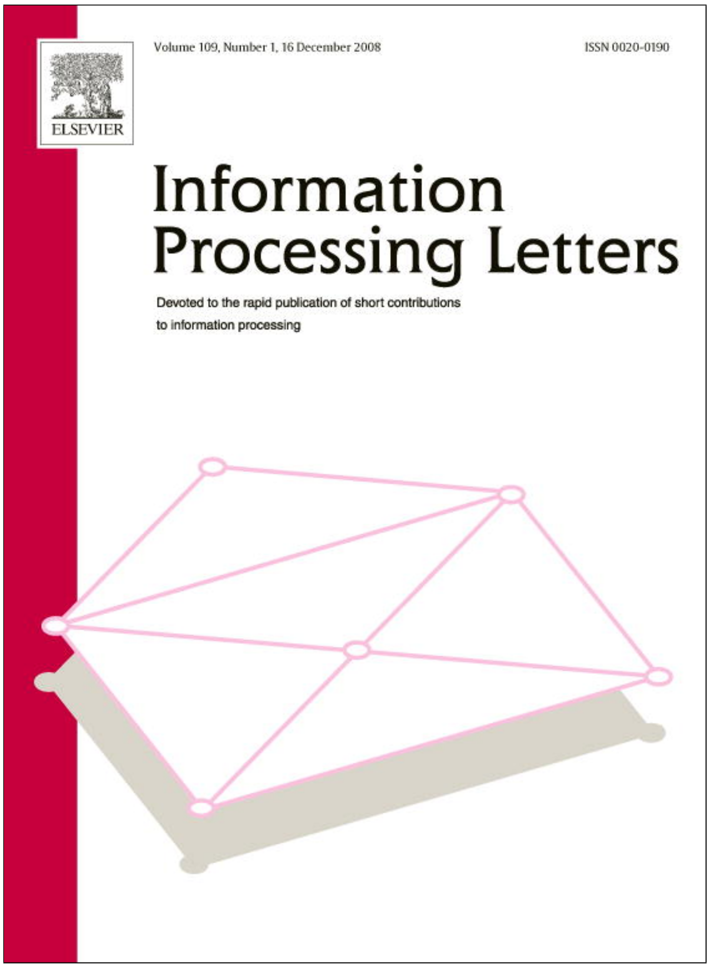
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A note on the online First-Fit algorithm for coloring k-inductive graphs

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ABSTRACT

In a FOCS 1990 paper, S. Irani proved that the First-Fit online algorithm for coloring a graph uses at most $O(k\log n)$ colors for k-inductive graphs. In this note we provide a very short proof of this fact.

1. Introduction

An online graph coloring algorithm A, is an algorithm that receives as input, the set of vertices of a graph in an arbitrary order in time steps. In each time step, the algorithm A is revealed a new vertex v of the graph together with all of the edges that connect v with the set of vertices that were revealed in previous time steps. The algorithm must assign v a color that is distinct from all colors assigned to neighbors of v that were already revealed. Once v is assigned a color, this color cannot be changed in the future. For a graph G, the *performance* of A, denoted $A(G)$ is the maximum number of colors used by A, where the maximum is taken over all possible permutation inputs of the vertices of G. The *competitive ratio* of A for a given class of graphs $\mathcal{C}$ is simply $\max_{G\in\mathcal{C}} A(G)/\chi(G)$, where $\chi(G)$ is the chromatic number of G.

In this note we focus on the class of k-inductive graphs. A graph $G=(V,E)$ is called *k-inductive* (or *k-degenerate*) if any subgraph $H\subset G$ contains a vertex of degree at most k. Equivalently, it is k-inductive if and only if there exists a bijective numbering of its vertices with $\{1,\dots,n\}$ (where $n=|V|$) such that every vertex has at most k neighbors with lower numbers. Indeed, given such a numbering, let $U\subset V$ and let H' be a subgraph of G with vertex set U. Then obviously the vertex of U with the largest number has at most k neighbors in H. In the other direction, one can number the vertices recursively by choosing a vertex in V with degree at most k, number it with "n" and continue recursively on the subgraph spanned by $V\setminus\{v\}$.

It is easy to see (using induction) that the chromatic number of such graphs is at most $k+1$. Inductive graphs are an important sub-class of graphs. For example, it easily follows from Euler formula that planar graphs are 5-inductive and bipartite planar graphs are 3-inductive. It is easily seen that a chordal graph G is $\chi(G)$-inductive.

