On vertex rankings of graphs and its relatives

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ABSTRACT

A vertex ranking of a graph is an assignment of ranks (or colors) to the vertices of the graph, in such a way that any simple path connecting two vertices of equal rank, must contain a vertex of a higher rank. In this paper we study a relaxation of this notion, in which the requirement above should only hold for paths of some bounded length l for some fixed l. For instance, already the case $l = 2$ exhibits quite a different behavior than proper coloring. We prove upper and lower bounds on the minimum number of ranks required for several graph families, such as trees, planar graphs, graphs excluding a fixed minor and degenerate graphs.

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1. Introduction

A *vertex-ranking* of a graph $G = (V, E)$ with k colors, is a k-coloring $c : V \to [k]$ such that for any $u, v \in V$, if $c(u) = c(v)$, then any simple path between u, v contains a vertex w with $c(w) > c(u)$ (note that this implies that the coloring is proper). The minimum k such that G admits a vertex-ranking with k colors is called the *vertex ranking chromatic number* (abbreviated as vr-number), and is denoted by $\chi_{\mathrm{vr}}(G)$. In this paper we introduce a natural relaxation of vertex-ranking to the case where the above requirement holds only for paths of length bounded by l. Already for the case $l = 2$ this notion becomes interesting and non-trivial:

Definition 1.1. Let $G = (V, E)$ be a simple graph. A *unique-superior* coloring (abbreviated as *us-coloring*) of G with k colors is a function $c : V \to [k]$, such that c is a proper coloring, and furthermore, for every length two simple path (u, v, w), if $c(u) = c(w)$, then $c(v) > c(u)$. The minimum k for which a graph G admits a us-coloring with k colors is called the *unique-superior chromatic number* of G (abbreviated *us-number*), and is denoted by $\chi_{\mathrm{us}}(G)$.

More generally, in an *l-vertex ranking* of G with k colors, every simple path of length at most l between two vertices of the same color contains a vertex of higher color. The minimum k for which G admits an l-vertex ranking is called the *l-vr-number* of G, and is denoted by $\chi_{\mathrm{l\text{-}vr}}(G)$.

Note that trivially we have:

$$\chi(G) \le \chi_{\mathrm{us}}(G) \le \chi_{\mathrm{vr}}(G).$$

The vr-number is a well studied notion, with both theoretical and practical applications. It has been used for VLSI layout design, Cholesky matrix factorization and scheduling assembly problems (see, e.g., [16,18,20]). Recently, Even and Smorodinsky [11] showed a strong connection between vertex ranking of graphs and some online hitting set problems

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on graphs. Specifically, they showed that the competitive ratio of the online hitting set problem for subsets inducing a connected subgraph of a given graph $G = (V, E)$ equals the vr-number of the graph G. Efficient polynomial time algorithms to determine the vr-number of graphs have been found for several families of graphs: Schäffer showed this for trees [22] and Deugun et al. for permutation graphs [9]. For other families of graphs, such as chordal graphs, bipartite and co-bipartite graphs, to name a few, the problem was shown to be NP-hard [10,4]. Nevertheless, computing the vertex ranking of a graph can be done in $\approx 1.96^n$ time [12] (where $n = |V|$), it can also be approximated up to a factor of $O(\log^2 n)$ in polynomial time [5], and admits a linear time algorithm whenever the vr-number is constant [21] (i.e., it is fixed parameter tractable).

As mentioned above, for every graph G, $\chi(G) \leq \chi_{us}(G) \leq \chi_{vr}(G)$. However, these parameters can be arbitrarily far apart already for bipartite graphs. Indeed, consider for example the family of trees. It is a well known fact that every tree T is bipartite i.e., $\chi(T) = 2$. However, it is not a difficult exercise to show that for every path P of length n we have $\chi_{vr}(P) = \Omega(\log n)$. See e.g., [17,23]. More generally, consider the family of planar graphs. By the famous Four-Color Theorem [3] we have $\chi(G) \leq 4$ for every planar graph G. Katchalski et al. [17] proved that for any planar graph G with n vertices, $\chi_{vr}(G) = O(\sqrt{n})$. They also showed that this bound is best possible, i.e., that for any integer n there is a planar graph G with n vertices such that $\chi_{vr}(G) = \Omega(\sqrt{n})$. Their proof extends to any fixed minor-free family. That is, a family of graphs excluding a fixed subgraph H as a minor (see below for details). Another family of graphs for which the chromatic number and the vr-number can be far apart, is the family of 2-degenerate graphs (or more generally k-degenerate graphs). A graph is called d-degenerate if every subgraph has a vertex of degree at most d. It is well known and an easy fact to prove that a d-degenerate graph is always $d + 1$-colorable. However, as is shown in this paper, for any integer n there are 2-degenerate graphs G with n vertices such that even their us-number is arbitrarily large.

