A Difficulty in the Concept of Social Welfare

KENNETH J. ARROW

Oren Roth - Ben-Gurion University, Beer-Sheva, Israel
1 Introduction
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3 Conclusion
Definition of the problem

- We passed couple of weeks in our seminar, let say we want to take all the lectures we saw already...
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- We passed couple of weeks in our seminar, let say we want to take all the lectures we saw already...
- and give some order on them (preferences)
So we will ask all the students in the seminar (and Paz as well) to give us an order on all of the lectures.
Definition of the problem

- So we will ask all the students in the seminar (and Paz as well) to give us an order on all of the lectures.
- We want some mechanism to take all those orders and combine it to one aggregate order.
Definition of the problem

We will want to keep some basic rules:
Definition of the problem

We will want to keep some basic rules:
- Unanimity

Example:

Student1:  >  >  
Student2:  >  >  
Student3:  >  >  
aggregate  >  >  >
Definition of the problem

We will want to keep some basic rules:

- Unanimity
- Independence of irrelevant alternatives
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The problem Formalization

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Formalization
Assumption: $n$ voters give their complete ranking on set $A$ of alternatives
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- Input to the aggregator/voting rule is $(<1, <2, \ldots, <n)$
Assumption: $n$ voters give their complete ranking on set $A$ of alternatives

- $L$ the set of linear orders on $A$ (permutation).
- Each voter $i$ provides $<_i$ in $L$.
- Input to the aggregator/voting rule is $(<_1, <_2, \ldots, <_n)$.
- Our Goal: find a function $W : L^n \to L$, (called a social welfare function) that aggregates voters preference into a common order.
Every Social Welfare Function $W$ over a set $A$ of at least 3 candidates, If it satisfies:
Arrows Impossibility Theorem

Every Social Welfare Function $W$ over a set $A$ of at least 3 candidates, if it satisfies:

- **Unanimity**
  
  $W(<, <, \ldots, <) = <$
  
  for all $<$ in $L$
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- **Independence of irrelevant alternatives**
  Let $(<_1, <_2, \ldots, <_n)$ and $(<_1', <'_2, \ldots, <'_n)$ s.t.
  $$W(<_1, <_2, \ldots, <_n) = <$$
  and
  $$W(<_1', <'_2, \ldots, <'_n) = W'$$
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  if $\forall i \ a <_i b \iff a <'_i b$ therefore
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Every Social Welfare Function $W$ over a set $A$ of at least 3 candidates, if it satisfies:

- **Unanimity**
  \[ W(<, <, \ldots, <) = < \]
  for all $<$ in $L$

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  Let $(<_1, <_2, \ldots, <_n)$ and $(<_1', <_2', \ldots, <_n')$ s.t.
  \[ W(<_1, <_2, \ldots, <_n) = < \text{ and } W(<'_1, <'_2, \ldots, <'_n) = W' \]
  if $\forall i \ a <_i b \iff a <'_i b$ therefore
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- **Then it is dictatorial:**
Every Social Welfare Function $W$ over a set $A$ of at least 3 candidates, If it satisfies:

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  $W(<, <, \ldots, <) = <$
  
  for all $<$ in $L$

- **Independence of irrelevant alternatives**
  
  Let $(<_1, <_2, \ldots, <_n)$ and $(<_1', <_2', \ldots, <_n')$ s.t.
  
  $W(<_1, <_2, \ldots, <_n) = <$ and $W(<>_1', <_2', \ldots, <'n) = W'$
  
  if $\forall i \ a <_i b \iff a <'_i b$ therefore
  
  $a < b \iff a' < b$

- **Then it is dictatorial:**
  
  there exists a voter $i$ where
  
  $W(<_1, <_2, \ldots, <_n) = <_i$
Every Social Welfare Function $W$ over a set $A$ of at least 3 candidates, if it satisfies:

- **Unanimity**
  
  $W(\prec, \prec, \ldots, \prec) = \prec$
  
  for all $\prec$ in $L$

- **Independence of irrelevant alternatives**

  Let $(\prec_1, \prec_2, \ldots, \prec_n)$ and $(\prec'_1, \prec'_2, \ldots, \prec'_n)$ s.t.
  
  $W(\prec_1, \prec_2, \ldots, \prec_n) = \prec$ and $W(\prec'_1, \prec'_2, \ldots, \prec'_n) = W'$

  if $\forall i \ a \prec_i b \iff a \prec'_i b$ therefore

  $a \prec b \iff a' \prec b$

- **Then it is dictatorial:**

  there exists a voter $i$ where

  $W(\prec_1, \prec_2, \ldots, \prec_n) = \prec_i$

  for all $\prec_1, \prec_2, \ldots, \prec_n$ in $L$
Proof
Questions?
Conclusion and open problems

Thanks