1-\ln 2}{\sqrt{k}}}. \tag{9}$$

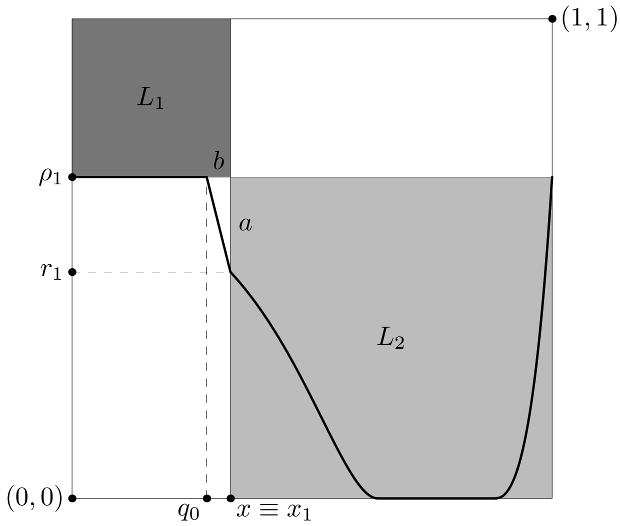

Figure 1: An example of a maximal curve for a 2-group policy.

4 An optimal two-group non-uniform policy

We are looking for a 2-group boarding policy that would be optimal for a given congestion factor k. We do so by assuming the optimal policy to have the form $F_2^* = (0, x, 1)$ for some unknown x, generalising the computations from [3, Sec. 5.2] to the non-uniform case, finding $T(F_2^*, k)$ as a function of x, and then minimising it over x.

Figure 1 shows the situation for a typical value of x. The maximal curve φ is drawn in black. Darker shade indicates higher joint probability density $p(q, r)$.

Given x, we let L be the length of the maximal curve. We first consider the non-degenerate case when the maximal curve spans both square cells L_1

and L_2 as shown in the figure.

In this case, as shown in [3, Sec. 5.2], the maximum length curve must consist of a horizontal line segment between $(0, 1-x)$ and $(q_0, 1-x)$, for some $0 \leq q_0 \leq 1$, then a straight-line segment sloping down to (x, r_1) with the maximum possible legitimate slope, i.e., $-k$, and then the maximal curve in the lower-right square L_2, ending at $(1, 1-x)$, as shown in Figure 1. The three possibilities for the portion of the curve which is contained in L_2 are shown (in a rescaled version) in figure 2. Each segment of the maximal curve φ must either be of the form in equation (4) or be a boundary component of a cell. The three cases shown in Figure 2 correspond to the case in which φ restricted to L_2 has no boundary component (I), is tangent to the boundary (II), or contains a boundary component (III). The latter possibility, where a boundary component exists, is the one shown in Figure 1.

Let L_1, L_2 denote the length of the maximal curve in the corresponding cells, scaled to the unit square size. Then

$$L = \sqrt{x} L_1 + \sqrt{1-x} L_2, \tag{10}$$

where

$$L_1 = \sqrt{k} \cdot \frac{q_0}{x}. \tag{11}$$

Next, we compute r_1 as a function of (q_0, x). As $\frac{a}{b} = k$; $a = 1 - x - r_1$; $b = x - q_0$, we have

$$r_1 = 1 - x - a = 1 - x - kb = 1 - x - k(x - q_0) = 1 - (k+1)x + kq_0.$$

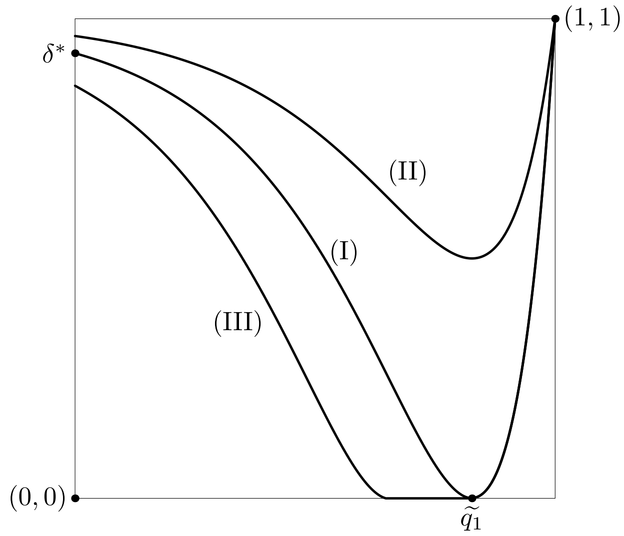

Figure 2: Three possible cases for the maximal curve in L_2.

Now let us compute L_2. Consider the second cell scaled to the unit size, the maximal curve enters the square at

$$r = \frac{r_1}{1-x}. \tag{12}$$

To be consistent with the notation of [3, eq. (22)] and with [3, Sec. 5] in general, we shall denote the right-hand side of (12) by δ, and seek to express L via x, δ. (See Figure 2.)

$$\delta(1-x) = r_1 = 1 - (k+1)x + kq_0, \tag{13}$$

$$kq_0 = \delta(1-x) + (k+1)x - 1. \tag{14}$$

Therefore

$$q_0 = \frac{1}{k}\left(\delta(1-x) + (k+1)x - 1\right). \tag{15}$$

Substituting (15) into (11) we obtain

$$L_1 = \sqrt{k} \cdot \frac{1}{xk} \left(\delta(1-x) + (k+1)x - 1\right) = \frac{\delta(1-x) + (k+1)x - 1}{x\sqrt{k}}. \quad (16)$$