1.1. Related notions of coloring

The notion of vertex-ranking is in itself a special case of the more general notion of *unique-maximum* coloring of hypergraphs.[1] A *unique-maximum* coloring (abbreviated *um-coloring*) of a hypergraph H is a coloring of its vertices such that for every hyperedge $e \in E$, the highest color of a vertex of e is unique in e. Then, for a given graph $G = (V, E)$, a vertex ranking of G is exactly a um-coloring of the hypergraph $H = (V, E')$, where E' is the family of all subsets of vertices that form a simple path in G.

A slightly different variant of the notion of um-coloring is that of *conflict-free coloring*. This notion is a relaxation of um-coloring. In a conflict-free coloring of a hypergraph, the restriction is that for every hyperedge there is some color that occurs exactly once in that hyperedge. This coloring has been studied in many settings for the last decade, see the survey [23] and the references therein.

There are other types of coloring that have been studied in the literature, which are more relaxed than us-coloring, yet stricter than proper coloring. Two such colorings are *acyclic coloring* and *star coloring*. An acyclic coloring of a graph G is a proper coloring, such that for every cycle of G, at least three colors appear in the cycle. Equivalently, it is a proper coloring such that every subgraph induced by the union of two color classes is acyclic. A star coloring of a graph G is a proper coloring, for which every path of length 4 in G uses at least 3 colors. Equivalently, it is a proper coloring such that every subgraph induced by the union of two color classes is a collection of stars. The star chromatic number and the acyclic chromatic number of G, $\chi_s(G)$ and $\chi_a(G)$, respectively, are defined in the obvious manner. It is easy to see that we have the following hierarchy: let G be a graph, then $\chi(G) \leq \chi_a(G) \leq \chi_s(g) \leq \chi_{us}(g) \leq \chi_{vr}(G)$. Grünbaum [14] showed that, for every planar graph G, $\chi_a(G) \leq 9$, and conjectured that $\chi_a(G) \leq 5$ for every planar graph, a conjecture which Borodin [6] later proved (this bound is known to be tight). A few years later, Nešetřil and Ossona de Mendez [8] showed that the acyclic chromatic number is bounded for every minor-excluding family. Albertson et al. [2] proved that for every graph G: $\chi_s(G) \leq 2\chi_a(G)^2 + \chi_a(G)$, which means that the star chromatic number is also bounded for graphs excluding a fixed minor.

1.2. Our results

We provide lower and upper bounds on the us-number and the l-vr-number of several graph families. For trees we show a tight bound of $\Theta(\log n / \log \log n)$ on the us-number, where n is the number of vertices. Thus already for trees the us-number can be far apart both from the chromatic number and the vr-number. For planar graphs, and more generally, graphs excluding a fixed minor, we show a nearly tight upper bound of $O(\log n)$ on the us-number and $O(l \cdot \log n)$ on the l-vr-number. For the family of d-degenerate graphs we show a lower bound of $\Omega(n^{1/3})$, and an upper bound of $O(\sqrt{n})$ on the us-number.

2. Preliminaries

All graphs mentioned in this paper are simple and undirected. For a graph $G = (V, E)$ we denote by $\Delta(G)$ (respectively, $\delta(G)$) its maximum (resp. minimum) degree. For a subset $V' \subseteq V$, $G[V']$ is the induced subgraph on V', and $G \setminus V'$ is the

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[1] A hypergraph $H = (V, E)$, has vertices V and hyperedges $E \subseteq 2^V$.

graph induced on $V \setminus V'$. For a subgraph H of G, we may write $G \setminus H$ which means $G \setminus V(H)$. For a collection of subgraphs S with $V' = \bigcup_{H \in S} V(H)$, we may abuse notation and write $G \setminus S$ rather than $G \setminus V'$. For a graph $G = (V, E)$, a *subdivision* of an edge $\{u, v\} \in E$ means adding a new vertex w on the edge which is connected only to u and to v. In other words, it is the graph $G' = (V \cup \{w\}, E \cup \{u, w\} \cup \{w, v\} \setminus \{u, v\})$.

Definition 2.1. A graph $G = (V, E)$ is called *d-degenerate* if for every $V' \subseteq V$, $\delta(G[V']) \leq d$.

Let $\mathbb{G}_d$ denote the family of *d-degenerate graphs*. It is easy to see that $\chi(G) \leq d + 1$ for every $G \in \mathbb{G}_d$. The following propositions are well known, we include proofs for completeness.

