Mayer Goldberg Numbers

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October 1999

Abstract

In this paper we analyze the so-called Mayer Goldberg numbers which contain all possible finite information encoded in given bases.

1 Introduction

Consider the following binary number:

\[
\text{mg}(2) = 0.10001101000001010011100111101110000001\ldots
\]

What's special about it? Taking a closer look, we discover its underlying structure:

\[
\text{mg}(2) = 0.1\underbrace{00}_{\text{chunk 1}}\underbrace{01}_{\text{chunk 2}}\underbrace{10 11}_{\text{chunk 3}}\underbrace{000 001 010 011}_{\text{chunk 4}}\underbrace{100 101 110 111}_{\text{chunk 5}}\ldots
\]

This number, when encoded in binary, is a concatenation of all words over \{0, 1\} in increasing order of their value in base-2. Thus, it "contains" within itself binary encodings of all possible deterministic structures.

2 Computation

In order to represent \text{mg}(2) algebraically, we shall compute it via summation. For this purpose the following formula will prove helpful:

\[
\sum_{k=1}^{m} k x^k = \frac{(mx - m - 1)x^{m+1} + x}{(x - 1)^2}
\] (2.1)

It immediately follows from derivation of \(\sum_{k=0}^{m} x^k\) by \(x\).

Let us count the digits in \text{mg}(2) from left to right, starting with 0. Then, the index of the leftmost digit in chunk \(n \geq 1\) is:

\[
\text{offs}(n) = \sum_{k=1}^{n-1} k 2^k = (n - 2)2^n + 2
\] (2.2)
Now \( mg(2) \) can be rewritten as

\[
mg(2) = 2^{-offs(1)} \cdot 0.1 \\
+ 2^{-offs(2)} \cdot 0.010111 \\
+ 2^{-offs(3)} \cdot 0.00101011001101111 \\
\ldots
\]

\[
= \sum_{n=1}^{\infty} \frac{\text{chunk}(n)}{2^{offs(n)}} \\
= \frac{1}{4} \sum_{n=1}^{\infty} \frac{\text{chunk}(n)}{2^{(n-2)2^n}}
\]

Where

\[
\text{chunk}(n) = \frac{1}{2^{n-1}} \sum_{k=1}^{2^n-1} k2^{-nk}
\]

\[
= \frac{1}{2^{n-1}} \sum_{k=1}^{2^n-1} k(2^{-n})^k
\]

\[
= \frac{(2^{-n})2^n((2^n-1)2^{-n} - 2^n) + 2^{-n}}{(2^n - 1)^2} \quad \text{by (2.1)}
\]

\[
= 2 \cdot \frac{2^{-n} - (2^n + 2^{-n} - 1)2^{-n2^n}}{2^n + 2^{-n} - 2}
\]

Finally resulting in

\[
mg(2) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{-n} - (2^n + 2^{-n} - 1)2^{-n2^n}}{(2^n + 2^{-n} - 2)2^{(n-2)2^n}}
\] (2.5)

First 51 digits of \( mg(2) \) in decimal representation:

\[
mg(2) = 0.5527742345589730476839735242380246111017997632137\ldots
\]

3 Generalization

Mayer Goldberg number for base-2 can be easily generalized for other bases:

\[
mg(\beta) = \beta^{1 - \frac{\beta}{\beta - 1}} \sum_{n=1}^{\infty} \frac{\beta^{-n} - (\beta^n + \beta^{-n} - 1)\beta^{-n2^n}}{(\beta^n - 1)^2} \quad \text{(3.1)}
\]

For example, the number for base-4, corresponding to all possible information encoded on a 4-hole punch-tape is:

\[
mg(4) = 0.42194444444444435373914553438965151883420879\ldots
\]

Note that \( mg(3) > mg(2) \).
4 Further Exploration

Here we shall ask how many such numbers are — how many numbers, when represented in base $\beta$, possess the property that they “contain” all words in \{0, \ldots, \beta - 1\}.

Easy to see, for each base $\beta \geq 2$ there are $\aleph_1$ such numbers. This can be shown by transposing words in the chunks of $m_g(\beta)$, for example.

However, in my opinion it is very unlikely that most of the numbers maintain this property.\footnote{Like they do with probability of 1, for example, in respect to equal representation of all digits in any base.} I don’t even think that this property is shared over bases in general, except for special cases like $\beta_1 = k_1 \beta_2$. 