Regarding the domain of L, note that r_1 cannot drop below the line with slope of $-k$, so even if the transition from the L_1 to the L_2 cell is made as early as possible (i.e., for $x = 0$), nevertheless $r_1 \geq 1 - x - xk$, and thus, for any $x \in (0, 1)$, the second parameter, δ, is constrained by $\delta \in (\frac{1-x-xk}{1-x}, 1) \cap (0, 1)$. I.e.,

$$1 > \delta > \begin{cases} \frac{1-x-xk}{1-x}, & 0 < x < \frac{1}{k+1}, \\ 0, & \frac{1}{k+1} \leq x < 1. \end{cases} \quad (17)$$

Let us compute the δ for the pivotal (tangent) case. We do it by applying to the general solution (4) the boundary conditions

$$\begin{cases} r(0) = \delta &= c_1 + c_2 + 1, \\ r(\widetilde{q_1}) = 0 &= c_1 e^{2k\widetilde{q_1}} + c_2 e^{k\widetilde{q_1}} + 1, \\ r'(\widetilde{q_1}) = 0 &= 2kc_1 e^{2k\widetilde{q_1}} + kc_2 e^{k\widetilde{q_1}}, \\ r(1) = 1 &= c_1 e^{2k1} + c_2 e^{k1} + 1. \end{cases} \quad (18)$$

Notation 2. Denote $X \equiv e^{kx}$, i.e., $Q = e^{kq}$, $Q_0 = e^{kq_0}$, etc.

Under this notation,

$$r' = \frac{dr}{dq} = 2kc_1 e^{2kq} + kc_2 e^{kq} = 2kc_1 Q^2 + kc_2 Q.$$

Exploring the sign of r' in the unit square, we note that $r'(0+) < 0$ as r

must go from r_0 down to 0. Note that $kQ > 0$, so $r' = 0$ at

$$Q = -\frac{c_2}{2c_1}, \tag{19}$$

i.e., the solution is tangent to the axis $q = 0$ at

$$\widetilde{q_1} \equiv q = \frac{1}{k} \ln\left(-\frac{c_2}{2c_1}\right). \tag{20}$$

Also, from (18),

$$-\frac{c_2}{2c_1} = \frac{e^k}{2},$$

and thus

$$\begin{aligned} c_1 &= \frac{1-\delta}{e^k - 1}, && (21)\\ c_2 &= \frac{-e^k}{e^k - 1}(1-\delta). && (22) \end{aligned}$$

So

$$\widetilde{q_1} = \frac{1}{k} \ln\left(\frac{e^k}{2}\right) = \frac{1}{k}(\ln e^k - \ln 2) = \frac{1}{k}(k - \ln 2) = 1 - \frac{\ln 2}{k}.$$

Plugging c_1, c_2, and $\widetilde{q_1}$ into (18), we have

$$\begin{aligned} r(\widetilde{q_1}) = 0 = c_1 \left(\frac{e^k}{2}\right)^2 + c_2 \left(\frac{e^k}{2}\right) + 1 = \\ = \frac{1-\delta}{e^k - 1} \cdot \frac{e^{k2}}{4} - \frac{e^k}{e^k - 1}(1-\delta)\frac{e^k}{2} + 1 = \\ = (1-\delta) \cdot \frac{e^{2k}}{e^k - 1}\left(\frac{1}{4} - \frac{1}{2}\right) + 1 \end{aligned}$$

and

$$(1-\delta) \cdot \frac{e^{2k}}{4(e^k - 1)} = 1,$$

which gives

$$\delta^* \equiv \delta = 1 - \frac{4(e^k - 1)}{e^{2k}} = \\ = \frac{1}{e^{2k}}(e^{2k} - 4e^k + 4) = \frac{1}{e^{2k}}(e^k - 2)^2 = \\ = (1 - 2e^{-k})^2.$$

Thus, for $\delta \geq \delta^*$, by (5) and (21)

$$L_2 = \left(e^k - 1\right)\sqrt{\frac{c_1}{k}} = \left(e^k - 1\right)\sqrt{\frac{1-\delta}{e^k - 1}\cdot\frac{1}{k}} = \sqrt{\frac{(1-\delta)(e^k - 1)}{k}}. \tag{23}$$

Here in case of the equality the solution is depicted by the case (I), and in case of strict inequality — the case (II) of Figure 2.

This reproduces the computations behind [3, eq. (23)]; note the difference by the factor of $\sqrt{2}$ due to the difference of the square size — we work with unit square, while the original computation was done for a 2×2 square.

Likewise, as per [3, eq. (24)], in the remaining case (see Figure 2, case (III)) of $\delta < \delta^*$ we have

$$L_2 = \sqrt{\frac{1}{k}}\left(\sqrt{\delta} + \ln\left(1 - \sqrt{\delta}\right) + k + 1 - \ln 2\right). \tag{24}$$

We shall denote by $L_2(\delta)$ the maximal length of a legitimate curve in the unit square, constrained to pass through the boundary point $(0, \delta)$. The length $L_2(\delta)$ was computed, in equations (23) and (24), to be

$$L_2(\delta) = \sqrt{\frac{1}{k}} \cdot \begin{cases} \sqrt{(1-\delta)(e^k - 1)}, & \delta \geq \delta^*; \\ \sqrt{\delta} + \ln\left(1 - \sqrt{\delta}\right) + k + 1 - \ln 2, & \delta < \delta^*. \end{cases} \tag{25}$$

Combining (10), (16), and (25), we get

$$\begin{aligned}
L(\delta) &= \sqrt{x}\cdot\frac{\delta(1-x)+(k+1)x-1}{x\sqrt{k}} + \sqrt{1-x}\cdot L_2(\delta) = \\
&= \frac{\delta(1-x)+(k+1)x-1}{\sqrt{kx}} + \\
&\qquad + \sqrt{\frac{1-x}{k}}\cdot\begin{cases}\sqrt{(1-\delta)(e^k-1)}, & \delta\ge\delta^*;\\ \sqrt{\delta}+\ln\left(1-\sqrt{\delta}\right)+k+1-\ln 2, & \delta<\delta^*.\end{cases}
\end{aligned} \tag{26}$$

