Weak $\frac{1}{r}$-nets for moving points.*

Alexandre Rok[1] and Shakhar Smorodinsky †[2]

1 **Department of Mathematics, Ben-Gurion University of the NEGEV, Be'er Sheva 84105, Israel.**
rok@math.bgu.ac.il

2 **Department of Mathematics, Ben-Gurion University of the NEGEV, Be'er Sheva 84105, Israel.**
shakhar@math.bgu.ac.il

Abstract

In this paper, we extend the weak $\frac{1}{r}$-net theorem to a kinetic setting where the underlying set of points is moving polynomially with bounded description complexity. We establish that one can find a kinetic analog N of a weak $\frac{1}{r}$-net of cardinality $O(r^{\frac{d(d+1)}{2}}\log^d r)$ whose points are moving with coordinates that are rational functions with bounded description complexity. Moreover, each member of N has one polynomial coordinate.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems, G.2.1 Combinatorics, G.2.2 Graph Theory

Keywords and phrases Hypergraphs, Weak ϵ-nets.

Digital Object Identifier 10.4230/LIPIcs.SoCG.2016.59

1 Introduction and Preliminaries

This paper deals with weak $\frac{1}{r}$-nets for convex sets. It is a central notion in discrete geometry. We initiate the study of kinetic weak $\frac{1}{r}$-nets, and extend the classical weak $\frac{1}{r}$-net theorem to a kinetic setting. Our main motivation is the recent result of De Carufel et al. [4] on kinetic hypergraphs.

Before presenting our results, we need a few definitions and well known facts: A pair $(X, \mathcal{S})$, where $\mathcal{S} \subset P(X)$, is called a *set system* or a *hypergraph*. A subset $A \subset X$ is called *shattered* if $\mathcal{S}|_A = 2^A$. The largest size of a shattered subset from X with respect to $\mathcal{S}$ is called the VC-*dimension* of $(X, \mathcal{S})$. The concept of VC-dimension has its roots in statistics. It first appeared in the paper of Vapnik and Chervonenkis in [10]. Nowadays, this notion plays a key role in learning theory and discrete geometry. Given a set system $(X, \mathcal{S})$, we say that $Y \subset X$ is a *strong* $\frac{1}{r}$*-net* if for each $S \in \mathcal{S}$ with $|S| > |X|/r$ we have $S \cap Y \neq \emptyset$. Based on the concept of VC-dimension, Haussler and Welzl provided a link to strong nets by proving that any set system with VC-dimension d has a strong $\frac{1}{r}$-net of size $O(dr\log r)$ [7].

The intersection of all convex sets contaning $X \subset \mathbb{R}^d$, denoted by conv(X), is called *the convex hull* of X. The *affine hull* of a finite set X, denoted by aff(X), is the intersection of all affine subspaces containing X. It is well known that $\text{aff}(X) = \{\sum_{i=1}^n \alpha_i x_i : \sum_{i=1}^n \alpha_i = 1 \text{ and } x_i \in X\}$. A set of points $X = \{x_1, \ldots, x_n\}$ is said to be affinely independent if for each $1 \leq i \leq n$ we have $x_i \notin \text{aff}(X \setminus \{x_i\})$. We refer to the convex hull of an affinely independent

* Work was partially supported by Grant 1136/12 from the Israel Science Foundation
† Work by this author was partially supported by Swiss National Science Foundation Grants 200020144531 and 200021-137574.

© Alexandre Rok and Shakhar Smorodinsky;
licensed under Creative Commons License CC-BY

set S as $(|S|-1)$-*dimensional simplex spanned by* S. A simplex S is *spanned by* P if it arises from some subset of P.

1.1 Weak $\frac{1}{r}$-nets

We now study the notion of weak $\frac{1}{r}$-net in a kinetic setting. Let us first recall the concept of weak $\frac{1}{r}$-net in the static case.

▶ **Definition 1 (Weak $\frac{1}{r}$-net).** *Let $P \subset \mathbb{R}^d$ be a finite set of points and $r \geq 1$. A set $N \subset \mathbb{R}^d$ is said to be a weak $\frac{1}{r}$-net for P if every convex set containing $> \frac{1}{r}|P|$ points of P also contains a point of N.*

The following theorem is one of the major milestones in modern discrete geometry:

▶ **Theorem 2 (Weak $\frac{1}{r}$-net Theorem [1, 5, 8]).** *Let $r \geq 1$ and $d \geq 1$ an integer. Then there exists a least integer $f(r,d)$ such that for every finite set $P \subset \mathbb{R}^d$ there is a weak $\frac{1}{r}$-net of size at most $f(r,d)$.*

The existence of $f(r,d)$ was first proved by Alon et al. [1] with the bounds $f(r,2) = O(r^2)$ and $f(r,d) = O(r^{(d+1)(1-\frac{1}{s_d})})$ for $d \geq 3$, where s_d tends to 0 exponentially fast. Later, better bounds on $f(r,d)$ for $d \geq 3$ were obtained by Chazelle at al. in [5], who showed that $f(r,d) = O(r^d \log^{b_d} r)$, where b_d is roughly $2^{d-1}(d-1)!$. The current best known upper bound for $d \geq 3$ due to Matoušek and Wagner [8] is $f(r,d) = O(r^d \log^{c(d)} r)$, where $c(d) = O(d^2 \log d)$, and $f(r,2) = O(r^2)$ [1]. The best known lower bound was provided by Bukh, Matoušek, and Nivasch [3], who showed that $f(r,d) = \Omega(r \log^{d-1} r)$ for $d \geq 2$.

Recently, some interesting connections were found between strong and weak nets. In particular, Mustafa and Ray [9] showed how one can construct weak $\frac{1}{r}$-nets from strong $\frac{1}{r}$-nets. They obtained a bound of $O(r^3 \log^3 r)$ in $\mathbb{R}^2$, $O(r^5 \log^5 r)$ in $\mathbb{R}^3$, and $O(r^{d^2} \log^{d^2} r)$ for $d \geq 4$ on the size of weak $\frac{1}{r}$-nets.

A kinetic framework: The problem of finding strong $\frac{1}{r}$-nets has been recently considered in a kinetic setting by De Carufel et al. [4]. Their work and extensive research in the static case motivates us to consider the problem of weak $\frac{1}{r}$-net in a kinetic setting.

Let us define this setting: The dimension $d \geq 1$ is assumed to be fixed. A *moving point* is a function from $\mathbb{R}_+$ to $\mathbb{R}^d \cup \{\emptyset\}$ for some $d \geq 1$. A *point p moving in* $\mathbb{R}^d$ is simply a moving point whose codomain is $\mathbb{R}^d \cup \{\emptyset\}$ and such that $p(t) \in \mathbb{R}^d$ for some $t \geq 0$. In this paper, we are interested in the case where this function is polynomial or rational, i.e., each coordinate is a polynomial or a rational function. If one of the coordinates is not defined for some t, then the moving point is not defined at t. For simplicity, we often use the term *point* for a moving point if there is no confusion. In what follows, the dimension d is assumed to be fixed. For a set P of moving points and a "time" $t \in \mathbb{R}_+$, we denote by $P(t)$ the set $\{p(t) | p \in P\}$. We say that a set P of moving points in $\mathbb{R}^d$ has *bounded description complexity* β if for each point $p(t) = (p_1(t), \ldots, p_d(t))$, each $p_i(t)$ is a rational function with both numerator and denominator having degree at most β.