Proposition 2.2. *Let $G = (V, E) \in \mathbb{G}_d$ be a graph with n vertices. Then there is an ordering of the vertices $(v_1, \ldots, v_n)$, so that for every $1 \leq i \leq n$, the vertex v_i has at most d neighbors v_j with $j < i$.*

Proof. By induction: let $v \in V$ be a vertex of degree at most d. By the induction hypothesis, the vertices of $G \setminus v$ can be ordered $(v_1, \ldots, v_{n-1})$ so that for all $1 \leq i \leq n - 1$, v_i has at most d neighbors to its left. Put $v_n = v$ and obviously the order $(v_1, \ldots, v_n)$ satisfies the desired property. $\square$

Proposition 2.3. *Let $G = (V, E)$ be a graph, $|V| = n$, $|E| = m$, and let $U \subseteq V$ be a subset of vertices of degree at least d. Then $|U| \leq 2m/d$.*

Proof. By summing the degrees of the vertices of G, we have:

$$2m = \Sigma_{v \in V} d(v) \geq \Sigma_{v \in U} d(v) \geq d|U|,$$

so that $|U| \leq 2m/d$. $\square$

The *square* of a graph $G = (V, E)$, denoted by $G^2 = (V, E')$, is the graph on the vertex set V with $\{u, v\} \in E'$ if and only if $d_G(u, v) \in \{1, 2\}$ where $d_G(u, v)$ denotes the length of a shortest path between u and v. Observe that a proper coloring of G^2 is a us-coloring of G (because there are no paths of length two whose end-points may get the same color).

3. The us-number of trees

In this section we focus on the us-number of trees, and provide sharp asymptotic bounds.

Theorem 3.1. *For every tree T on n vertices,*

$$\chi_{us}(T) = O\left(\frac{\log n}{\log \log n}\right).$$

Theorem 3.2. *For any integer n there exists a tree T with n vertices such that $\chi_{us}(T) = \Omega(\frac{\log n}{\log \log n})$.*

3.1. Upper bound

Let $k > 3$ be some integer. It suffices to show that if a tree T with n vertices has us-number k (namely, $\chi_{us}(T) = k$), then $n \geq \Omega((k - 2)!)$. Indeed, if $n \geq (k - 2)!$, it is easy to see (e.g., using Stirling's approximation) that $k = O(\frac{\log n}{\log \log n})$. To this end, define for each $1 \leq i \leq k - 1$ a boolean function p_i on the set of all rooted trees as follows: for a tree T with root r, $p_i(T, r) = 1$ if and only if for every two colors $l \leq i$, $m \leq k - 1$, there is a us-coloring of T with $k - 1$ colors, so that r has color l, and none of its children have color m. Because to have $p_i(T, r) = 1$ it must be that T admits a us-coloring with $k - 1$ colors, we have the following:

Observation 1. *If $\chi_{us}(T) = k$, then $p_i(T, r) = 0$ for every i and every $r \in V(T)$.*

Define the function $f : [k - 1] \to \mathbb{N}$ by

$$f(i) = \min\{n : \text{ there exists a tree } T \text{ on } n \text{ vertices with root } r \in V(T) \text{ such that } p_i(T, r) = 0\}.$$

By Observation 1, if $\chi_{us}(T) = k$, then $n \geq f(k - 1)$, thus proving the following lemma will conclude the proof.

Lemma 3.3. $f(k - 1) > (k - 2)!$.

Proof. We show that for every $1 \leq i \leq k - 3$,
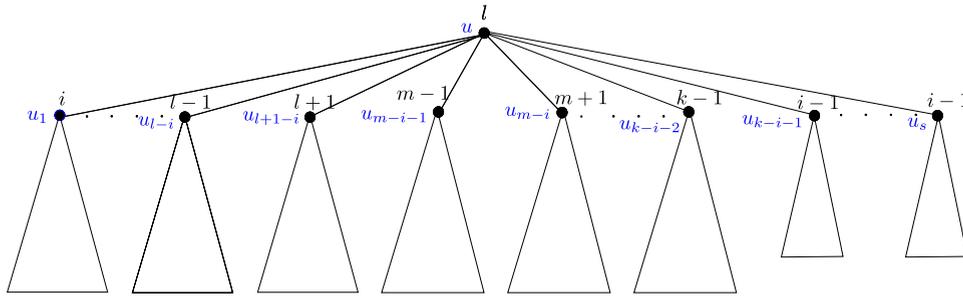
$$f(i) \geq 1 + (k - i - 1)f(i - 1).$$