Given $x\in(0,1)$, we seek to maximise L as a function of δ. For this, we need to explore the derivative

$$\frac{dL}{d\delta} = \frac{1-x}{\sqrt{kx}} + \sqrt{\frac{1-x}{k}}\cdot\begin{cases}-\frac{\sqrt{e^k-1}}{2\sqrt{1-\delta}}, & \delta\ge\delta^*;\\ \frac{1}{2\sqrt{\delta}}-\frac{1}{2\left(1-\sqrt{\delta}\right)\sqrt{\delta}}, & \delta<\delta^*.\end{cases}$$

As

$$\begin{aligned}
\frac{1}{2\sqrt{\delta}} - \frac{1}{2\left(1-\sqrt{\delta}\right)\sqrt{\delta}} &= \frac{1}{2\sqrt{\delta}}\left(1-\frac{1}{1-\sqrt{\delta}}\right) = \\
&= \frac{1}{2\sqrt{\delta}}\cdot\frac{1-\sqrt{\delta}-1}{1-\sqrt{\delta}} = \frac{-1}{2\left(1-\sqrt{\delta}\right)},
\end{aligned}$$

we have

$$\frac{dL}{d\delta} = \frac{1-x}{\sqrt{kx}} - \frac{1}{2}\cdot\sqrt{\frac{1-x}{k}}\cdot\begin{cases}\frac{\sqrt{e^k-1}}{\sqrt{1-\delta}}, & \delta\ge\delta^*;\\ \frac{1}{1-\sqrt{\delta}}, & \delta<\delta^*.\end{cases} \tag{27}$$

Thus for $\delta \geq \delta^*$

$$
\begin{aligned}
\frac{dL}{d\delta} = 0 \iff \\
\iff \frac{1}{\sqrt{1-\delta}} = \frac{1-x}{\sqrt{kx}} \cdot 2 \cdot \sqrt{\frac{k}{(1-x)(e^k-1)}} = 2\sqrt{\frac{1-x}{x(e^k-1)}} \iff \\
\iff 1-\delta = \frac{x(e^k-1)}{4(1-x)} \iff \\
\iff \delta = \frac{4-3x-e^k x}{4(1-x)},
\end{aligned}
$$

while for $\delta < \delta^*$

$$
\begin{aligned}
\frac{dL}{d\delta} = 0 \iff \\
\iff \frac{1-x}{\sqrt{x}} = \frac{1}{2\left(1-\sqrt{\delta}\right)} \cdot \sqrt{1-x} \iff \\
\iff 2\left(1-\sqrt{\delta}\right) = \sqrt{\frac{x}{1-x}} \iff \\
\iff \sqrt{\delta} = 1 - \frac{1}{2} \cdot \sqrt{\frac{x}{1-x}} \iff \\
\iff \delta = 1 - \sqrt{\frac{x}{1-x}} + \frac{x}{4(1-x)} \iff \\
\iff \delta = \frac{4(1-x) - 4\sqrt{x(1-x)} + x}{4(1-x)} \iff \\
\iff \delta = \frac{4-3x-4\sqrt{x-x^2}}{4(1-x)}.
\end{aligned}
$$

So,

$$
\frac{dL}{d\delta} = 0 \iff \begin{cases} \delta_1 \equiv \delta = \frac{4-3x-e^k x}{4(1-x)}, & \delta \geq \delta^*; \\ \delta_2 \equiv \delta = \frac{4-3x-4\sqrt{x-x^2}}{4(1-x)}, & \delta < \delta^*. \end{cases}
$$

The second derivative, from differentiating (27) an additional time, is for $\delta \geq \delta^*$

$$\frac{d^2L}{d\delta^2} = -\frac{1}{2} \cdot \sqrt{\frac{(1-x)(e^k-1)}{k}} \cdot \left(\frac{1}{\sqrt{1-\delta}}\right)' =$$

$$= -\frac{1}{2} \cdot \sqrt{\frac{(1-x)(e^k-1)}{k}} \cdot \frac{-1}{2\,(1-\delta)^{3/2}} \cdot (-1),$$

and for $\delta < \delta^*$

$$\frac{d^2L}{d\delta^2} = -\frac{1}{2} \cdot \sqrt{\frac{1-x}{k}} \cdot \left(\frac{1}{1-\sqrt{\delta}}\right)' = -\frac{1}{2} \cdot \sqrt{\frac{1-x}{k}} \cdot \frac{-1}{\left(1-\sqrt{\delta}\right)^2} \cdot \frac{-1}{2\sqrt{\delta}}.$$

Thus for any $x, \delta \in (0,1)$ and $k > 0$

$$\frac{d^2L}{d\delta^2} < 0. \tag{28}$$

In our quest for the optimal policy, the following remains to be done:

i. For any given x

1) Explore when the roots of the derivative $\frac{dL}{d\delta}$ satisfy the corresponding constraints (with respect to δ^* and (17)), and thus are points of maximum of L by (28). We shall call such roots *admissible*.

2) When (i1) is not satisfied, find the relevant boundary values of $L(\delta)$, out of $L\left(\max\left\{0, \frac{1-x-xk}{1-x}\right\}\right)$, $L(1)$, and $L\left(\max\left\{\delta^*, \frac{1-x-xk}{1-x}\right\}\right)$.

3) Take maximum from the admissible maximum points, or (for the inadmissible) the relevant boundary points of (i2) instead.