We say that the function h with domain $\mathbb{R}_+$ is a *moving affine subspace* if for some integer k and any $t \geq 0$, $h(t)$ is an affine subspace of dimension k or the emptyset. In the case $h(t)$ is not always equal to the emptyset, we also say that such a h has *dimension* k. If the dimension is 1 or $d-1$ we refer to the corresponding moving affine subspaces as *moving line* and *moving hyperplane*, respectively. For simplicity, we often write *moving subspace* instead of moving affine subspace. We now introduce some notations to define affine subspaces. We say that $\tilde{h}$ is given by $x_1 = p_1, \ldots, x_k = p_k$ if $\tilde{h} = \{x \in \mathbb{R}^d : \text{for } 1 \leq i \leq k,\ x_i = p_i\}$.

Analogously, we say that a moving affine subspace h is given by $x_1 = p_1, \dots, x_k = p_k$, where each p_i is a point moving in $\mathbb{R}$, if $h(t)$ is given by $x_1 = p_1(t), \dots, x_k = p_k(t)$. Similarly to moving points, if a moving subspace h is given by $x_1 = p_1, \dots, x_k = p_k$, where each p_i is a point moving in $\mathbb{R}$, and $p_i(t)$ is not defined for some $t \geq 0$, then $h(t)$ is not defined.

Finally, for a set $P = \{p_1, \dots, p_n\}$ of points moving in $\mathbb{R}^d$ and a vector space $V \subset \mathbb{R}^d$, we say that $P' = \{p'_1, \dots, p'_n\}$ is a *projection of P onto V* if $p'_i(t) = \text{proj}_V(p_i(t))$ for all $t \geq 0$.

▶ **Definition 3 (Kinetic Weak $\frac{1}{r}$-net).** *Given a set P of n points moving in $\mathbb{R}^d$, we say that a set of moving points N is a kinetic weak $\frac{1}{r}$-net for P if for any $t \in \mathbb{R}_+$ and any convex set C with $C \cap P(t) > n/r$ we have $C \cap N(t) \neq \emptyset$.*

We sometimes abuse the notation and write *net* or *weak net* instead of kinetic weak net. In order to establish our result regarding kinetic weak nets, we need the following natural general position assumption on the set P of moving points: We assume that for any $t \geq 0$ the affine hull of any d-tuple of points in $P(t)$ is a hyperplane, but no $d + 2$ points of $P(t)$ are contained in a hyperplane. The latter can easily be relaxed to no $c(d) \geq d + 2$ points in a hyperplane.

Under these assumptions, we prove the following theorem that could be viewed as a generalization of Theorem 2:

▶ **Theorem 4 (Kinetic Weak $\frac{1}{r}$-net Theorem).** *For every pair of integers $d \geq 1$, β and every $r \geq 1$, there exist a least integer $c(r, d, \beta)$ and $g(d, \beta)$ such that for every finite set P of points moving in $\mathbb{R}^d$ with description complexity β there is a kinetic weak $\frac{1}{r}$-net of cardinality at most $c(r, d, \beta)$ and description complexity $g(d, \beta)$. Moreover, for fixed d and β and $r \geq 2$, we have $c(r, d, \beta) = O(r^{\frac{d(d+1)}{2}} \log^d r)$.*

Furthermore, in the case where the points of P move polynomially, the moving points of the kinetic weak $\frac{1}{r}$-net have one polynomial coordinate. This is an important advantage of our construction as many naturally defined moving points, obtained by intersecting moving affine spaces, have no polynomial coordinates.

2 Weak $\frac{1}{r}$-net in a Kinetic Setting

2.1 Points moving in $\mathbb{R}$

In a kinetic setting, one needs to capture the combinatorial changes occurring with time. The concept of *kinetic hypergraph* defined below was introduced in [4] by De Carufel et al.

▶ **Definition 5 (Kinetic Hypergraph).** *Let P be a set of points moving in $\mathbb{R}^d$ with bounded description complexity and let $\mathcal{R}$ be a set of ranges. We denote by $(P, \mathcal{S})$ the kinetic hypergraph of P with respect to $\mathcal{R}$. Namely, $S \in \mathcal{S}$ if and only if there exists an $R \in \mathcal{R}$ and a "time" $t \in \mathbb{R}_+$ such that $S(t) = R \cap P(t)$. We sometimes abuse the notation, and denote by $(P, \mathcal{R})$ the kinetic hypergraph $(P, \mathcal{S})$.*

Figure 1 illustrates the concept of a kinetic hypergraph for $d = 1$ and $\mathcal{R}$ being the family of intervals. De Carufel et al. [4] also established the following important lemma to investigate strong $\frac{1}{r}$-nets in a kinetic setting.

▶ **Lemma 6 (De Carufel et al. [4]).** *Let $\mathcal{R}$ be a collection of semi-algebraic sets in $\mathbb{R}^d$, each of which can be expressed as a Boolean combination of a constant number of polynomial equations and inequalities of maximum degree c, where c is some constant. Let P be a family of points moving polynomially in $\mathbb{R}^d$ with bounded description complexity. Then the kinetic hypergraph of P with respect to $\mathcal{R}$ has bounded* VC*-dimension.*

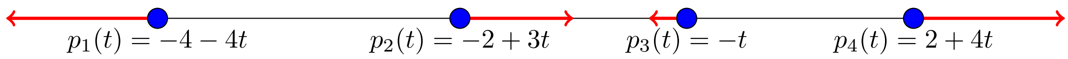

Figure 1 A family $P = \{p_1, p_2, p_3, p_4\}$ of points moving linearly along the the real line. One can easily see that the kinetic hypergraph of P with respect to intervals is $(P, 2^P \setminus \{\{p_1, p_2, p_4\}, \{p_1, p_3, p_4\}, \{p_1, p_4\}\})$.

Unfortunately, this is not enough for our purposes, since we need to assume that the moving points can be described with coordinates which are rational functions. However, by following a similar scheme it is not hard to prove the lemma below:

▶ **Lemma 7.** *Let P be a finite set of points moving in $\mathbb{R}$ with bounded description complexity, and let $\mathcal{K} = (P, \mathcal{S})$ be the kinetic hypergraph of P with respect to intervals. Then the* VC*-dimension of $\mathcal{K}$ is $O(1)$.*

We start by defining the concept of *primal shatter function.*

▶ **Definition 8.** *Let P be a finite set. For a set system $X = (P, \mathcal{S})$ the primal shatter function $\pi_X : \{1, \ldots, |P|\} \to \mathbb{N}$ is defined by*

$$\pi_X(m) = \max_{A \subset P:\ |A|=m} |\{A \cap S : S \in \mathcal{S}\}|.$$

First, we establish a link between the primal shatter function and the VC-dimension of a set system. The lemma below is folklore:

▶ **Lemma 9.** *Let P be a finite set, and $X = (P, \mathcal{S})$ be a set system such that $\pi_X(m) \leq cm^k$ (for $k \geq 2$ say), where c is some constant. Furthermore, let d be the* VC*-dimension of X. Then $d = O(k \log k)$.*

Proof. If $d = 0$, then there is nothing to show. Otherwise, $\pi_X(d)$ is defined, and we easily see that $c \geq 2$ since $\pi_X(1) = 2$. Hence, by the definition of the primal shatter function and the lower bound on c, the following inequalities are satisfied $2^d \leq cd^k \leq (cd)^k$. This implies that $d \leq k \log cd$. Obviously, there is a $c' > 0$ (depending only on c) such that

$$c' d^{\frac{1}{2}} \leq \frac{d}{\log cd} \leq k.$$

Hence,

$$d \leq k \log \frac{c}{c'^2} k^2 = k \log \frac{c}{c'^2} + 2k \log k = O(k \log k).$$

◀

By some pretty elementary arguments one can establish the lemma below.