The First-Fit (FF) algorithm is an online algorithm that colors every input vertex with the smallest possible integer. We consider the First-Fit greedy algorithm for coloring graphs online. The performance of this algorithm for restricted classes of graphs has been analyzed in several papers (see, e.g., [1–3]). S. Irani considered the class of k-inductive graphs (also known as k-degenerate graphs). In [2], Irani showed that First-Fit uses at most $O(k\log n)$ colors and thus proving that the competitive ratio of this algorithm for the class of k-inductive graphs is $O(\log n)$. In [2] it is also shown that this bound is tight in the sense that any online coloring algorithm has competitive ratio $\Omega(\log n)$ on the class of k-inductive graphs.

In this note, we provide an alternative analysis for the performance of First-Fit on k-inductive graphs. Surpris-

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ingly, our analysis is very short. We believe that our analysis is also much simpler than the one provided in [2].

2. A simple analysis of online First Fit

Let $G = (V, E)$ be a k-degenerate graph with n vertices and let FF denote the First-Fit online coloring algorithm. Fix an input permutation $\sigma = \{v_1, v_2, \ldots, v_n\}$. For a vertex $v \in V$, let $FF_\sigma(v)$ denote the color assigned to v by FF when running on input σ. Let $C_i \subset V$ denote the set of vertices that are assigned colors greater than or equal to i by FF with input σ. Formally:

$$C_i = \{v \in V \mid FF_\sigma(v) \geqslant i\}.$$

We need the following easy facts on k-degenerate graphs:

Lemma 2.1. *Let G be a simple k-degenerate graph and let $H \subset G$ be a subgraph of G. Let $e(H)$ and $v(H)$ denote the number of edges and of vertices of H, respectively. Then*:

(i) $e(H) \leqslant k \cdot v(H)$.
(ii) *The number of vertices in H with degree at least $4k$ is at most $\frac{v(H)}{2}$.*

Proof. (i) follows directly from the definition as we can order the vertices from left to right such that each vertex in H has at most k neighbors to its left. The number of edges is exactly the sum of the sizes of "left" neighbors of vertices of H. (ii) If there would be more than $v(H)/2$ vertices with degree at least $4k$, the sum of degrees would be more than $2kv(H)$ so the number of edges would be more than $kv(H)$, a contradiction to (i). □

The fact that algorithm FF uses at most $O(k \log n)$ colors will be an immediate consequence of the following lemma:

Lemma 2.2. *For every integer $j \geqslant 2$, $|C_{4kj}| \leqslant |C_{4k(j-1)}|/2$.*

Proof. for an integer i, let G_i denote the subgraph of G induced by the vertices of C_i (i.e., the subgraph induced by all vertices that are colored by FF with a color greater than or equal to i). Note that $C_{4kj} \subset C_{4k(j-1)}$. Consider a vertex $v \in C_{4kj}$. Obviously, v is a vertex of $G_{4k(j-1)}$. It is easy to see that the degree of v in $G_{4k(j-1)}$ is at least $4k$. This follows from the fact that at the time that v was revealed to algorithm FF, if FF assigned v a color $c \geqslant 4kj$, then v has a neighbor (in $G_{4k(j-1)}$) that was already assigned a color i for every $4k(j-1) \leqslant i < 4kj$. By Lemma 2.1(ii), it follows that $|C_{4kj}| \leqslant |C_{4k(j-1)}|/2$. This completes the proof of the lemma. □

Theorem 2.3. *Let $G = (V, E)$ be a k-degenerate graph and let $c(V)$ be the coloring produced by the greedy algorithm FF on some input permutation σ of the vertices V. Then the number of colors used by FF is at most $4k(\log n + 1) - 1$.*

Proof. Using induction, Lemma 2.2 implies that

$$|C_{4k(\log n+1)}| \leqslant \frac{|C_0|}{2^{1+\log n}} = \frac{|C_0|}{2n}.$$

However $|C_0| = n$. So $|C_{4k(\log n+1)}| \leqslant \frac{1}{2}$. Hence $C_{4k(\log n+1)} = \emptyset$. Thus there are no vertices colored by FF with a color greater than $4k(\log n + 1) - 1$. This completes the proof of the theorem. □

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