4) Take maximum from the above and the maximum of the two cases when one of the two groups is completely ignored by the passengers.

ii. Minimize the result with respect to x, and compare to the single-group uniform case $T(F_1, k)$.

When is $\delta_1 \geq \delta^*$? Substituting δ_1, we have to solve

$$\frac{4 - 3x - e^k x}{4(1-x)} \geq \delta^*.$$

As $x < 1$, we can transform it as follows:

$$\begin{aligned} 4 - 3x - e^k x &\geq 4(1-x)\delta^*, \\ 4x\delta^* - 3x - e^k x &\geq 4\delta^* - 4, \\ x(4\delta^* - 3 - e^k) &\geq 4(\delta^* - 1). \end{aligned}$$

Compare the left-hand side to 0:

$$4\delta^* - 3 - e^k = 4(1 - 2e^{-k})^2 - 3 - e^k = 4(1 - 4e^{-k} + 4e^{-2k}) - 3 - e^k =$$
$$= 1 - 16e^{-k} + 16e^{-2k} - e^k \geq 0 \iff$$
$$\overset{*e^{2k}}{\iff} -e^{3k} + e^{2k} - 16e^k + 16 \geq 0. \quad (29)$$

Assuming $t \equiv e^k$ (which is strictly positive), we may transform (29) into an equivalent

$$-t^3 + t^2 - 16t + 16 = (t-1)(-t^2 - 16) = -(t-1)(t^2 + 16) \geq 0,$$

and therefore

$$e^k \leq 1,$$

and

$$k \leq 0,$$

which never holds, i.e., δ_1 is not an admissible root of the derivative. Thus, we shall be interested in the appropriate (i.e., the larger) boundary value out of $L\left(\max\left\{\delta^*, \frac{1-x-xk}{1-x}\right\}\right)$ and $L(1)$, instead of $L(\delta_1)$. By (28), the larger value of L will be achieved on the boundary of the admissible values closer to the non-admissible root, i.e., at $L\left(\max\left\{\delta^*, \frac{1-x-xk}{1-x}\right\}\right)$.

When is $\delta_2 < \delta^*$?

$$\begin{aligned} \frac{4-3x-4\sqrt{x-x^2}}{4(1-x)} &< \delta^*, \\ 4-3x-4\sqrt{x-x^2} &\overset{(0<x<1)}{<} 4(1-x)\delta^*, \\ \underbrace{4-3x+4(x-1)\delta^*}_{a} &< \underbrace{4\sqrt{x-x^2}}_{b}. \end{aligned}$$

Since for any $0 < x, \delta^* < 1$, both $a, b > 0$, the last inequality holds iff $a^2 < b^2 \iff x \in (x_1, x_2) \cap (0,1)$, where $x_1 \le x_2$ are the roots of $a^2 = b^2$. It turns out that both the roots

$$x_{1,2} = 1 - \frac{1}{5 \mp 8\sqrt{\delta^*} + 4\delta^*}$$

fall within $(0,1)$ for any $\delta^* \in (0,1)$, so δ_2 is an admissible root of the derivative whenever, given δ^*, the other input parameter x satisfies

$$1 - \frac{1}{5 - 8\sqrt{\delta^*} + 4\delta^*} < x < 1 - \frac{1}{5 + 8\sqrt{\delta^*} + 4\delta^*}. \qquad (30)$$

When is $\delta_2 > \max\left\{0, \frac{1-x-xk}{1-x}\right\}$? For δ_2 to be admissible it further remains to check the constraints of (17). For $x \ge \frac{1}{k+1}$ it reduces to the trivial $\delta_2 > 0$.

Now assume $x < \frac{1}{k+1}$. One needs to check that

$$\delta_2 > \frac{1-x-xk}{1-x},$$

and, in case it holds, δ_2 is admissible; otherwise, the boundary value at $\delta = \frac{1-x-xk}{1-x}$ should be considered instead of $L(\delta_2)$.

The above inequality holds if (and only if)

$$\frac{16}{17+8k+16k^2} < x. \tag{31}$$

The relevant boundary values of $L(\delta)$: From (26) we seek the values of $L(\delta)$ at $\max\left\{\delta^*, \frac{1-x-xk}{1-x}\right\}$ and $\max\left\{\delta_2, \frac{1-x-xk}{1-x}\right\}$:

$$L(\delta^*) \quad = \quad \frac{\delta^*(1-x)+(k+1)x-1}{\sqrt{kx}} + \sqrt{\frac{1-x}{k}} \cdot \sqrt{(1-\delta^*)(e^k-1)}.$$

$\max\left\{L\left(\max\left\{\delta^*, \frac{1-x-xk}{1-x}\right\}\right), L\left(\max\left\{\delta_2, \frac{1-x-xk}{1-x}\right\}\right), F_1\right\}$: Given x and k, we seek the maximum of L at the relevant boundary points, the remaining potentially admissible root of the derivative δ_2 (when it is indeed admissible, i.e., both (30) and (31) hold), and the single-group case F_1.

Whenever (30) holds, $L(\delta_2) \geq L(\delta^*)$, since δ_2 is the point of maximum on the interval $(0, \delta^*)$.

$L_{1'}$ — the case when only the first group is used: Consider the case when the maximal curve does not use both squares with non-zero support of the passenger distribution, but rather comes from the first group only. Then, within that square, it will be the maximal curve for F_1, with its length determined by (6), scaled accordingly, by (10):

$$L_{1'} = \sqrt{x} \cdot T(F_1, k).$$

$L_{2'}$ **— the case when just the second group is used:** Likewise, if the maximal curve only spans the second group, its length will be

$$L_{2'} = \sqrt{1-x} \cdot T(F_1, k).$$

4.1 The case of $k = 4$

We show how the calculations of the previous section can be applied specifically to the case $k = 4$ which we consider to be realistic. For this case, using the upper bound (9), we conclude that the possible boarding time savings given by any policy cannot exceed 20%.