Fig. 1. $s' = k - i - 2$, and $l \le m$. Not shown in the figure are the grandchildren of u, all of which have colors different from l.
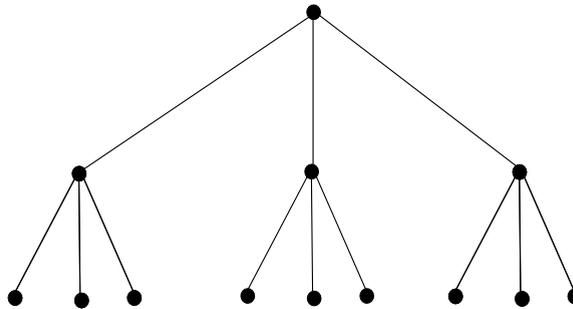

Fig. 2. T_3.

Let T be a tree with root u, such that $p_i(T, u) = 0$ and $|V(T)| = f(i)$. Denote by $u_1, \ldots, u_s$ the children of u, and by T_{u_j} the sub-tree of T rooted in u_j. Seeking contradiction, assume that $f(i) < 1 + (k - i - 1)f(i - 1)$. Then there are at most $k - i - 2$ trees T_{u_j} for which $p_{i-1}(T_{u_j}, u_j) = 0$, since the number of vertices in each such tree is at least $f(i - 1)$. By renaming we may assume that for $1 \le j \le s'$, $p_{i-1}(T_{u_j}, u_j) = 0$, and for $s' + 1 \le j \le s$, $p_{i-1}(T_{u_j}, u_j) = 1$, where $s' \le k - i - 2$.

Fix any $i \le l, m \le k - 1$. We demonstrate that there is a us-coloring c of T with at most $k - 1$ colors, so that $c(u) = l$ and $c(u_j) \ne m$ for all $j \in [s]$. First, for each $j > s'$, as $p_{i-1}(T_{u_j}, u_j) = 1$ there is a coloring c_j of T_{u_j} such that $c_j(u_j) = i - 1$ and $c_j(w) \ne l$ for every child w of u_j. As for $1 \le j \le s'$, we use the fact that T has the minimal number of vertices for which $p_i(T, u) = 0$, thus it must be that $p_i(T_{u_j}, u_j) = 1$. This implies that for every color d out of the color set $A = \{i, \ldots, k - 1\} \setminus \{l, m\}$ there exists a us-coloring c_j of T_{u_j} so that $c_j(u_j) = d$ and $c_j(w) \ne l$ for every child w of u_j. As $|A| = k - i - 2$, we can choose a color $d(j)$ such that $d(j) \ne d(j')$ for all $s' < j < j' \le s$. Finally, color the root u with color l. See Fig. 1 for an illustration. Then it can be verified that the obtained coloring c is a valid us-coloring of T satisfying the constraints imposed by the definition of $p_i(T, r)$. But this means that $p_i(T, u) = 1$, a contradiction. Using that recursive formula, we have $f(k - 3) > (k - 3)! f(1)$. One can check that $f(1) = k - 2$, so $f(k - 3) > (k - 2)!$, and of course also $f(k - 1) > (k - 2)!$. □

3.2. Lower bound

Next, we show that for every n, there exists a tree T on n vertices such that $\chi_{\text{us}}(T) = \Omega(\frac{\log n}{\log \log n})$. Let $T = T_k$ be the *complete k-ary tree with k levels*, i.e., a rooted tree of height $k - 1$, where each non-leaf has exactly k children, and all leaves have distance $k - 1$ from the root. Since $|V(T_k)| = \frac{k^k - 1}{k - 1}$ it follows that $k = \Omega(\frac{\log n}{\log \log n})$. The *level* of a vertex is its distance to the root, so the root is at level 0, its children are at level 1, and the leaves are at level $k - 1$. See Fig. 2 for an illustration with $k = 3$. We shall prove that $\chi_{\text{us}}(T_k) = k = \Omega(\frac{\log n}{\log \log n})$.

Lemma 3.4. *Let c be a us-coloring of T_k with k colors. Then for every $1 \le i \le k$, color i can be used only for vertices from the bottom i levels.*

Proof. The proof is by induction on i. The base case is for $i = 1$, and seeking contradiction, assume that $c(v) = 1$ and v is not a leaf. Note that v has exactly k children $u_1, \ldots, u_k$. Since $c(v) = 1$ and the coloring is proper, then $c(u_l) > 1$, for all $l \in [k]$. Furthermore, $c(u_l) \ne c(u_m)$ for every $1 \le l < m \le k$, since v is a common neighbor of u_l and u_m with the lowest color. Hence, at least $k + 1$ colors are used, a contradiction.

