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Hitting sets online and unique-max coloring☆

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ABSTRACT

We consider the problem of hitting sets online. The hypergraph is known in advance, and the hyperedges to be stabbed are input one-by-one in an online fashion. The online algorithm must stab each hyperedge upon arrival. The best known competitive ratio for hitting sets online by Alon et al. (2009) is $O(\log n \cdot \log m)$ for general hypergraphs, where n and m denote the number of vertices and the number of hyperedges, respectively.

In this paper we provide the following results:

1. We consider hypergraphs in which the union of two intersecting hyperedges is also a hyperedge. Our main result for such hypergraphs is as follows: We consider a recently studied hypergraph coloring notion referred to as "unique-maximum coloring" and show that the competitive ratio of the online hitting set problem is at most the unique-maximum chromatic number and at least this number minus one.

2. Given a graph $G = (V, E)$, let $H = (V, R)$ denote the hypergraph whose hyperedges are subsets $U \subseteq V$ such that the induced subgraph $G[U]$ is connected. We establish a new connection between the best competitive ratio for the online hitting set problem in H and a well studied graph coloring notion referred to as the "vertex ranking number" of G. This connection states that these two parameters are equal. Moreover, this equivalence is constructive. As a corollary, we obtain optimal online hitting set algorithms for many such hypergraphs including those realized by planar graphs, graphs with bounded tree width, trees, etc. This improves the best previously known general bound of Alon et al. (2009).

3. We also consider two geometrically defined hypergraphs. The first one is defined by subsets of a given set of n points in the Euclidean plane that are induced by half-planes. The second hypergraph is defined by subsets of a given set of n points in the plane induced by unit discs. For these hypergraphs, the competitive ratio obtained by Alon et al. is $O(\log^2 n)$. For both cases, we introduce an algorithm with $O(\log n)$-competitive ratio. We also show that any online algorithm for both settings has a competitive ratio of $\Omega(\log n)$, and hence our algorithms are optimal.

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1. Introduction

In the minimum hitting set problem, we are given a hypergraph (X, R), where X is the ground set of vertices and R is a set of hyperedges. The goal is to find a "small" cardinality subset $S \subseteq X$ such that every hyperedge is stabbed by S, namely, every hyperedge has a nonempty intersection with S.

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The minimum hitting set problem is a classical NP-hard problem [15], and remains hard even for geometrically induced hypergraphs (see [12] for several references). A sharp logarithmic threshold for hardness of approximation was proved by Feige [11] (see also [20]). On the other hand, the greedy algorithm achieves a logarithmic approximation ratio [14,19,7]. Better approximation ratios have been obtained for several geometrically induced hypergraphs using specific properties of the underlying hypergraphs [12,17,2]. Other improved approximation ratios are obtained using the theory of VC-dimension and ε-nets [4,10,8]. Much less is known about online versions of the hitting set problem.

In this paper, we consider an online setting in which the hypergraph is given in the beginning, but the hyperedges that need to be stabbed are introduced one by one. Upon arrival of a new hyperedge, the online algorithm may add vertices (from X) to the hitting set so that the hitting set also stabs the new hyperedge. However, the online algorithm may not remove vertices from the hitting set. We use the competitive ratio for our analysis, a classical measure for the performance of online algorithms [23,3].

Alon et al. [1] considered the online set-cover problem for arbitrary hypergraphs. In their setting, the hypergraph is known in advance, but the vertices are introduced one by one. Upon arrival of an uncovered vertex, the online algorithm must choose a hyperedge that covers the vertex. Hence, by interchanging the roles of hyperedges and vertices, the online set-cover problem and the online hitting-set problems are equivalent. The online hitting-set algorithm presented by Alon et al. [1] achieves a competitive ratio of $O(\log n \log m)$ where n and m are the number of vertices and the number of hyperedges respectively. Note that if $m \geq 2^{n/\log n}$, the analysis of the online algorithm only guarantees that the competitive ratio is $O(n)$; a trivial bound because one may simply choose one vertex to stab each hyperedge.