By (6), the random boarding time is $T(F_1, 4) = 2.5 - \frac{\ln 2}{2} \approx 2.153426409$

Out of the 2-group split possibilities, one needs to take $L(\delta_2)$ as long as (30) and (31) simultaneously hold, i.e., when x is within

$$(\underbrace{\sim 0.005338746}_{x_1}, \underbrace{\sim 0.939095941}_{x_2}) \cap (\frac{16}{305}, 1) = (\frac{16}{305}, x_2). \tag{32}$$

Otherwise $L(\max\left\{\delta^*, \frac{1-x-xk}{1-x}\right\})$ should be preferred.

$$\max\left\{\delta^*, \frac{1-x-xk}{1-x}\right\} = \delta^* \iff 0.017662599 \approx \frac{e^4-1}{e^8+e^4-1} < x < 1,$$

so we have to consider $\max\{L(\delta^*), L_{1'}, L_{2'}\}$ for

$$x \in (\frac{e^4-1}{e^8+e^4-1}, \frac{16}{305}) \cap (x_2, 1),$$

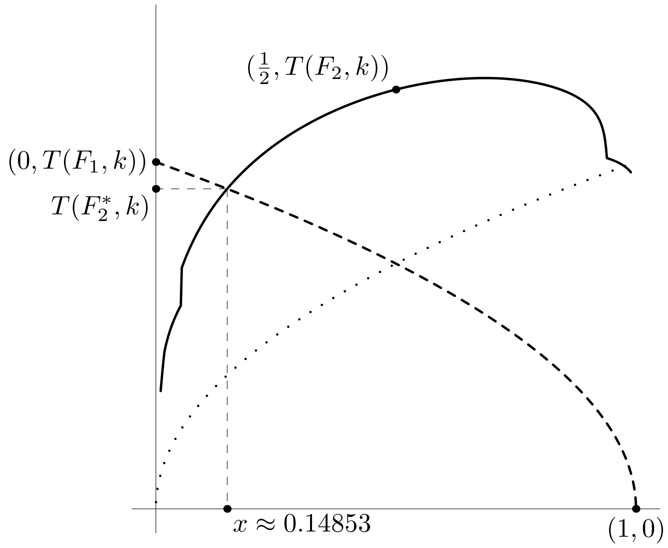

Figure 3: L (solid), $L_{1'}$ (dotted), and $L_{2'}$ (dashed) as functions of x, determining the optimal non-uniform 2-group policy $F_2^* = (0, x, 1)$ for $k = 4$, and the relative magnitude of the boarding times under three various 2-group policies: F_2^* (best), random, and uniform F_2 (worst).

and $L(\frac{1-x-xk}{1-x})$ for $x < \frac{e^4-1}{e^8+e^4-1}$.

As one can see in Figure 3, substituting the appropriate δ for each of the above subintervals of $(0, 1)$ and concatenating the resulting segments gives a combined function $L(x)$ which intersects both $L_{1'}$ and $L_{2'}$ once each.

Numerical search by x for the intersection in the interval (32) shows that the minimum of $\max\{L(\delta_2), L_{1'}, L_{2'}\}$ is achieved at $x \approx 0.148531234$ and equals $L(\delta_2, x) = \sqrt{1-x} \cdot T(F_1, 4) \approx 1.987075623$. (Note that this already beats F_1).

The interval $(0, \frac{16}{305})$ can be excluded immediately from consideration since already beyond its upper bound the maximum was hit by a single-group passenger behaviour, which will only give worse results if x is further decreased, since $\sqrt{1-x}$ is decreasing on $(0, 1)$.

Numerical search for the other intersection for $x \geq x_2$ discovers another, slightly worse, minimum of $\max\{L(\delta^*), L_{1'}, L_{2'}\}$ at $x \approx 0.975885874$, where $L(\delta^*) = \sqrt{x} \cdot T(F_1, 4) \approx 2.127303971$.

Thus, when $k = 4$, the optimal 2-group policy is achieved at

$$x \approx 0.148531234,$$

which saves

$$1 - \frac{L_{2'}}{T(F_1, 4)} \approx 0.077249348 \approx 8\%$$

of the boarding time over the random boarding.

This exceeds the expectations of [1], which conjectured that no policy (irrespectively of the number of boarding groups) will yield a time reduction of more than 5%.

4.2 The cases of $k = 4/3, 2, 3$

In a similar way, we found the optimal two-group policy and the ratio of its boarding time to the time under the random policy. Inspired by [3, Table 1], we summarise the optimal two-group policy data, along with the ratio of the boarding times under the uniform two-group policy and the random one, in Table 1 further down in Section 6.

5 An optimal m-group non-uniform policy

5.1 Plan: induction on m

We would like to compute the optimal m-group non-uniform policy and to measure its efficiency relatively to $T(F_1, k)$. That is, we shall find the split points between the row groups, and the corresponding maximal curve length under this policy, similar to what we have done with 2 groups.

We shall describe a recursive computation of the optimal policy, proving along the correctness of the algorithm, i.e., the optimality of the found results, by induction on m.

Most of the techniques and the constructions developed for the 2-group policy will be re-used, either during the induction base step ($m = 1$), or during the transition (from $m-1$ to m for $m>1$).

For the smallest m such that $T_m = T_{m+1}$, there is no reason to add further groups beyond m. Thus, in our construction of an m-group policy we shall assume that the corresponding $m-1$-group constructions are all non-degenerate (i.e., the maximal curve actually spans all the $m-1$ cells in the partition).

Notation 3. We denote by $L_1^{(m)}(z)$ the length of the maximal curve in a m-cell partition of the unit square constrained by the exit point $(z, 0)$.