▶ **Lemma 10.** *Let P be a set of $n \geq 1$ points moving in $\mathbb{R}$ with bounded description complexity β. Then the number of hyperedges in the kinetic hypergraph $\mathcal{K} = (P, \mathcal{S})$ with respect to intervals is at most $c_\beta n^4$ for some $c_\beta > 0$.*

It is easy to see that the bound on the number of hyperedges above is also valid for induced hypergraphs having at least one vertex. Consider an induced hypergraph $(X, \mathcal{S}|_X)$ of $(P, \mathcal{S})$, and let $A = S \cap X$ be a hyperdge of $(X, \mathcal{S}|_X)$ arising from some $S \in \mathcal{S}$. By definition, there is an interval $[a, b]$ and a $t \geq 0$ such that $P(t) \cap [a, b] = S(t)$. We now show that $[a, b] \cap X(t) = A(t)$. Clearly, $A(t) \subset [a, b]$ otherwise for some $a \in A$ we have $a(t) \notin S(t)$

implying $a \notin S$, hence $A(t) \subset [a,b] \cap X(t)$. Let us prove that $[a,b] \cap X(t) \subset A(t)$. Take an $x(t) \in X(t) \cap [a,b]$, then clearly $x \in S$ implying $x \in S \cap X = A$, so $x(t) \in A(t)$.

This proves that the induced hypergraph $(X, \mathcal{S}|_X)$ is contained in the kinetic hypergraph of X with respect to intervals, hence the bound of Lemma 10 holds for induced hypergraphs that have at least one vertex.

Proof of Lemma 7. The lemma is an immediate corollary of Lemma 9 combined wirh Lemma 10 and the reasoning above. ◀

Together with the well known strong $\frac{1}{r}$-net theorem mentioned in Section 1, Lemma 7 implies:

▶ **Lemma 11.** *Let P be a finite set of points moving in $\mathbb{R}$ with bounded description complexity. Then the kinetic hypergraph of P with respect to intervals has a strong $\frac{1}{r}$-net (for $r \geq 2$ say) of size $O(r \log r)$.*

For technical reasons, we need the two lemmas above without any general position assumption. Hence, for any $t \geq 0$ more than two moving points from P can coincide at t. Later on, we shall use Lemma 11 in order to find weak $\frac{1}{r}$-nets in a kinetic setting.

2.2 Points moving in $\mathbb{R}^d$

The proof of Theorem 12 below is inspired by a construction from Chazelle et al. [6].

The arguments we use are also valid when the set P consists of points with bounded description complexity. However, as explained in the first section, when the motion is polynomial the construction we present has an important feature: One coordinate is a polynomial. In particular, when $d = 2$ the construction below gives a kinetic weak $\frac{1}{r}$-net N of size only $O(r^3 \log^2 r)$ and the first coordinate of each point in N is a polynomial. Note that in the static setting, the best known upper bound on the function $f(r, 2)$, defined in Section 1, is $O(r^2)$, so our bound is only an $O(r \log^2 r)$ factor of it.

We recall the general position assumption made in Section 1: Given a set of moving points P in $\mathbb{R}^d$, for any $t \geq 0$ the affine hull of any d-tuple of points in $P(t)$ is a hyperplane, and no $d + 2$ points of $P(t)$ are contained in a hyperplane.

▶ **Theorem 12 (Weak $\frac{1}{r}$-net in a Kinetic Setting).** *Let P be a set of n points moving polynomially in $\mathbb{R}^d$ with bounded description complexity β. Then there exists a kinetic weak $\frac{1}{r}$-net (for $r \geq 2$ say) N of size $O(r^{\frac{d(d+1)}{2}} \log^d r)$ and bounded description complexity. Moreover, the first coordinate of each point of N is a polynomial.*

Proof. The case $d = 1$ is implied by Lemma 11, so we can assume that $d \geq 2$. The method below works for $n \geq cr$, where c is a sufficiently large constant whose existence is proved later. If $n < cr$, then the theorem holds trivially, since one defines the kinetic weak $\frac{1}{r}$-net to be P.

We start by defining N and other structures we need throughout the proof. Later, we show that N is indeed a kinetic weak $\frac{1}{r}$-net for P. The claims regarding the size and the description complexity of N will follow easily from its definition. First, we need to introduce the concept of *moving subspace of step j* for $1 \leq j \leq d$. It will be some specific moving subspace of dimension $d - j$. Moreover, a moving subspace of step $i + 1$ arises from some moving subspace of step i, hence these structures will be defined iteratively. In what follows, we use parameters $\lambda_1, \ldots, \lambda_d$ with $0 < \lambda_i \leq 1$, whose values are specified later.

Call the projection of P onto x_1-axis P_1. Note that P_1 has description complexity β. Choose a strong $\frac{\lambda_1}{r}$-net N_1 for the kinetic hypergraph of P_1 with respect to intervals. Lemma

11 guarantees that one can select N_1 with $|N_1| \leq b_1 r/\lambda_1 \log r/\lambda_1$, where b_1 depends on β. For each point p of N_1, we consider the moving hyperplane such that at any $t \geq 0$ it is orthogonal to x_1-axis and passes through $p(t)$. The moving affine subspaces of step 1 are exactly these moving hyperplanes arising from N_1.

The construction of moving subspaces of step at least 2 is more involved. Assume that we have constructed the moving affine subspaces up to step j satisfying $1 \leq j \leq d-1$. For each moving subspace h of step j, we define F_h to be the set consisting of moving points $p^{h,X}$ for all $(j+1)$-tuples X of P. The position of $p^{h,X}$ at $t \geq 0$ is given by $p^{h,X}(t) = \text{aff}(X(t)) \cap h(t)$ if this intersection contains a single point. A moving point $p^{h,X}$ is not necessarily uniquely defined, but this not a problem for our purposes. One can define it with description complexity $f(j+1)$ for some increasing function $f : \{1, \dots, d\} \to \mathbb{N}$ such that $f(1) = \beta$. The technical proof of this fact is provided later in Lemma 17.

Next, for each moving subspace h of step j call the projection of F_h onto x_{j+1}-axis P_h. Note that P_h also has description complexity $f(j+1)$. Choose a strong $\frac{\lambda_{j+1}}{r^{j+1}}$-net N_h for the kinetic hypergraph of P_h with respect to intervals. Again, Lemma 11 ensures that one can select N_h with $|N_h| \leq b_{j+1} r^{j+1}/\lambda_{j+1} \log r^{j+1}/\lambda_{j+1}$, where b_{j+1} depends on $f(j+1)$.

If N_h consists of $q_1, \dots, q_s$, then the moving affine subspaces of step $j+1$ *induced* by h are $\tilde{h}_i$ given by $x_1 = x_{h,1}, \dots,\ x_j = x_{h,j},\ x_{j+1} = q_i$ for $1 \leq i \leq s$, where $x_{h,k}$ is the moving point giving the k-th coordinate of h. The set of moving subspaces of step $j+1$ is the union of moving subspaces induced by h among all moving subspaces h of step j.