Unique-maximum coloring. We need a recently studied hypergraph coloring notion. Let $H = (X, R)$ be a hypergraph. A coloring $c : X \rightarrow \{1, \dots, k\}$ is a *unique-max coloring* or *UM-coloring H* if, for each hyperedge $r \in R$, exactly one vertex in r is colored by the color $\max_{x \in r} c(x)$. Let $\chi_{um}(H)$ denote the least integer k for which H admits a UM-coloring with k colors. The notion of UM-coloring was first introduced and studied in [24,9]. This notion attracted many researchers and has been the focus of many research papers both in the computer science and mathematics communities. Such colorings arise in the context of frequency assignment to cellular antennae, in RFID protocols, and several other fields. (cf., [25]).

Let $G = (V, E)$ be a simple graph. An *ordered coloring* (also a *vertex ranking*) of G is a coloring of the vertices $\chi : V \rightarrow \{1, \dots, k\}$ such that whenever two vertices u and v have the same color i then every simple path between u and v contains a vertex with color greater than i. Such a coloring has been studied before and has several applications. It was studied in the context of VLSI design [22] and in the context of parallel Cholesky factorization of matrices [18]. The vertex ranking problem is also interesting for the Operations Research community. It has applications in planning efficient assembly of products in manufacturing systems [13]. See also [16,21].

The vertex ranking coloring is yet another special form of UM-coloring. Given a graph G, consider the hypergraph $H = (V, E')$ where a subset $V' \subseteq V$ is a hyperedge in E' if and only if V' is the set of vertices in some simple path of G. It is easily observed that an ordered coloring of G is equivalent to a UM-coloring of H.

Relation between unique-maximum coloring and the competitive ratio. We consider the competitive ratio for the hitting set problem as a property of the underlying hypergraph. Namely, the competitive ratio of a hypergraph $H = (X, R)$ is the competitive ratio of the best deterministic online algorithm for the hitting set problem for H. We say that a hypergraph is union-closed if the union of two intersecting hyperedges is always a hyperedge. Our main result (Theorem 7) shows a new connection between the competitive ratio of a union-closed hypergraph H and the minimum number of colors required to color H in a unique-max coloring. In fact, we present "black box" reductions that construct an online hitting set algorithm from a unique-max coloring, and vice-versa.

Applications. Three applications of the main result are presented. The first application is motivated by the following setting. Consider a communication network $G = (V, E)$. This network is supposed to serve requests for virtual private networks (VPNs). Each VPN request is a subset of vertices that induces a connected subgraph in the network, and requests for VPNs arrive online. For each VPN, we need to assign a server (among the nodes in the VPN) that serves the nodes of the VPN. Since setting up a server is expensive, the goal is to select as few servers as possible.

This application can be abstracted by considering the hypergraph H whose hyperedges are all subsets of vertices of a given graph G that induce a connected subgraph. This hypergraph captures the online problem in which the adversary chooses subsets $V' \subseteq V$ such that the induced subgraph $G[V']$ is connected, and the algorithm must stab the subgraphs. A direct consequence of the observation that every unique-max coloring of H is a vertex ranking of G implies that the competitive ratio of H equals the vertex ranking number of G. This application leads to improved optimal competitive ratios for graphs that admit (hereditary) small balanced separators (see Table 1).

Two more classes of hypergraphs are obtained geometrically as follows. In both settings we are given a set X of n points in the plane. In one hypergraph, the hyperedges are intersections of X with half planes. In the other hypergraph, the hyperedges are intersections of X with unit discs. Although these hypergraphs are not union-closed, we present an online algorithm for the hitting set problem for points in the plane and unit discs (or half-planes) with an optimal competitive ratio of $O(\log n)$. The competitive ratio of this algorithm improves the $O(\log^2 n)$-competitive ratio of Alon et al. by a logarithmic factor.

An application for points and unit discs is the selection of access points or base stations in a wireless network. The points model base stations and the disc centers model clients. The reception range of each client is a disc, and the algorithm has to select a base station that serves a new uncovered client. The goal is to select as few base stations as possible.

Table 1
A list of several graph classes with small separators ($n = |V|$) for which an optimal competitive ratio is obtained.