Under this notation, we rewrite (11) as

$$L_1^{(1)}(z) = \sqrt{k} \cdot z. \tag{33}$$

For our recursive construction, we shall also reuse $L_2(\delta)$ as defined in (25).

Using it, from $L_1^{(m-1)}$ (with corresponding m normalized row numbers determining the split points of the partition into the row groups), we shall be able to construct $L_1^{(m)}$ (as groundwork for the next stage, if need be) and T_m (the actual result for m groups), with the explicit partitioning. Comparing T_m against T_{m-1}, we can find out whether further partitioning (into $m+1$ groups) makes sense; by measuring T_m against T_1, we can gauge the improvement of m-group policy with respect to the uniform case F_1.

5.2 Induction base: $m = 1$

By definition

$$
\begin{aligned}
F_1^* &\equiv F_1, \\
T_1 &= T(F_1^*, k) \equiv T(F_1, k); \\
L_1^{(1)}(z) &= \sqrt{k} \cdot z \quad \text{(seen as (33))}.
\end{aligned}
$$

Also

$$
\begin{aligned}
\frac{dL_1^{(1)}(z)}{dt} &= \sqrt{k} \cdot \frac{dz}{dt}, && (34) \\
\frac{d^2 L_1^{(1)}(z)}{dt^2} &= \sqrt{k} \cdot \frac{d^2 z}{dt^2}. && (35)
\end{aligned}
$$

5.3 The corner length $L_C(r_0, q_0)$

We seek a legitimate curve r that enters the unit square at $(0, r_0)$ and exits it at $(q_0, 0)$, that maximises the length functional $L(r)$, as well as the achieved maximal length, which we call *corner length* and denote $L_C(r_0, q_0)$.

Plugging the initial conditions for the general solution (4), we get

$$
\begin{cases}
r_0 = c_1 + c_2 + 1, \\
0 = c_1 e^{2kq_0} + c_2 e^{kq_0} + 1.
\end{cases}
$$

Using the Notation 2, we rewrite the above system as

$$
\begin{cases}
r_0 = c_1 + c_2 + 1, \\
0 = c_1 Q_0^2 + c_2 Q_0 + 1.
\end{cases} \tag{36}
$$

Recalling (19), we substitute $Q_0 = -\frac{c_2}{2c_1}$ into (36), in order to get the solution tangent to the axis $r = 0$:

$$\begin{cases} r_0 &= c_1 + c_2 + 1, \\ 0 &= c_1 \left(-\frac{c_2}{2c_1}\right)^2 + c_2 \left(-\frac{c_2}{2c_1}\right) + 1. \end{cases}$$

Transforming the second equation above, we get

$$0 = \frac{c_2^2}{4c_1} - \frac{c_2^2}{2c_1} + 1 \iff 0 = c_2^2 - 2c_2^2 + 4c_1 \iff 4c_1 = c_2^2 \iff c_1 = \frac{c_2^2}{4}.$$

(Observe that thus $c_1 > 0, c_2 < 0$, as $Q > 0$, so $c_2 = -2\sqrt{c_1}$). Denote $c_1 = x^2$. Then $c_2 = -2x$, and

$$x^2 - 2x + (1 - r_0) = 0,$$

so

$$x = 1 \pm \sqrt{r_0}. \tag{37}$$

We need to decide which of the two x-s to take: both are positive as $(1 - r_0) > 0$.

$$c_1 = x^2 = 1 + r_0 \pm 2\sqrt{r_0}, \tag{38}$$

$$c_2 = -2x = -2 \mp 2\sqrt{r_0};$$

$$Q_0 = -\frac{c_2}{2c_1} = \frac{2x}{2x^2} = \frac{1}{x} = \frac{1}{1 \pm \sqrt{r_0}}.$$

Thus,

$$q_0 = \frac{1}{k} \ln \frac{1}{1 \pm \sqrt{r_0}}.$$

We seek $q_0 > 0$, therefore $\frac{1}{x} > 1$, i.e., in the (37) we take the "minus" case, i.e.,

$$\begin{aligned} Q^* \equiv Q_0 &= \frac{1}{1-\sqrt{r_0}}, \quad (39)\\ q^* \equiv q_0 &= \frac{1}{k}\ln\left(\frac{1}{1-\sqrt{r_0}}\right). \quad (40)\end{aligned}$$

So $r'(q) < 0$ for $0 \leq q < q^*$ and $r'(q) > 0$ for $q > q^* > 0$; the value of the length functional $L(r)$ for this solution (we denote this value $L_C(r_0, q_0)$) is

$$(Q_0 - 1)\sqrt{\frac{c_1}{k}}$$

in the former case, and

$$(Q^* - 1)\sqrt{\frac{c_1^*}{k}} + (q_0 - q^*)\sqrt{k}$$

in the latter. Here c_1 corresponds to the solution (5), as found from (4), that exits the unit square via q_0, and c_1^* is the c_1 constant from (38) corresponding to the tangent solution (i.e., again, the "minus" case), the one that satisfies $r(q^*) = r'(q^*) = 0$.

Given r_0 and q_0 (hence Q_0) find the corresponding c_1, c_2 from the system (36):

$$\begin{aligned} c_1 &= r_0 - 1 - c_2,\\ 0 &= (r_0 - 1 - c_2)Q_0^2 + c_2 Q_0 + 1\\ &= (r_0 - 1)Q_0^2 + c_2 Q_0(1 - Q_0) + 1. \end{aligned}$$

Thus

$$c_2 = \frac{(1-r_0)Q_0^2 - 1}{Q_0(1-Q_0)}; \tag{41}$$

$$c_1 = \frac{(r_0-1)Q_0(1-Q_0) + (r_0-1)Q_0^2 + 1}{Q_0(1-Q_0)} = \frac{(r_0-1)Q_0 + 1}{Q_0(1-Q_0)}. \tag{42}$$

Given r_0 and q_0 and thus c_1 and c_2 from (42) and (41), we want to check when the corresponding curve (4) is legitimate.