We define the kinetic weak $\frac{1}{r}$-net N to be the union of the moving subspaces of step d. This makes sense, since the moving subspaces of step d have each coordinate specified by some function, so those are moving points. The size of N is at most

$$b_1 \frac{r}{\lambda_1} \log \frac{r}{\lambda_1} b_2 \frac{r^2}{\lambda_2} \log \frac{r^2}{\lambda_2} \dots b_d \frac{r^d}{\lambda_d} \log \frac{r^d}{\lambda_d} = O(r^{\frac{d(d+1)}{2}} \log^d r).$$

Moreover, for each $v = (v_1, \dots, v_d)$ of N, the moving point v_i has description complexity $f(i)$. Since f is an increasing function, the moving point v has description complexity $f(d)$.

We start by briefly outlining main ideas of the proof for $d \geq 3$. The case $d = 2$ is much easier, and does not require the inductive step presented below.

Let $t \geq 0$ and let C be a convex set containing $> n/r$ points of $P(t)$. We start by showing that if one chooses an appropriate value for λ_1, then for some moving subspace h of step 1 the set $h(t)$ intersects "a lot" of segments spanned by $C \cap P(t)$.

Next, the inductive step comes. We assume that λ_i were defined up to some $1 \leq j \leq d-2$, and some moving subspace h of step j (of dimension $d-j$) is such that $h(t)$ intersects a "large" number of j-simplices spanned by $C \cap P(t)$. We start finding a static affine subspace s contained in $h(t)$ of dimension $d-j-1$ such that s intersects a "large" number of $(j+1)$-simplices spanned by $C \cap P(t)$. These $(j+1)$-simplices are obtained from the j-simplices intersecting $h(t)$. Then we show that with an appropriate choice of λ_{j+1}, there are two moving subspaces h_1, h_2 of step $j+1$ induced by h such that $h_1(t)$ and $h_2(t)$ are "close" to s, and therefore at least one of them also intersects a "large" number of $(j+1)$-simplices spanned by $C \cap P(t)$, which completes the inductive step.

This way, we establish that one can define λ_i for $1 \leq i \leq d-1$, so that for some moving line l of step $d-1$ "a lot" of $(d-1)$-simplices spanned by $C \cap P(t)$ are intersected by $l(t)$. In particular, from the definition of F_l the segment $C \cap l(t)$ is such that for "many" moving points $p \in F_l$ the point $p(t)$ belongs to it. Hence, the projection of $C \cap l(t)$ (call it I) onto x_d-axis leads to a "heavy" hyperedge in the kinetic hypergraph of P_l with respect to intervals (because P_l is the projection of F_l). For an appropriate choice of λ_d, there is a point q of the net N_l such that $q(t)$ must be in I. Finally, by construction of N the moving point whose first $d-1$ coordinates are given by l and the last one by q is in N, so $q(t)$ is in C and we are done.

We now proceed with a detailed proof. Let us show that the set N we defined is indeed a kinetic weak $\frac{1}{r}$-net for P for an appropriate choice of λ_i.

Let $t \geq 0$ and let C be any convex set containing at least n/r points from $P(t)$. It is sufficient to assume that C contains exactly n/r points of $P(t)$ (we choose any n/r points of $C \cap P(t)$, and disregard the remaining ones). We will define the parameters λ_i so that C must contain a point of $N(t)$. It is important to notice that these parameters do not depend on C or t.

For technical reasons, for $1 \leq j \leq d-1$ we also prove the existence of $\gamma_j n^{j+1}/r^{j+1}$ j-simplices spanned by $C \cap P(t)$ and intersecting $h(t)$ for some moving subspace h of step j exactly once in their relative interior, where $\gamma_j > 0$ are iteratively defined later. Clearly, this implies for each simplex above that the affine hull of the $j+1$ points of $P(t)$ spanning it intersects $h(t)$ exactly once as well. In particular, if $h(t)$ intersects $\gamma_j n^{j+1}/r^{j+1}$ j-simplices spanned by $C \cap P(t)$ once in their relative interior, then for at least $\gamma_j n^{j+1}/r^{j+1}$ points $p \in F_h$ we have $p(t) \in C \cap h(t)$. This implication is crucial for our purposes, and will be used in order to prove that N is a kinetic net once the parameters λ_i are specified.

We prove the existence of γ_j and define λ_j for $1 \leq j \leq d-1$ by induction. Then, we define λ_d and show that the values λ_j imply that N is a kinetic weak $\frac{1}{r}$-net.

▶ **Lemma 13.** *If $\lambda_1 = 1/4$ and $n > 4r(2d+2)$, then there exists a moving hyperplane h of step 1 such that $h(t)$ intersects at least $n^2/16r^2$ segments spanned by $C \cap P(t)$ once and in their relative interior.*

Proof. Among moving subspaces $\tilde{h}$ of step 1 with the property that $> n/4r$ points of $C \cap P(t)$ have a strictly smaller x_1-coordinate than the intersection of $\tilde{h}(t)$ with x_1-axis, choose a moving space with the smallest intersection point with x_1-axis at t and denote it by h. The moving subspace h exists, since the above defined set of moving subspaces is easily seen to be nonempty. Indeed, let z be the largest real such that at most $n/4r$ points of $C \cap P(t)$ have their x_1-coordinate in $]-\infty, z[$. Then, since from the general position assumption at most $d+1$ points of $C \cap P(t)$ can share the same x_1-coordinate, we deduce that there are at least $n/r - n/4r - d - 1 > n/4r$ points of $C \cap P(t)$ whose x_1-coordinate is in $]z, \infty[$. Since N_1 is a strong $\frac{1}{4r}$-net for the kinetic hypergraph of P_1 with respect to intervals, there should be a point $w \in N_1$ such that $w(t) \in]z, \infty[$ implying the existence of a moving subspace of step 1 whose intersection with x_1-axis at t is $w(t)$. Hence, the above set of moving subspaces is indeed nonempty, so h exists.

Let x denote the intersection point of $h(t)$ with x_1-axis. We now show that we also have at least $n/4r$ points of $C \cap P(t)$ having a strictly bigger x_1-coordinate than x. Indeed, using one more time the hypothesis that no $d+2$ points are contained in a hyperplane, we deduce that the number of points of $C \cap P(t)$ having their x_1-coordinate smaller or equal to x is at most $2n/4r + 2(d+1) < 3n/4r$. To see this, let $\tilde{h}(t)$ be a predecessor of $h(t)$, i.e., a moving hyperplane of step 1 at t whose intersection point with x_1-axis is the biggest one among those having an intersection point with x_1-axis strictly smaller than $h(t)$. The existence of such a hyperplane is again implied by the the definition of N_1. Indeed, we have $> n/4r$ points $p \in P_1$ such that $p(t) \in]-\infty, x[$, so there is a point $w \in N_1$ such that $w(t) \in]-\infty, x[$. Thus, there is a moving hyperplane of step 1 with its x_1-coordinate equal to $w(t)$, which implies the existence of $\tilde{h}(t)$. Similarly, by our choice of $\tilde{h}(t)$, it is easily seen that at most $n/4r$ points of $C \cap P(t)$ are strictly between $h(t)$ and $\tilde{h}(t)$. In summary, by the choice of h, at most $n/4r$ points of $C \cap P(t)$ have their x_1-coordinate strictly smaller than the intersection of $\tilde{h}(t)$ with