The induction hypothesis is that the statement holds for all $j < i$. Seeking contradiction, assume that $c(v) = i$ for some vertex $v \in V(T_k)$ which is not in the bottom i levels. Denote by $u_1, \ldots, u_k$ the children of v. We know that $c(u_l) > i - 1$ for every $1 \le l \le k$ by the induction hypothesis, since u_l is not in the bottom $i - 1$ levels. Moreover, for every $l \in [k]$,

since c is also a proper coloring then $c(u_l) \neq c(v) = i$, and we conclude that $c(u_l) > i$. This means that $c(u_l) \neq c(v)$ and $c(u_l) \neq c(u_m)$ for every $1 \leq l < m \leq k$. Again, this implies that $k + 1$ colors are used for $v, u_1, \ldots, u_k$, a contradiction. This completes the induction step, proving the lemma. $\square$

Now we are ready to prove that $\chi_{us}(T_k) = k$. It can be checked that $\chi_{us}(T_k) \leq k$, since the coloring which assigns every vertex of level i the color $k - i$ is a us-coloring of T_k. To see that $\chi_{us}(T_k) \geq k$, we note that applying Lemma 3.4 on the root r suggests that it can be assigned only colors which are at least k.

4. Planar graphs

In this section we provide a nearly sharp asymptotic upper-bound on the us-number and the *l*-vr-number of planar graphs.

Theorem 4.1. *For any planar graph G on n vertices, $\chi_{us}(G) = O(\log n)$.*

Before presenting the proof of Theorem 4.1, we give an informal sketch. Given a planar graph $G = (V, E)$, we begin by partitioning the set of vertices into $t = O(\log n)$ independent sets $V = V_1 \cup \cdots \cup V_t$ with the following property: for every $1 \leq i < j \leq t$, any vertex $v \in V_i$ has at most $O(1)$ neighbors in V_j. For each of the sets V_j we shall use a distinct palette of $O(1)$ colors, such that V_i will have smaller colors than V_j for all $1 \leq i < j \leq t$. We shall exhibit a coloring of each set V_j with the property that whenever two vertices $u, v \in V_j$ have a common neighbor in V_i (for $i < j$) then u and v will get distinct colors.

We need the following crucial lemma which provides a coloring with the required property for each of the V_j.

Lemma 4.2. *Let $G = (A \cup B, E)$ be a bipartite planar graph with bipartition A and B. Assume further that every vertex in A has degree at most b for some fixed integer b. Then there exists a coloring $c : B \to [6b]$ of the vertices in B, such that if $u, v \in B$ have a common neighbor (in A), then $c(u) \neq c(v)$.*

Proof. By a lemma that can be found in, e.g., [7], there exist a plane graph $G' = (B, E')$ with a set of faces F and a bijection $g : A \to F'$ where $F' \subset F$, that satisfies the following property: for any two vertices $x \in A, y \in B$ we have that $\{x, y\} \in E(G)$ if and only if y is incident in G' with the face $f = g(x)$. Our goal is to show that there exists a coloring of the vertices of G' with $6b$ colors, such that all vertices incident on a face $f \in F'$ get distinct colors. We define the auxiliary graph G'' by adding edges to G'. For every $f \in F'$ we place edges between any pair of vertices on f.

It remains to prove that $\chi(G'') \leq 6b$. We do this by showing that G'' is $(6b - 1)$-degenerate. Note that G'' is obtained from G' by adding cliques to a subset of the faces. Since the degree in G of any vertex in A is at most b, each of the faces in F' has at most b vertices on its boundary. As each clique of size b is contained in a union of $b - 1$ trees,[2] we have that G'' is the union of at most $b - 1$ planar graphs. The same argument holds for any subgraph of G'', thus, as the average degree of a planar graph is bounded by 6, the average degree in a union of $b - 1$ planar graphs is at most $6(b - 1)$, so G'' is indeed $(6b - 1)$-degenerate and so $6b$ colorable. $\square$

Proof of Theorem 4.1. We start by partitioning the vertices into $t = O(\log n)$ independent sets as follows: let $V' \subset V$ be the subset of vertices with degree less than 12. It is easy to see that $|V'| \geq n/2$. Indeed, by Euler's formula it follows that $|E(G)| \leq 3n-6$, and by Proposition 2.3, $|V \setminus V'| < 6n/12$ and hence $|V'| \geq n/2$. Since every planar graph is four-colorable [3], there exists a subset $V'' \subset V'$ of size at least $\frac{|V'|}{4} \geq n/8$ which is also independent. Put $V_1 = V''$ and continue recursively on the subgraph $G[V \setminus V_1]$. Since in every iteration we discard at least a 1/8 fraction of the remaining vertices, the final number of sets is at most $t = \log_{8/7} n + 1$.