To find the border case of the legitimate curves, we want to find a lightlike solution $r(q)$ that achieves a zero of the length functional $L(r)$, which in this case is expressed by (3). Thus, we are looking for r such that

$$r' + k(1-r) = 0,$$

which has the general solution

$$r(q) = 1 + e^{kq}c_1. \tag{43}$$

Given r_0, we are looking for the corresponding q_0. From the boundary condition $r(0) = r_0$, the solution we seek is

$$r(q) = 1 + e^{kq}(r_0 - 1).$$

Now where does this solution get to the axis $r = 0$?

$$1 + e^{kq_0}(r_0-1) = 0 \iff e^{kq_0}(r_0-1) = -1.$$

A necessary condition is that $r_0 < 1$, otherwise the curve never gets down to zero. As $Q_0 = \frac{1}{1-r_0}$, therefore

$$q_* \equiv q_0 = \frac{1}{k}\ln\left(\frac{1}{1-r_0}\right).$$

Thus

$$L_C(r,q) = \begin{cases} \sqrt{\frac{(Q-1)((1-r)Q-1)}{kQ}}, & q_* \le q < q^*; \\ \frac{(Q^*-1)(1-\sqrt{r})}{\sqrt{k}} + (q-q^*)\sqrt{k}, & q > q^*; \end{cases}$$

$$= \begin{cases} \sqrt{\frac{(Q-1)((1-r)Q-1)}{kQ}}, & q_* \le q < q^*; \\ \frac{kq+\sqrt{r}+\ln\left(1-\sqrt{r}\right)}{\sqrt{k}}, & q > q^*. \end{cases} \tag{44}$$

The other way around, given q_0 we find r_0 as follows. Say,

$$r(q_0) = 0 = 1 + e^{kq_0}c_1,$$

then

$$c_1 = \frac{-1}{e^{kq_0}} = -e^{-kq_0},$$

so the solution is

$$r(q) = 1 - e^{kq}e^{-kq_0},$$

which intersects the axis $q = 0$ at

$$r(0) = 1 - e^{-kq_0}. \tag{45}$$

5.4 Inductive transition: from $m-1$ to m

We shall mostly reuse the construction from Section 4. However, instead of L_1 (see Figure 1), computed over a non-partitioned group of the back rows, we shall have $L_1^{(m-1)}$. Again by doing a min-max by x and δ, we shall find the optimal split point x achieving the minimal worst-case boarding time.

From the corresponding value of δ we shall be able to recover q_0, and thus, by induction, obtain the remaining internal split points obtained along with $L_1^{(m-1)}(q_0/x)$. This will give us T_m. Then, using a similar construction and min-max analysis, we shall compose $L_1^{(m)}$ by appropriately combining $L_1^{(m-1)}(\cdot)$ and $L_C(\cdot,\cdot)$.

Given x, δ (and the corresponding q_0 as found from (15)), we generalise (16) to find the contribution of the upper-left $m-1$ cells in the m-cell partition we are about to construct to be

$$L_1^{(m-1)}\left(\frac{\delta(1-x)+(k+1)x-1}{kx}\right). \tag{46}$$

Thus, similarly to (26), we obtain the recursive formula for the function to min-max in order to compute T_m:

$$L^{(m)} = \sqrt{x} \cdot L_1^{(m-1)}\left(\frac{\delta(1-x)+(k+1)x-1}{kx}\right) + \sqrt{1-x} \cdot L_2(\delta). \tag{47}$$

Instead of (27), we now have a recursive formula for the derivative as well,

$$\begin{aligned} \frac{dL^{(m)}}{d\delta} = \frac{1-x}{k\sqrt{x}} \cdot \frac{d}{d\delta} L_1^{(m-1)}\left(\frac{\delta(1-x)+(k+1)x-1}{kx}\right) - \\ - \frac{1}{2} \cdot \sqrt{\frac{1-x}{k}} \cdot \begin{cases} \frac{\sqrt{e^k-1}}{\sqrt{1-\delta}}, & \delta \geq \delta^*; \\ \frac{1}{1-\sqrt{\delta}}, & \delta < \delta^*. \end{cases} \end{aligned} \tag{48}$$

From (33) (for the case where the critical path spans just the lower-right cell), and from (46) (for the case where the critical path has contributions both from $L_1^{(m-1)}$ and the lower-right corner), we find

$$L_1^{(m)}(z) = \min_{0<x<z} \max\left\{\sqrt{1-x}\cdot L_1^{(1)}\left(\frac{z-x}{1-x}\right),\right.$$
$$\max_\delta\left\{\sqrt{x}\cdot L_1^{(m-1)}\left(\frac{\delta(1-x)+(k+1)x-1}{kx}\right)+\right.$$
$$\left.\left.+\sqrt{1-x}\cdot L_C\left(\delta,\frac{z-x}{1-x}\right)\right\}\right\},$$

where the $\max_\delta$ is, by (17), taken over $\max\left\{0, \frac{1-x-xk}{1-x}\right\} < \delta$; the upper limit is now further restricted by $\delta < 1 - e^{-k\cdot\frac{z-x}{1-x}}$ according to (45).