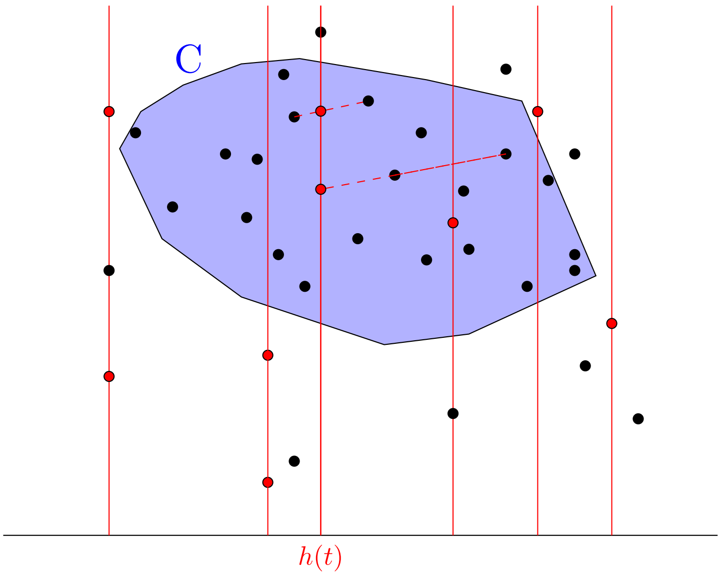

Figure 2 The elements from $P(t)$ and elements of $N(t)$ are black and red dots, respectively. The red lines are moving affine subspaces of step 1 at t. The line $h(t)$ splits $C \cap P(t)$ into two parts of cardinality $> n/4r$. At least one point from the net N induced by h must be in C at t.

x_1-axis, at most $n/4r$ are strictly between $h(t)$ and $\tilde{h}(t)$, and at most $d+1$ points lie on each of $h(t)$, $\tilde{h}(t)$.

Hence, both open halfspaces delimited by $h(t)$ contain $> n/4r$ points of $C \cap P(t)$. This means that at least $n^2/16r^2$ segments spanned by $C \cap P(t)$ intersect $h(t)$ once and in their relative interior, so the lemma follows. ◀

The lemma above implies that if we define $\lambda_1 = 1/4$, then we can set $\gamma_1 = 1/16$. If $d = 2$, then set $\lambda_2 = 1/8$. Indeed, let h be the moving subspace guaranteed by Lemma 13. By definition of F_h, there exist at least $n^2/16r^2 > \binom{n}{2}/8r^2 = |F_h|/8r^2$ points p of F_h such that $p(t) \in C \cap h(t)$. Since P_h is the projection of F_h onto x_2-axis, there exist $> |P_h|/8r^2$ points p' of P_h such that $p'(t)$ belongs to the projection of the segment $C \cap h(t)$ onto x_2-axis. Hence, since N_h is a strong $\frac{1}{8r^2}$-net for the kinetic hypergraph of P_h with respect to intervals, the projection of $C \cap h(t)$ onto x_2-axis must contain a point $v(t)$ of $N_h(t)$. By definition of N, the moving point $q = (x_{h,1}, v)$ is in N, so $C \cap N(t) \neq \emptyset$ and the case $d = 2$ follows, see Figure 2 for an illustration. Hence, one can assume that $d \geq 3$.

In higher dimensions the analysis requires more effort. We need the following lemma implicitly established by Chazelle et al. in [6]. For the sake of completeness, the technical proof is postponed to the end of this section.

▶ **Lemma 14 (Chazelle et al. [6]).** *Let $d \geq 3$ and $P \subset \mathbb{R}^d$ be a set of n/r points such that any d points of P are affinely independent. Assume that we have an affine subspace h given by $x_1 = a_1, \ldots, x_j = a_j$ with $1 \leq j \leq d-2$, and a set $\mathcal{S}$ of at least $\alpha_j n^{j+1}/r^{j+1}$ $(j+1)$-tuples of P with $\alpha_j > 0$ such that the corresponding simplices intersect h exactly once. Then given $n \geq 4(j+1)r/\alpha_j$, there is an $\alpha_{j+1} > 0$ and an affine subspace $x_1 = a_1, \ldots, x_j = a_j, x_{j+1} = a_{j+1}$ intersecting at least $\alpha_{j+1} n^{j+2}/r^{j+2}$ $(j+1)$-simplices spanned by $(j+2)$-tuples from P. Moreover, each such $(j+2)$-tuple has the form $\{p_1, \ldots, p_{j+1}\} \cup \{p_1, \ldots p_j, q_1\}$ for $\{p_1, \ldots p_{j+1}\}$, $\{p_1, \ldots p_j, q_1\} \in \mathcal{S}$. Finally, $a_{j+1} \in [\{p_1, \ldots p_{j+1}\}, \{p_1, \ldots p_j, q_1\}]$, where by* *abuse of notation* $\{p_1, \ldots p_{j+1}\}$ *is the projection of the intersection point of the corresponding* j*-simplex with* h *onto* x_{j+1}*-axis.*

Assume that we have defined λ_i, γ_i for $i \leq j$, where $1 \leq j \leq d-2$. Let h be a moving subspace of step j such that at least $\gamma_j n^{j+1}/r^{j+1}$ j-simplices spanned by $C \cap P(t)$ intersect $h(t)$ once in their relative interior. Let us assume that $n \geq 4(j+1)r/\gamma_j$. In what follows, we use the same notation as in the statement of Lemma 14. By this lemma (used with $\alpha_j = \gamma_j$, the affine subspace $h(t)$, and the set of points $C \cap P(t)$), we get a point a_{j+1} contained in at least $\alpha_{j+1}n^{j+2}/r^{j+2}$ intervals $[\{p_1(t), \ldots p_{j+1}(t)\}, \{p_1(t), \ldots p_j(t), q_1(t)\}]$ as in the statement of Lemma 14. This is true, because we distinguish two intervals that do not arise from the same pair of $(j+1)$-tuples. We sometimes refer to the projection $\{p_1(t), \ldots, p_{j+1}(t)\}$ as a vertex.

Set $J = \{x_{\tilde{h},j+1}(t) : \tilde{h}$ is a moving subspace induced by $h\}$. We recall that $x_{\tilde{h},j+1}(t)$ is the $j+1$-th coordinate of $\tilde{h}(t)$. Let y_1 be the biggest $a \in J$ smaller or equal to a_{j+1} (if no such a exists, take $-\infty$). Similarly, let y_2 be the smallest $a \in J$ bigger or equal to a_{j+1}(if no such a exists, take ∞). The following lemma shows that by an appropriate choice of λ_{j+1}, not many intervals as above can lie strictly between y_1 and y_2.

▶ **Lemma 15.** *If* $\lambda_{j+1} = 2\alpha_{j+1}/3(j+1)$*, then at most* $\alpha_{j+1}n^{j+2}/3r^{j+2}$ *intervals as above are contained in* $]y_1, y_2[$ *on* x_{j+1}*-axis.*