$G = (V, E)$	Competitive ratio	Previous result [1]	Lower bound on $\chi_{vr}(G)$
Path P_n	$\lfloor \log_2 n \rfloor + 1$	$O(\log^2 n)$	$\lfloor \log_2 n \rfloor + 1$
Tree	$O(\log(\text{diameter}(G)))$	$O(n)$	$\Omega(\log(\text{diameter}(G)))$(for P_n)
Tree-width d	$O(d \log n)$	$O(n)$	$\Omega(d \log n)$(for $G_{d,k}$ if $k = \text{poly}(n)$)
Planar graph	$O(\sqrt{n})$	$O(n)$	$\Omega(\sqrt{n})$(for square grid)

Organization. Definitions and notation are presented in Section 2. In Section 3, we study the special case of intervals on a line. The main result is presented in Section 4. We apply the main result to hypergraphs induced by connected subgraphs of a given graph in Section 5. An online algorithm for hypergraphs induced by points and half-planes is presented in Section 6. An online algorithm for the case of points and unit discs is presented in Section 7. We conclude with open problems.

Efficient computability. The study of online algorithms does not focus on efficient computation (e.g., polynomial time algorithms). Instead, the focus is on the effect of having to make decisions without knowing the future. We therefore do not elaborate on the running times of the presented online algorithms.

2. Preliminaries

The online minimum hitting set problem. Let $H = (X, R)$ denote a hypergraph, where R is a set of nonempty subsets of the ground set X. Members in X are referred to as *vertices*, and members in R are referred to as *hyperedges*. A subset $S \subseteq X$ *stabs* a hyperedge r if $S \cap r \neq \emptyset$. A *hitting set* is a subset $S \subseteq X$ that stabs every hyperedge in R. In the minimum hitting set problem, the goal is to find a hitting set with the smallest cardinality.

We present the online setting as a multiple round game between two players: an adversary and a coverer. Both players know the hypergraph $H = (X, R)$ before the game starts. In each round, the adversary picks a hyperedge $r \in E$. The coverer has to stab r by choosing a vertex $x \in r$. The coverer has to pay one dollar for choosing x if this is the first time x is chosen. If the coverer already chose the vertex x in a previous round, then the coverer can choose x again with zero cost. The goal of the coverer is to minimize the cost that he has to pay. The adversary knows all the vertices that have been chosen by the coverer, and may adapt his strategy to the strategy of the coverer. The adversary may stop the game at any time. This setting is an online setting because the coverer has to choose a vertex in each round without any knowledge about the hyperedges that will be picked by the adversary in future rounds. We refer to the coverer as the online algorithm.

We denote the hyperedge picked by the adversary in round i by r_i. The sequence of hyperedges picked by the adversary is denoted by σ. Let σ_i denote the prefix $\{r_1, \ldots, r_i\}$. Let x_i denote the vertex chosen by the coverer in round i. Let $C_i \triangleq \{x_1, \ldots, x_i\}$. Note that C_i is a hitting set with respect to the hyperedges in σ_i.

Fix a hypergraph H and an online deterministic algorithm ALG. For a finite input sequence $\sigma = \{r_i\}_{i=1}^{s}$, let $\text{OPT}(\sigma) \subseteq X$ denote a minimum cardinality hitting set for the hyperedges in σ. Let $\text{ALG}(\sigma) \subseteq X$ denote the hitting set computed by an online algorithm ALG when the input sequence is σ.

Definition 1. The *competitive ratio* of a deterministic online hitting set algorithm ALG with respect to a hypergraph H is defined by

$$\rho_H(\text{ALG}) \triangleq \max_{\sigma} \frac{|\text{ALG}(\sigma)|}{|\text{OPT}(\sigma)|}.$$

The *competitive ratio* of the hypergraph H is defined by

$$\rho_H \triangleq \min_{\text{ALG}} \rho_H(\text{ALG}).$$

Note that always $1 \leq \rho_H \leq n$, where n is the number of vertices of H. The definition of ρ_H can be viewed as a hypergraph property. It equals the best competitive ratio achievable by *any* online deterministic algorithm with respect to the hypergraph H.

Union-closed hypergraphs. We now define a notion that captures an important property of the hypergraph of intervals over collinear points.

Definition 2. A hypergraph $H = (X, R)$ is *union-closed* if it satisfies the following property:

$$\forall r_1, r_2 \in R : \; r_1 \cap r_2 \neq \emptyset \Rightarrow r_1 \cup r_2 \in R.$$

Unique-max colorings. Consider a hypergraph $H = (X, R)$ and a coloring $c : X \to \mathbb{N}$. For a hyperedge $r \in R$, let $c_{\max}(r) \triangleq \max\{c(x) \mid x \in r\}$. Similarly, $c_{\min}(r) \triangleq \min\{c(x) \mid x \in r\}$.