Next, we color each of the sets V_i by applying Lemma 4.2 on the bipartite graph induced by $A = \bigcup_{j=1}^{i-1} V_j$ and $B = V_i$. Observe that, by construction, every vertex $y \in A$ has at most 11 neighbors in B. Thus, there exists a coloring of V_i with 66 colors, such that if $x, z \in B$ have a common neighbor $y \in A$, then x, z get different colors. The total number of colors used for V is at most $66t = O(\log n)$.

To complete the proof of the theorem, we need to show that the coloring c obtained in this manner is a us-coloring. Indeed, the coloring is proper since each V_i is an independent set, and each V_i has different colors than V_j for $i \neq j$. Thus, it is only left to check that if we have a path (x, y, z) for which $c(x) = c(z)$, then $c(y) > c(x)$. Since $c(x) = c(z)$ it must be that $x, z \in V_i$ for some $1 \leq i \leq t$, and by Lemma 4.2 it also holds that $y \in V_j$ for some $j \geq i$ (as otherwise the assertion of the lemma is violated for this triple). Recall that V_i is an independent set, so in fact $j > i$, and then $c(y) > c(x)$ by construction. $\square$

The *l*-vr-number of planar graphs. We have the following generalization of Theorem 4.1 to the *l*-vr-number of planar graphs.

Theorem 4.3. *Let l be some fixed positive integer. Then for any $n > 0$ and any planar graph $G = (V, E)$ with n vertices, $\chi_{l\text{-}vr}(G) \leq \lceil 3(l + 1) \log n \rceil$. Furthermore, computing such a coloring can be done in polynomial time.*

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[2] If the vertices of the clique are $v_1, \ldots, v_b$, then the tree T_i for $1 \leq i \leq b - 1$, will consist of all edges from v_i to the other vertices.

The proof is omitted, since it is similar in spirit to the upcoming proof of Theorem 5.4. Though it uses the planar path-separator of [24], which provides a very good control on the constant.

5. Minor-excluded graph families

In this section we extend the results of Section 4 to families of graphs excluding some fixed minor. We begin by stating a bound on the us-number of such graphs, which generalizes Theorem 4.1.[3] As the proof is similar to that of Theorem 4.1, we omit it.

Theorem 5.1. *For any graph G with n vertices excluding K_r as a minor,*

$$\chi_{us}(G) = O((r\sqrt{\log r})^3 \cdot \log n).$$

We now present the most general theorem, and prove a bound on the *l*-vr number of graphs excluding a fixed minor. We shall use *path separators* of minor-free graphs due to [1], which extend the planar separators of [19,24].

Definition 5.2. A graph $G = (V, E)$ on *n* vertices is *s-path separable* if there exist an integer *t* and a separator *S* such that:

1. $S = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \mathcal{P}_t$, where for each $0 \le i \le t$, $\mathcal{P}_i$ is a collection of shortest paths in the graph $G \setminus (\bigcup_{0 \le j < i} \mathcal{P}_j)$.
2. $\sum_{i=0}^{t} |\mathcal{P}_i| \le s$, that is, the total number of paths in *S* is at most *s*.
3. Each connected component of $G \setminus S$ is *s*-path separable and has at most $n/2$ vertices.

Theorem 5.3 ([1]). *Every H-minor-free graph is s-path separable for some $s = s(H)$, and an s-path separator can be computed in polynomial time.*

Now we are ready to prove the following theorem. (We remark that Theorem 5.1 provides a far better dependence on the size of the excluded minor for the case $l = 2$, and Theorem 4.3 attains a better constant for planar graphs.)