Proof. By contradiction, assume that $\geq \alpha_{j+1}n^{j+2}/3r^{j+2}$ intervals are contained in $]y_1, y_2[$. In what follows, we distinguish two vertices arising from different $(j+1)$-tuples. Counted with multiplicities, there are at least $2\alpha_{j+1}n^{j+2}/3r^{j+2}$ vertices $\{p_1(t), \ldots, p_{j+1}(t)\}$ in $]y_1, y_2[$. Each vertex $\{p_1(t), \ldots, p_{j+1}(t)\}$ is counted at most $(j+1)n/r$ times, since there are at most $j+1$ choices of $\{p_{i_1}(t), \ldots, p_{i_j}(t)\} \subset \{p_1(t), \ldots, p_{j+1}(t)\}$ and at most n/r choices for $q(t)$ so that $[\{p_1(t), \ldots, p_{j+1}(t)\}, \{p_{i_1}(t), \ldots, p_{i_j}(t), q(t)\}]$ is an interval as above. Hence, there are at least $\geq 2\alpha_{j+1}n^{j+1}/3(j+1)r^{j+1}$ distinct vertices in $]y_1, y_2[$, a contradiction with the value of λ_{j+1}. To see this, we recall that each vertex $\{p_1(t), \ldots, p_{j+1}(t)\}$ is the projection of $p^{h,\{p_1,\ldots,p_{j+1}\}}(t)$ onto x_{j+1}-axis for $p^{h,\{p_1,\ldots,p_{j+1}\}} \in F_h$. Since the number of vertices $\{p_1(t), \ldots, p_{j+1}(t)\}$ in $]y_1, y_2[$ is at least $\geq 2\alpha_{j+1}n^{j+1}/3(j+1)r^{j+1}$, the number of $p^{h,\{p_1,\ldots,p_{j+1}\}} \in F_h$ such that the projection of $p^{h,\{p_1,\ldots,p_{j+1}\}}(t)$ onto x_{j+1}-axis is in $]y_1, y_2[$ is obviously also $\geq 2\alpha_{j+1}n^{j+1}/3(j+1)r^{j+1}$. Hence, by definition of P_h the number of $p \in P_h$ such that $p(t) \in]y_1, y_2[$ is at least

$$\frac{2\alpha_{j+1}n^{j+1}}{3(j+1)r^{j+1}} = \frac{\lambda_{j+1}n^{j+1}}{r^{j+1}} > \frac{\lambda_{j+1}\binom{n}{j+1}}{r^{j+1}} = \frac{\lambda_{j+1}|P_h|}{r^{j+1}}.$$

Thus, since N_h is a strong $\frac{\lambda_{j+1}}{r^{j+1}}$-net for the kinetic hypergraph of P_h with respect to intervals, there should be a point $w \in N_h$ such that $w(t)$ is in $]y_1, y_2[$. This means that there is a moving affine subspace induced by h whose x_{j+1}-coordinate at t $w(t)$ is strictly between y_1 and y_2, which contradicts the definition of y_1 or y_2. ◀

Let us set $\lambda_{j+1} = 2\alpha_{j+1}/3(j+1)$. By the pigeonhole principle and the lemma above, y_1 or y_2 belongs to at least $\alpha_{j+1}n^{j+2}/3r^{j+2}$ intervals as above (say w.l.o.g. y_1). Let us denote by h_1 a moving subspace induced by h such that the x_{j+1}-coordinate of $h_1(t)$ is y_1. Thus, at least $\alpha_{j+1}n^{j+2}/3r^{j+2}$ $(j+1)$-simplices spanned by $C \cap P(t)$ intersect $h_1(t)$. One needs to be careful, since some of these simplices may intersect $h_1(t)$ more than once or not in their relative interior. However, assuming that $n \geq c_{\alpha_j/3}r$, where $c_{\alpha_j/3}$ is as in Lemma 16, one can apply this lemma to conclude that at least $\alpha_{j+1}n^{j+2}/6r^{j+2}$ of them intersect $h_1(t)$ only once and in their relative interior. Hence, setting $\gamma_{j+1} = \alpha_{j+1}/6$ completes the induction.

Note that we still need to define λ_d. Let us set $\lambda_d = \gamma_{d-1}$. It remains us to see that the resulting N is a kinetic weak $\frac{1}{r}$-net for P. From the definition of $\gamma_{d-1} = \lambda_d$, we know that some affine subspace $h(t)$ where h is a moving space of step $d-1$, i.e., a moving line of step $d-1$, must intersect at least $\lambda_d n^d/r^d > \lambda_d \binom{n}{d}/r^d = \lambda_d |F_h|/r^d$ $(d-1)$-simplices spanned by $C \cap P(t)$ once in their relative interior. By definition of F_h, this implies that there exist $> \lambda_d |F_h|/r^d$ points p of F_h such that $p(t)$ belongs to the segment $C \cap h(t)$. Since P_h is the projection of F_h onto x_d-axis, there exist $> \lambda_d |P_h|/r^d$ points p' of P_h such that $p'(t)$ belongs to the projection of the segment $C \cap h(t)$ onto x_d-axis. Hence, since N_h is a strong $\frac{\lambda_d}{r^d}$-net for the kinetic hypergraph of P_h with respect to intervals, the projection of $C \cap h(t)$ onto x_d-axis must contain a point $v(t)$ of $N_h(t)$. By definition of N, the moving point $q = (x_{h,1}, \ldots, x_{h,d-1}, v)$ is in N and obviously belongs to C. Thus, N is a kinetic weak $\frac{1}{r}$-net for P, and the theorem follows.

◀

We now establish the remaining technical lemmas.

▶ **Lemma 16.** *Let $1 \leq j \leq d-1$ and $P \subset \mathbb{R}^d$ be a set of n/r points such that no $d+2$ of them lie in a hyperplane. Assume that we have a set $\mathcal{S}$ of $\alpha n^{j+1}/r^{j+1}$ $(j+1)$-tuples from P such that the convex hull of each of them intersects a given affine subspace V of dimension $d-j$. Then there exists c_α such that if $n \geq c_\alpha r$, then there are at least $\alpha n^{j+1}/2r^{j+1}$ $(j+1)$-tuples from $\mathcal{S}$ such that their convex hulls intersect V exactly once and in their relative interior.*

Proof. We can assume that $\alpha > 0$, otherwise there is nothing to show. Assume that the convex hulls of at least $\alpha n^{j+1}/2r^{j+1}$ $(j+1)$-tuples from $\mathcal{S}$ intersect the affine subspace V more than once or on their relative boundary. We will show that for $n \geq c_\alpha r$, where c_α is large enough, we obtain a contradiction. When the convex hull of a $(j+1)$-tuple A intersects V more than once, one can take two intersection points x_1 and x_2 with the affine subspace V and follow the line passing through x_1, x_2 until the relative boundary of conv(A) is intersected. Hence, since the line through x_1,x_2 is in V, in both cases the relative boundary of conv(A) must be intersected. Clearly, this means that there is a subset of j points from A whose convex hull intersects V. Each such j-tuple can be counted at most n/r times. Hence, there are at least $\alpha n^j/2r^j$ distinct j-tuples arising from elements of $\mathcal{S}$ as above.

We define $\mathcal{S}_j$ to be the set of j-tuples above, i.e., those whose convex hulls intersect V. Set $\gamma_j = \alpha/2$. If $j \geq 2$, then in order to obtain a contradiction we consider the following iterative procedure. Assume that $\mathcal{S}_i$ was defined for some $2 \leq i \leq j$ and contains at least $\gamma_i n^i/r^i$ i-tuples whose convex hulls intersect V. We say that $\mathcal{S}_i$ is *good* if it has a subset of at least $\gamma_i n^i/2r^i$ i-tuples, denoted by $\mathcal{G}_i$, such that the convex hull of no $(i-1)$-tuples which are $(i-1)$-subsets of the i-tuples from $\mathcal{G}_i$ intersects the affine space V. Otherwise, we say that the set $\mathcal{S}_i$ is *bad*, and define $\mathcal{S}_{i-1}$ to be the set of $(i-1)$-tuples whose convex hulls intersect V and each of them is contained in some i-tuple from $\mathcal{S}_i$. Clearly, the size of S_{i-1} is at least $\gamma_i n^{i-1}/2r^{i-1}$, since an $(i-1)$-tuple can appear in at most n/r i-tuples of S_i. Finally, we set $\gamma_{i-1} = \gamma_i/2$. For some i the procedure must stop with a good $\mathcal{S}_i$. Indeed, if we had to compute $\mathcal{S}_1$, then this means that we have a set of points from P of cardinality at least $\gamma_1 n/r$ such that each point belongs to V. This means that for n large enough $(n \geq (d+2)r/\gamma_1)$, we get a set of at least $d+2$ points contained in V. That is, an affine subspace of dimension at most $d-1$, a contradiction.