For small enough z it might be possible for a policy to set $x > z$ and still be able to just use the scaled contribution from the upper-left $m-1$ cells. Thus, if $z < \frac{1}{k+1}$, then $L_1^{(m)}(z)$ is defined as the minimum of the above and

$$\min_{z<x<1-kz}\left\{\sqrt{x}\cdot L_1^{(m-1)}\left(\left(z-\frac{1-x}{k}\right)/x\right)\right\}.$$

Finally, the optimal time is given by

$$T_m = \min_{0<x<1}\max\left\{\sqrt{x}\cdot T_{m-1}, \sqrt{1-x}\cdot T_1, \max_\delta L^{(m)}\right\}, \tag{49}$$

where the $\max_\delta$ is, by (17), taken over the interval

$$\max\left\{0, \frac{1-x-xk}{1-x}\right\} < \delta < 1.$$

Conjecture 1. *For $k \geq 4$, the curve $\max_\delta L^{(m)}$ will be a piecewise concave function of x; the layout of the 3 curves under the maximum in* (49) *will resemble Figure 3, therefore just looking for the two intersections and taking the lowest will produce the optimum.*

k	T_2/T_1	$T(F_2,k)/T_1$	x
4/3	0.770	0.876	0.40700
2.00	0.840	1.031	0.29436
3.00	0.894	1.147	0.20152
4.00	0.923	1.209	0.14853

Table 1: The optimal non-uniform F_2^* and its time T_2 vs. F_2

6 Numerical results

In Table 1 we give a numerical comparison of the boarding times between the optimal non-uniform two-group policy $F_2^* = (0, x, 1)$, the uniform two-group policy F_2, and random boarding F_1, for various congestion factors. The times for F_2^* were computed using the method described in Section 4, and $T(F_2, k)$ was found from (7). They were normalised according to F_1^* as given by (6).

One can see that the optimal policy achieves visible savings over the random boarding with congestion as high as $k = 4$, whereas the uniform policy, consistently with the results of [3], only gives visible savings below $k = 2$.

The last column in the table shows the optimal partitioning point x defining the optimal policy $F_2^* = (0, x, 1)$.

Table 2 summarises the results of looking for the optimal non-uniform

k	m	T_m/T_1	$T(F_m,k)/T_1$	F_m^*		
2	2	0.840	1.031	0.29436		
2	3	0.792	1.106	0.11030	0.37226	
2	4	0.774	1.187	0.04477	0.15009	0.40041
4	2	0.923	1.209	0.14853		
4	3	0.911	1.397	0.02526	0.17004	
4	4	0.909	1.565	0.00439	0.02953	0.17368

Table 2: The optimal non-uniform F_m^* and its time T_m vs. F_m.

m-class policy for typical modern airplanes, i.e., with $k = 4$. Similarly to Table 1, it measures T_m against both F_1, and the uniform policy F_m with the same number of groups. Again, the last columns list the inner points $\rho_{m-1}, \ldots, \rho_1$ of the partition comprising F_m^*. This time, we relied on (8) to compute the $T(F_m, k)$ and on the recursion described in Section 5 to obtain T_m.

From this table we conclude that, for $k = 4$, further partitioning beyond $m = 4$ is impractical — recall that modern airplanes' passenger load is at most several hundred, therefore, until a 10^{-4}-th of the number of passengers means a meaningful statistic quantity of people, one should not implement boarding policies with more than 4 groups.

Looking also at the data obtained for $k = 2$, the experimental results lead to the following conjecture.

Conjecture 2. *The first inner partition point ρ_{m-1} in the optimal non-uniform policy F_m^* decreases with the number of groups m, by the factor of over k for each additional group, for any $k > 1$.*

Also, since the additional savings with each additonal boarding group diminish, it is probably worthless to increase m beyond 3 — the increase from $m = 3$ to $m = 4$ brings just an additional .002 fraction of the random boarding time in savings. Indeed, since a typical modern airplane random boarding time is on the order of tens of minutes (see [3, p.26]), these savings will be under ten seconds — probably less than the time needed to make the announcement inviting to board the passengers from the extra group.

6.1 Methods of computation and validation

The numerical computations behind Table 1 were done by finding x for the two intersections seen in the Figure 3. Three different numerical computation packages were used, with the results agreeing beyond the accuracy given in this work.

The results for $m > 2$ were obtained by straightforward implementation of the recursion described in the Section 5. Plotting the curves determining T_3 supported the Conjecture 1. The algorithm implementation was validated by checking that its brute force min-max solving produced the same results as the ones obtained semi-numerically for $m = 2$.

The results are in agreement with [3] and [4].

7 Further work

The following milestones yet remain on the road to complete understanding of optimal back-to-front airplane boarding policies:

- Validate the two conjectures.
- Explore the convergence rate of T_m.
- Solve the recursion (49) and express T_m in closed form.

Our numerical algorithm implementation running time is exponential in m, and thus does not yield an answer for T_m for $m > 4$ within realistic (several hours) time. For further experimental data, if needed as a hint to solve the above theoretical questions, one could implement the algorithm using dynamic programming, building an adaptive mesh for $L_1^{(m)}$ for successive m; while probably resulting in less precision, this would improve the speed.

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תקציר

נחקרת בעיית המציאה של מדיניות אופטימלית להעלאת נוסעים למטוס החל בירכתי המטוס וכלה בקדמתו, תוך שימוש במודל המתמטי של בכמט ושות'. פתרון אנליטי-נומרי משולב מוצג עבור $m = 2$ קבוצות העלאה. עבור m גדול יותר מתואר חישוב רקורסיבי שהורכב באופן דומה, אשר מייצר מדיניות אופטימלית. ע"י יישום נומרי התקבלו סוגי מדיניות אופטימלית עבור $m = 3, 4$; תכנון מטוסי הנוסעים של ימינו הופך הגדלה נוספת של m לחסר תועלת מעשית.

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מגיסטר למדעים

מדעי המחשב

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אוניברסיטת בן-גוריון בנגב

כסלו תשס״ט | באר שבע | דצמבר 2008