Theorem 5.4. *Let $G = (V, E)$ be a graph on n vertices that excludes H as a minor, then $\chi_{l\text{-}vr}(G) \le s(l+1) \log n$, where $s = s(H)$ is a constant that depends only on H. Moreover, such a coloring can be computed in polynomial time.*

Proof. Let *S* be an *s*-path separator of *G* for some $s = s(H)$, as guaranteed by Theorem 5.3. As each connected component of $G \setminus S$ has at most $n/2$ vertices (and of course still excludes *H* as a minor), we assume inductively that there exists an *l*-bounded vertex ranking for it using $s(l+1)\log(n/2) = s(l+1)\log n - s(l+1)$ colors. Note that we use the same colors for different components. We will use additional $s(l+1)$ colors for the vertices of the paths of *S*, each of these new colors will be higher than each color used for $V \setminus S$. Each of the paths in *S* will have its own $l+1$ colors, in such a way that paths in $\mathcal{P}_j$ will have higher colors than paths in $\mathcal{P}_i$ for all $0 \le j < i \le t$. Note that the second property of Definition 5.2 guarantees that the total number of paths is at most *s*, so we have enough colors. To color a path $Q = (v_1, \ldots, v_t)$ by the $l+1$ colors $c_0, c_1, \ldots, c_l$, define that the color of v_i is $c_{i \pmod{l+1}}$. If a certain vertex belongs to more than one path, it will keep the highest color assigned to it.

Next we prove the validity of the coloring. To this end, consider a path *P* of length at most *l*. If it is the case that *P* is contained in a connected component of $G \setminus S$, the induction hypothesis guarantees that the coloring is valid, so consider the case that *P* contains a vertex of some path in *S*. Let *i* be the minimal such that there exists $Q \in \mathcal{P}_i$ with $P \cap Q \ne \emptyset$. By the definition of the coloring, the highest color in *P* will be one of the colors given to a path in $\mathcal{P}_i$. W.l.o.g assume *Q* is the path in $\mathcal{P}_i$ such that the highest color in *P* comes from a vertex of *Q*. The proof will be concluded once we show that any color in $P \cap Q$ is unique. Seeking contradiction, assume that there are two vertices $u, v \in P \cap Q$ that were assigned the same color. By the first property of Definition 5.2, *Q* is a shortest path of $G' = G \setminus (\bigcup_{0 \le j < i} \mathcal{P}_j)$, so the distance between u, v in G' must be at least $l+1$. However, the minimality of *i* suggests that *P* is also contained in G', and as *P* is a path of length at most *l*, it cannot contain both u, v. $\square$

6. Degenerate graphs

In this section we focus on the *us-number* of the family of *d*-degenerate graphs. This family is a generalization of $\mathcal{M}_r$, and it could seem plausible that the us-number will be at most polylogarithmic in the number of vertices for degenerate graphs as well. We show that no such bound exists, by showing that for any integer *n* there exists a 2-degenerate graph *G* with *n* vertices satisfying $\chi_{us}(G) = \Omega(n^{1/3})$. We also prove an upper bound $\chi_{us}(G) = O(d\sqrt{n})$ for any $G \in \mathbb{G}_d$ with *n* vertices. These bounds demonstrate the large difference between us-coloring and proper coloring. We also show a large gap between the us-number and vr-number, by proving that there exists a graph $G \in \mathbb{G}_3$ with *n* vertices such that $\chi_{vr}(G) = \Omega(n)$.

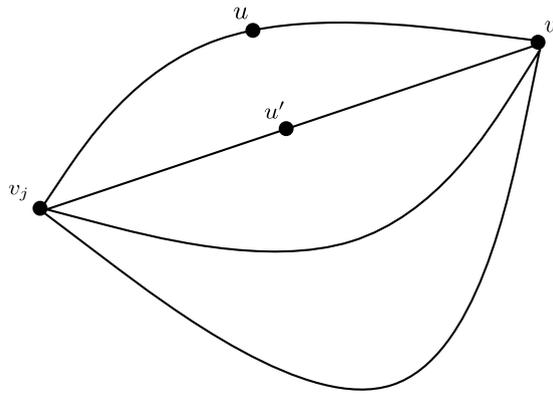[3] Recall that planar graphs are in particular K_5-free.

Fig. 3. If $c(v_i) = c(v_j)$, then $c(u) \neq c(u')$ for every two subdividing vertices u and u'.

6.1. Upper bound on the us-number of degenerate graphs

In order to bound the us-number we will need the following lemma, that provides a bound on the (usual) chromatic number of the squared graph G^2, for degenerate bounded degree graphs.