Hence, we can assume that S_i is good for some $i \geq 2$. Let $\mathcal{G}_i$ be as above. Define a graph G whose vertices are the different $(i-1)$-tuples each contained in some i-tuple from $\mathcal{G}_i$. For each i-tuple from $\mathcal{G}_i$ choose two different $(i-1)$ subsets and connect them by an edge. The number of edges is at least $\gamma_i n^i/2r^i$, since an edge determines the i-tuple it arises from.

Clearly, there is a vertex of degree at least $\gamma_i n^i/2r^i\binom{n/r}{i-1} \geq \gamma_i n/2r$. Take one such $(i-1)$-tuple $\{p_1,\dots,p_{i-1}\}$. This means that the affine space given by aff$(V,p_1,\dots,p_{i-1})$ of dimension at most $d-1$ contains at least $i-1+\gamma_i n/2r$ points, i.e., $p_1,\dots,p_{i-1}$ and the points of the union of all neighbours of $\{p_1,\dots,p_{i-1}\}$ in G. Indeed, let p be the intersection point of conv$(\{p_1,\dots,p_i\}$ with V, where p_i belongs to some neighbour of $\{p_1,\dots,p_{i-1}\}$ in G. We show that aff$(\{p_1,\dots,p_{i-1},p\})$=aff$(\{p_1,\dots,p_{i-1},p_i\})$. If p_i is in aff$(\{p_1,\dots,p_{i-1}\})$, then the equality is clear. If not, then aff$(\{p_1,\dots,p_{i-1},p\})$ has dimension strictly bigger than aff$(\{p_1,\dots,p_{i-1}\})$ while being contained in aff$(\{p_1,\dots,p_{i-1},p_i\})$, so the equality holds. Hence, for n large enough $(n \geq (d+1)2r/\gamma_i)$ we get a contradiction, since strictly more than $d+2$ points are in the affine subspace aff$(\{V,p_1,\dots,p_{i-1}\})$ whose dimension is at most $d-1$, in particular, the points are contained in a hyperplane.

◀

▶ **Lemma 17.** *Let P be a set of points moving polynomially in $\mathbb{R}^d$ with bounded description complexity β. Let $\{p_1,\dots,p_{j+1}\}$ be a $(j+1)$-tuple from P and h some moving affine subspace of step j, as defined in the proof of Theorem 12. Then one can define a moving point p such that for each $t \geq 0$ when the intersection of* aff$(\{p_1(t),\dots,p_{j+1}(t)\})$ *and $h(t)$ is a single point, it is equal to $p(t)$. Moreover, p has description complexity $f(j+1)$, where $f:\{1,\dots,d\} \to \mathbb{N}$ is some increasing function with $f(1)=\beta$.*

Proof. The case where for each $t \geq 0$ the intersection of aff$(\{p_1(t),\dots,p_{j+1}(t)\})$ and $h(t)$ is empty or contains more than one point is trivial, since one can define p to be static.

Hence, one can assume that for some $t \geq 0$ the intersection above contains a single point. We prove the lemma by induction on the step. Observe that the function defining the first coordinate of a moving subspace of step i is obtained by projection of some point from P, hence has description complexity $\beta = f(1)$.

Asssume that the lemma holds for moving points arising from moving subspaces of step at most $j-1$, where $0 \leq j-1 \leq d-2$. Let $p_1,\dots,p_{j+1}$ be any $(j+1)$-tuple of points from P and h any moving subspace of step j and given by $x_1 = x_{h,1},\dots,x_j = x_{h,j}$. Then it follows from the definition of $x_{h,i}$ (see Theorem 12), the induction hypothesis, and the observation above that $x_{h,i}$ has description complexity $f(i)$. Assume $h(t)$ and aff$(\{p_1(t),\dots,p_{j+1}(t)\})$ intersect in a unique point $p(t)$. Then we can write $p(t) = \alpha_1(t)p_1(t)+\ldots+\alpha_{j+1}(t)p_{j+1}(t)$ and from the general position assumption the points $p_1(t),\dots,p_{j+1}(t)$ are affinely independent, so a point of aff$(\{p_1(t),\dots p_{j+1}(t)\})$ is uniquely determined by an affine combination of the points $p_i(t)$. An immediate consequence from the unicity of $\alpha_i(t)$ is the following matricial equivalence:

$$\begin{pmatrix} [p_1(t)]_1 & \dots & [p_{j+1}(t)]_1 \\ \vdots & & \vdots \\ [p_1(t)]_j & \dots & [p_{j+1}(t)]_j \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1(t) \\ \vdots \\ \vdots \\ \alpha_{j+1}(t) \end{pmatrix} = \begin{pmatrix} x_{h,1}(t) \\ \vdots \\ x_{h,j}(t) \\ 1 \end{pmatrix}$$

$$\iff$$

$$\begin{pmatrix} \alpha_1(t) \\ \vdots \\ \vdots \\ \alpha_{j+1}(t) \end{pmatrix} = \begin{pmatrix} [p_1(t)]_1 & \dots & [p_{j+1}(t)]_1 \\ \vdots & & \vdots \\ [p_1(t)]_j & \dots & [p_{j+1}(t)]_j \\ 1 & \dots & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_{h,1}(t) \\ \vdots \\ x_{h,j}(t) \\ 1 \end{pmatrix}$$

It follows from the Cramer's rule that the moving point α_i, whose position at any $t \geq 0$ is $\alpha_i(t)$ given by the equation above, has description complexity depending only on j and $f(j)$. Hence, the moving point p whose position at t is $\alpha_1(t)p_1(t) + \ldots + \alpha_{j+1}(t)p_{j+1}(t)$ also has description complexity depending only on j and $f(j)$ that we denote by $f(j+1)$ (w.l.o.g. $f(j+1) \geq f(j)$). This completes the proof. ◀

Proof of Lemma 14. Define the hypergraph on P whose hyperedges are the different $(j+1)$-tuples of $\mathcal{S}$. Iteratively remove a j-tuple A from $\binom{n/r}{j}$ and remove the $(j+1)$-tuples containing it from $\mathcal{S}$ if the number of the remaining elements from $\mathcal{S}$ containing A is at most $\alpha_j n^{j+1}/2r^{j+1}\binom{n/r}{j}$. Call $\mathcal{S}'$ the remaining set of $(j+1)$-tuples. This procedure cannot remove more than $\alpha_j n^{j+1}/2r^{j+1}$ hyperedges, so the resulting hypergraph is not empty and each j-tuple contained in some element from $\mathcal{S}'$ is contained in

$$> \frac{\alpha_j n^{j+1}}{2r^{j+1}\binom{n/r}{j}} \geq \frac{\alpha_j n}{2r} = \frac{\alpha' n}{r}$$

elements from $\mathcal{S}'$, where we set $\alpha' = \alpha_j/2$.

