OUTLINE

Introduction
History and basic concepts
Fuzzy sets and fuzzy logic
Fuzzy clustering
Fuzzy inference
Fuzzy systems
Application examples
"So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality."

Albert Einstein. Geometry and Experience
Fuzzy Logic

Mathematical formalism for representing imprecise knowledge in a human–like way.

- Knowledge representation
- Mathematically formal
- Deals with imprecision
some history...

Multi–valued logic

Jan Lukasiewicz. Three–valued logic (1920)
Other multi–valued logics:
Kurt Gödel, John von Neumann, Donald Kleene

Proto–fuzzy Sets

Max Black. Vagueness, a logic analysis (1937)

Fuzzy Logic

Lotfi Zadeh. Fuzzy Sets (1965)
Sentence: It is possible that tomorrow will rain
Negation: It is possible that tomorrow will not rain

Paradoxes: This sentence is false. True or false?
Falakros paradox: (belongs to the "sorites paradox" class)

Would you describe a man with one hair on his head as bald? Yes.
Would you describe a man with two hairs on his head as bald? Yes.
You must refrain from describing a man with ten thousand hairs on his head as bald, so where do you draw the line?
Max Black’s sets vs actual fuzzy sets

Based on degree of usage

Bald ↔ Number of hairs ↔ Hairy

Based on (subjective) degree of truth

Carlos Andrés Peña–Reyes
Logic Systems Laboratory – Swiss Federal Institute of Technology Lausanne
Lotfi Zadeh: Father of modern fuzzy logic (1965)

Key concept: Partial membership ($\mu$)

<table>
<thead>
<tr>
<th>Height</th>
<th>Crisp</th>
<th>Fuzzy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.60</td>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>1.65</td>
<td>0</td>
<td>0.06</td>
</tr>
<tr>
<td>1.70</td>
<td>0</td>
<td>0.22</td>
</tr>
<tr>
<td>1.75</td>
<td>1</td>
<td>0.50</td>
</tr>
<tr>
<td>1.80</td>
<td>1</td>
<td>0.78</td>
</tr>
<tr>
<td>1.85</td>
<td>1</td>
<td>0.94</td>
</tr>
<tr>
<td>1.90</td>
<td>1</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Fuzzy numbers and fuzzy quantities

... about 4 kg of beans ...

... between 5 and 8 weeks ...

As they assign a membership value $\mu$ to a given real value $x$, they are called membership functions

Carlos Andrés Peña–Reyes
Logic Systems Laboratory – Swiss Federal Institute of Technology Lausanne
Fuzzy-set operations

Emptiness:
Sets with no members (i.e. $\forall x, \mu(x) = 0$)

Complement:
How much do items not belong

Containment:
What groups belong to other groups

Intersection:
How much are items in BOTH sets

Union:
How much are items in EITHER sets
Containment: What groups belong to other groups

i.e. each element on the subset belong to the larger set

\[ B \subseteq A \iff \forall x, \mu_B(x) \leq \mu_A(x) \]
Complement: How much do items not belong?

i.e. how far are they from full membership?

\[ \mu_{\overline{F}}(x) = 1 - \mu_F(x) \]
Intersection: How much are items in both sets?

i.e. the degrees of membership both sets share

\[ \mu_{A \cap B}(x) = \min\{ \mu_B(x), \mu_A(x) \} \]
**Union:** How much are items in either set?

\[
\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}
\]
Equivalence between sets and logic

Sets

Membership
John’s height belongs to "Tall"

Complement

Intersection

Union

Logic

Truth
John is tall

Negation

And

Or

Carlos Andrés Peña–Reyes
Logic Systems Laboratory – Swiss Federal Institute of Technology Lausanne
1–D vs 2–D fuzzy logic operations

1–D: Temperature is cool AND warm
1–D vs 2–D fuzzy logic operations (2)

2–D: Temperature is Cool AND Pressure is High
Temperature is Warm and Pressure is Ok
Fuzzy Clustering

Cluster analysis: To partition a given set of data into clusters having the following properties:

Intra–cluster homogeneity
Data in the same cluster, as similar as possible

Inter–cluster heterogeneity
Data in different clusters, as different as possible

Carlos Andrés Peña–Reyes
Logic Systems Laboratory – Swiss Federal Institute of Technology Lausanne
How do humans explain their decisions?

* By enumerating some (partially) fullfilled conditions

Why did you choose this car?

– It is big enough for my family, but not too much...
– Its gas consumption is not high, but the car is still fast...
– I prefer clear colors, but not white...
– This mark is well known...
How do humans explain their decisions?