Lemma 6.1. *Let $G \in \mathbb{G}_d$ be a graph with maximum degree Δ. Then $\chi(G^2) \leq (2\Delta + 1)d + 1$.*

Proof. Take an ordering of the vertices of G, $V = (v_1, \ldots, v_n)$ as in Proposition 2.2. We color the vertices greedily in order. Assume by induction that we have a proper coloring of $G^2[\{v_1, \ldots, v_{i-1}\}]$ using at most $(2\Delta + 1)d + 1$ colors. Assign to v_i the lowest possible color c_i. We claim that $c_i \leq (2\Delta + 1)d + 1$. Indeed, since the maximum degree is Δ there can be at most Δ indices $h > i$ such that v_h is a neighbor of v_i. By the property of Proposition 2.2, each v_h can have at most d neighbors v_j with $j < h$, so there are at most Δd paths of the form (v_i, v_h, v_j) with $j < i < h$. On the other hand, v_i itself has at most d neighbors v_j with $j < i$, each of these has at most Δ neighbors, so there are at most Δd paths of the form (v_i, v_j, v_g) with $j < i, g < i$. We conclude that there are at most $2\Delta d$ vertices among $v_1, \ldots, v_{i-1}$ that are of distance 2 from v_i, and at most d of distance 1, therefore v_i has an available color in $[(2\Delta + 1)d + 1]$ such that the coloring of $G^2[\{v_1, \ldots, v_i\}]$ is proper. □

Theorem 6.2. *Let $G \in \mathbb{G}_d$ be a graph with n vertices. Then $\chi_{us}(G) = O(d\sqrt{n})$.*

Proof. Put $U = \{v \in V : d(v) < \sqrt{n}\}$, and let $G[U]$ be the subgraph of G induced by U. Notice that $G[U] \in \mathbb{G}_d$ and $\Delta(G[U]) < \sqrt{n}$, so by Lemma 6.1, $\chi_{us}(G[U]) \leq \chi(G[U]^2) \leq (2\sqrt{n} + 1)d$. Assume that we have such a coloring for U, and it remains to complete the coloring for the large degree vertices $V \setminus U$. For every vertex $v \in V \setminus U$ use a distinct color larger than $(2\sqrt{n} + 1)d$. By Proposition 2.3, $|V \setminus U| \leq \frac{2dn}{\sqrt{n}} = 2d\sqrt{n}$, so in total we have used at most $d(4\sqrt{n} + 1)$ colors. □

6.2. Lower bound on the us-number of degenerate graphs

Theorem 6.3. *For every n there is a 2-degenerate graph $G = (V, E)$ with n vertices, such that $\chi_{us}(G) > n^{1/3}$.*

Proof. Fix some $k \in \mathbb{N}$, and let G be the graph obtained by taking the complete graph K_k, replicating each edge k times and then subdividing each edge. Notice that G is, indeed, a 2-degenerate graph, since every vertex subdividing an edge has degree 2, and once all these vertices are removed we are left only with isolated vertices. We have $|V| = k + k\binom{k}{2} < k^3$. We will prove that $\chi_{us}(G) \geq k$. Seeking contradiction, assume that $c : V \to \{1, \ldots, m\}$ is a unique-superior coloring of G, with $m < k$. Then, by the pigeon-hole principle, there must be two distinct vertices among the original k vertices of K_k, v_i and v_j, such that $c(v_i) = c(v_j)$. Hence, for every vertex u subdividing an edge between v_i and v_j, $c(u) > c(v_i)$. But notice that it must hold that $c(u) \neq c(u')$ for every two vertices u and u' subdividing two distinct edges between v_i and v_j, as otherwise the path (u, v_i, u') will violate the requirement of a legal us-coloring. Since there are k vertices subdividing two distinct edges between v_i and v_j, we must use at least k distinct colors for them, which yields a contradiction. See Fig. 3 for an illustration. □

6.3. The vr-number of 3-regular graphs

Here we show that the vr-number of a bounded degree graph can be arbitrarily larger than its us-number. In particular, we show that there exists a 3-regular graph G with n vertices such that $\chi_{vr}(G) = \Omega(n)$. Denote by pw(G) the *path-width* of G, by tw(G) the *tree-width* of G, and by $h(G)$ the *vertex expansion* of G. See [5,13] for the definitions of these notions. We need the following known facts:

$$\text{pw}(G) \geq \text{tw}(G). \tag{1}$$

Lemma 6.4 ([5]). *For every graph G:*

$$\chi_{vr}(G) \geq pw(G).$$

Lemma 6.5 ([13]). *For every graph G:*

$$tw(G) \geq h(G) \cdot |V(G)|/4.$$

Remark 6.6. There exist a constant $h > 0$ and an infinite family $\mathcal{F}$ of 3-regular graphs, such that for all $G \in \mathcal{F}$, $h(G) \geq h$ (see, e.g., [15]).

Combining (1), Lemmas 6.4 and 6.5, we obtain that the family $\mathcal{F}$ of Remark 6.6 satisfies for all $G \in \mathcal{F}$:

$$\chi_{vr}(G) \geq pw(G) \geq tw(G) \geq h(G) \cdot |V(G)|/4 = \Omega(|V(G)|).$$

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