We now project the intersections of simplices corresponding to $(j+1)$-tuples from $\mathcal{S}'$ with h onto the x_{j+1}-axis. For the sake of simplicity, the projection of the intersection point induced by the tuple $\{p_1, \ldots, p_{j+1}\}$ will still be denoted by $\{p_1, \ldots, p_{j+1}\}$. Two projections $\{p_1, \ldots, p_{j+1}\}$ and $\{q_1, \ldots, q_{j+1}\}$ give an interval of *type 1* if there is a sequence $\{p_1, \ldots, p_{j+1}\}$, $\{p_1, \ldots, p_j, q_1\}, \ldots,$ $\{q_1, \ldots, q_{j+1}\}$, where each member of the sequence is an element of $\mathcal{S}'$ and the points $p_1, \ldots, p_{j+1}, q_1, \ldots, q_{j+1}$ are all distinct.

The following procedure gives a lower bound on the number of such intervals (we distinguish two intervals arising from different pairs of $(j+1)$-tuples): Choose any $\{p_1, \ldots, p_{j+1}\}$ in $\mathcal{S}'$. Take any q_1 such that $\{p_1, \ldots, p_j, q_1\}$ is in $\mathcal{S}'$ with q_1 different from p_{j+1}. Then take any q_2 such that q_2 is different from p_j, p_{j+1} and $\{p_1, \ldots, p_{j-1}, q_1, q_2\}$ is in $\mathcal{S}'$ etc. The lower bound below follows

$$\frac{|\mathcal{S}'|(\alpha' n/r - j - 1)^{j+1}}{2(j+1)!} \geq \frac{|\mathcal{S}'|(\alpha' n/2r)^{j+1}}{2(j+1)!}$$

given $\alpha' n/2r \geq j+1$. Indeed, starting from $\{p_1, \ldots, p_{j+1}\}$ an interval $[\{p_1, \ldots, p_{j+1}\}$, $\{q_1, \ldots, q_{j+1}\}]$ is counted at most once for each permutation of $q_1, \ldots, q_{j+1}$. Thus from the one dimensional selection lemma, see [2], we know that there exists a point a_{j+1} contained in at least

$$\frac{|\mathcal{S}'|^2[(\alpha' n/2r)^{j+1}/2(j+1)!]^2}{4|\mathcal{S}'|^2} = \frac{1}{4}\frac{[(\alpha' n/2r)^{j+1}]^2}{[2(j+1)!]^2} = \frac{\alpha'' n^{2j+2}}{r^{2j+2}}$$

intervals, where we set $\alpha'' = \alpha'^{2j+2}/2^{2j+6}[(j+1)!]^2$.

Clearly, if a point is contained in an interval $[\{p_1, \ldots, p_{j+1}\}, \{q_1, \ldots, q_{j+1}\}]$, it must also be contained in some interval $[\{p_1, \ldots p_s, q_1, \ldots q_{j-s+1}\}, \{p_1, \ldots p_{s-1}, q_1, \ldots q_{j-s+2}\}]$. This latter kind of intervals is refered to as *type 2*. Moreover, an interval of type 2 can be counted at most $(j+1)(jn/r)^j$ times. Indeed, there are at most $j+1$ possible positions for such an interval in a chain as above (used to define type 1 intervals), at most j possibilities of choosing a point that is replaced in a $(j+1)$-tuple while a subchain is extended, and at most n/r candidates to replace such a point. Hence, a_{j+1} is contained in at least $\alpha'' n^{2j+2}/r^{2j+2}(j+1)(jn/r)^j = \alpha''' n^{j+2}/r^{j+2}$ intervals of type 2, where $\alpha''' = \alpha''/(j+1)j^j$.

Each interval of type 2 containing the point a_{j+1} corresponds to a $(j+1)$-simplex spanned by P intersecting the affine subspace given by $x_1 = a_1, \ldots, x_{j+1} = a_{j+1}$. Finally, it is easy to see that a spanned $(j+1)$-simplex arises from at most $(j+2)(j+1)$ intervals of type 2.

Hence, there exist at least $\alpha''' n^{j+2}/(j+2)(j+1)r^{j+2}$ $(j+1)$-simplices arising from intervals of type 2 pierced by a_{j+1}, and the lemma follows. ◀

3 Open problems

This paper naturally leads to some questions. Can we restrict ourselves to points moving polynomially in order to find a kinetic net? More precisely:

▶ **Problem 1.** *Let $d \geq 2, \beta$ be integers and $r \geq 1$. Is there a pair $c(d, \beta, r)$, $g(d, \beta)$ such that for any finite set P of points moving polynomially with bounded description complexity β in $\mathbb{R}^d$ there exists a kinetic weak $\frac{1}{r}$-net for P of cardinality at most $c(d, \beta, r)$ and description complexity $g(d, \beta)$ whose points move polynomially?*

Let $d \geq 1, \beta$ be fixed integers and $c(d, \beta, r)$ be as in theorem 4. We didn't prove any lower bound on $c(d, \beta, r)$, so the current best lower bounds coincide with those in the static case which are $\Omega(r \log^{d-1} r)$, see [3]. This leads to the following research direction.

▶ **Problem 2.** *Close the gap between the lower and upper bounds on $c(d, \beta, r)$.*

References

1. N. Alon, I. Bárány, Z. Füredi, and D.J. Kleitman. Point selections and weak e-nets for convex hulls. *Combinatorics, Probability & Computing*, 1:189–200, 1992.
2. B. Aronov, B. Chazelle, and H. Edelsbrunner. Points and triangles in the plane and halving planes in space. *Discrete & Computational Geometry*, 6:435–442, 1991. URL: `http://dx.doi.org/10.1007/BF02574700`, `doi:10.1007/BF02574700`.
3. B. Bukh, J. Matoušek, and G. Nivasch. Lower bounds for weak epsilon-nets and stair-convexity. *Israel Journal of Mathematics*, 182(1):199–228, 2011.
4. J.L. De Carufel, M. Katz, M. Korman, A. van Renssen, M. Roeloffzen, and S. Smorodinsky. On kinetic range spaces and their applications. *CoRR*, abs/1507.02130, 2015. URL: `http://arxiv.org/abs/1507.02130`.
5. B. Chazelle, H. Edelsbrunner, M. Grigni, L.J. Guibas, M. Sharir, and E. Welzl. Improved bounds on weak epsilon-nets for convex sets. *Discrete & Computational Geometry*, 13:1–15, 1995.
6. B. Chazelle, H. Edelsbrunner, M. Grigni, L.J. Guibas, M. Sharir, and E. Welzl. Improved bounds on weak epsilon-nets for convex sets. *Discrete & Computational Geometry*, 13:1–15, 1995. URL: `http://dx.doi.org/10.1007/BF02574025`, `doi:10.1007/BF02574025`.
7. D. Haussler and E. Welzl. epsilon-nets and simplex range queries. *Discrete & Computational Geometry*, 2:127–151, 1987. URL: `http://dx.doi.org/10.1007/BF02187876`, `doi:10.1007/BF02187876`.
8. J. Matoušek and U. Wagner. New constructions of weak epsilon-nets. *Discrete & Computational Geometry*, 32(2):195–206, 2004.
9. N.H. Mustafa and S. Ray. Weak epsilon-nets have basis of size o(1/epsilon log(1/epsilon)) in any dimension. *Comput. Geom.*, 40(1):84–91, 2008. URL: `http://dx.doi.org/10.1016/j.comgeo.2007.02.006`, `doi:10.1016/j.comgeo.2007.02.006`.
10. A. Ya. Chervonenkis V. N. Vapnik. On the uniform convergence of relative frequencies of events to their probabilities. *Theory of Probability and its Applications*, 16:264–280, 1971.