* By proposing some (fuzzy) behavior rules

How do you decide to push or to release your car’s accelerator?

Release if car is fast(er than desired) and speed is constant or rising,
push if car is slow(er) and speed is constant or slowing down;
else do nothing.

<table>
<thead>
<tr>
<th>Slowing</th>
<th>Fast</th>
<th>Ok</th>
<th>Slow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slow</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>Slow</td>
<td>P</td>
<td>N</td>
<td>P</td>
</tr>
<tr>
<td>Constant</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>Rising</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>Rising</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>Rising</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
</tbody>
</table>
Linguistic or "fuzzy" variables

Name: The "thing" being qualified (e.g. speed)

Labels: "Linguistic" values or adjectives (e.g. slow, fast)

Membership functions: assign membership values to real, measured values.
Fuzzy Inference System

- Knowledge base
  - Database
  - Rule base
- Fuzzifier
- Inference Engine
- Defuzzifier
Inverted Pendulum Problem

State of the System

\[ S = \{ x, v, \theta, w \} \]

Forces:

\[ F = \{-10, 10\} \]
Inverted Pendulum Control: Rules

IF angle is zero AND angular velocity is zero THEN speed is zero
IF angle is zero AND angular velocity is pos_low THEN speed is pos_low
IF angle is neg_low AND angular velocity is zero THEN speed is neg_low

<table>
<thead>
<tr>
<th>speed</th>
<th>NH</th>
<th>NL</th>
<th>Z</th>
<th>PL</th>
<th>PH</th>
</tr>
</thead>
<tbody>
<tr>
<td>NH</td>
<td>NH</td>
<td>NL</td>
<td>Z</td>
<td>PL</td>
<td>PH</td>
</tr>
<tr>
<td>NL</td>
<td>NL</td>
<td>Z</td>
<td>PL</td>
<td>PH</td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td>NH</td>
<td>NL</td>
<td>Z</td>
<td>PL</td>
<td>PH</td>
</tr>
<tr>
<td>PL</td>
<td>Z</td>
<td>PL</td>
<td>PH</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PH</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Carlos Andres Pena Reyes
Inverted Pendulum Control: Variables

Input 1: Angular velocity

Input 2: Angle

Output: Speed

Carlos Andres Pena Reyes
Inverted Pendulum Control: Input measures
Inverted Pendulum Control: Inference 1 (Fuzzyfication)

IF angle is zero AND angular velocity is zero THEN speed is zero

We realize that our actual value belongs to the fuzzy set "zero" to a degree of 0.75.
Inverted Pendulum Control: Inference 2 (Fuzzyfication)

IF angle is zero AND angular velocity is zero THEN speed is zero.

We realize that our actual value belongs to the fuzzy set "zero" to a degree of 0.75.

We realize that our actual value belongs to the fuzzy set "zero" to a degree of 0.4.
Inverted Pendulum Control: Inference 3 (Implication)

IF angle is zero AND angular velocity is zero THEN speed is zero
Inverted Pendulum Control: Inference 4 (Aggregation)
Inverted Pendulum Control: Inference 5 (Defuzzyfication)
Inverted Pendulum Control: Response
Fuzzy Inference System

Knowledge base

Database
Rule base

Fuzzifier
Inference Engine
Defuzzifier

Carlos Andrés Peña-Reyes
Logic Systems Laboratory - Swiss Federal Institute of Technology Lausanne
3 types of fuzzy rule consequents

IF angle is zero AND angular velocity is pos_low THEN speed is pos_low

Mamdani–type: membership functions

Sugeno–type: linear function of inputs

Singleton–type constant values

pos_low = 0.1*angle + 0.2* velocity + 10

pos_low = 10
Fuzzy temperature controller

Carlos Andrés Peña–Reyes
Logic Systems Laboratory – Swiss Federal Institute of Technology Lausanne
Mamdani-type fuzzy temperature controller

Carlos Andrés Peña-Reyes
Logic Systems Laboratory - Swiss Federal Institute of Technology Lausanne
Sugeno-type fuzzy temperature controller

PB = -0.1T + 1
PS = -0.01T + 0.5
Z = 0
NS = -0.01T - 0.1P - 0.5
NB = -0.1T - 0.5P - 1

Carlos Andrés Peña-Reyes
Logic Systems Laboratory - Swiss Federal Institute of Technology Lausanne
Singleton-type fuzzy temperature controller

Carlos Andrés Peña-Reyes
Logic Systems Laboratory - Swiss Federal Institute of Technology Lausanne
Comparison among the three systems

Carlos Andrés Peña-Reyes

Logic Systems Laboratory - Swiss Federal Institute of Technology Lausanne