Definition 3. A coloring $c : X \to \mathbb{N}$ is a *unique-max coloring* of $H = (X, R)$ if, for every hyperedge $r \in R$, there is a unique vertex $x \in r$ for which $c(x) = c_{\max}(r)$.

Similarly, a coloring is unique-min if, for every hyperedge r, exactly one vertex $x \in r$ is colored $c_{\min}(r)$.

The *unique-max-chromatic number* of a hypergraph H, denoted by $\chi_{um}(H)$, is the smallest natural number k for which H admits a unique-maximum coloring that uses only k colors.

Vertex ranking. We define a coloring notion for graphs known as vertex ranking [16,21].

Definition 4. A *vertex ranking* of a graph $G = (V, E)$ is a coloring $c : V \to \mathbb{N}$ that satisfies the following property. For every pair of distinct vertices x and y and for every simple path P from x to y, if $c(x) = c(y)$, then there exists an internal vertex z in P such that $c(x) < c(z)$.

The *vertex ranking number* of G, denoted $\chi_{\rm vr}(G)$, is the least integer k for which G admits a vertex ranking that uses only k colors.

A vertex ranking of a graph $G = (V, E)$ is also a unique-max coloring of the hypergraph $H = (V, P)$, where P denotes the set of all simple paths in G. The converse is also true (see Proposition 17).

A vertex ranking of a graph G is also a proper coloring of G since adjacent vertices must be colored by different colors. On the other hand, a proper coloring is not necessarily a vertex ranking as is easily seen by considering a path graph P_n. This graph admits a proper coloring with 2 colors but this coloring is not a valid vertex ranking. In fact, $\chi_{\rm vr}(P_n) = \lfloor \log_2 n \rfloor + 1$, as proved in the following proposition that has been proven several times [13,16,9].

Proposition 5. $\chi_{\rm vr}(P_n) = \lfloor \log_2 n \rfloor + 1$.

Proof. Consider a vertex ranking c of P_n in which the highest color is used once to split the path into two disjoint paths as evenly as possible. The number of colors $f(n)$ satisfies the recurrence $f(1) = 1$ and

$$f(n) \le 1 + f\left(\left\lceil \frac{n-1}{2} \right\rceil\right).$$

It is easy to verify that $f(n) \le 1 + \lfloor \log_2 n \rfloor$. For a matching lower bound, consider a coloring and the vertex with the highest color. Note that the highest color appears uniquely in P_n. This vertex separates the path into two disjoint paths colored by one color less. The length of one of these paths must be at least $\lceil \frac{n-1}{2} \rceil$. Hence $f(n) \ge 1 + f(\lceil \frac{n-1}{2} \rceil)$ and therefore $f(n) \ge 1 + \lfloor \log_2 n \rfloor$, as required. □

3. Special case: hitting set for intervals on the line

As a warm-up, we consider the hypergraph $H = (X, R)$ of intervals over n collinear points defined by:

$$X \triangleq \{1, 2, \ldots, n\}$$
$$R \triangleq \{[i, j] \mid 1 \le i \le j \le n\}.$$

The competitive ratio of the online hitting-set algorithm of Alon et al. [1] for the hypergraph of intervals over n collinear points is $O(\log |X| \cdot \log |R|) = O(\log^2 n)$. In this section we prove a better competitive ratio for this specific hypergraph.

Proposition 6. $\rho(H) = \lfloor \log_2 n \rfloor + 1$.

Proof. We begin by proving the lower bound $\rho(H) \ge \lfloor \log_2 n \rfloor + 1$. The adversary generates the sequence $\sigma \triangleq \{r_i\}$ of hyperedges to be stabbed. Let $\{C_i\}_i$ denote the chain of hitting sets computed by the algorithm. The first hyperedge consists of all the points, namely, $r_1 = X$. In every step, the next hyperedge r_{i+1} is chosen (by the adversary) to be a larger interval in $r_i \setminus C_i$, namely, $|r_{i+1}| \ge \frac{|r_i|-1}{2}$. While r_i is not empty, the adversary forces the algorithm to stab each hyperedge by a distinct point. In fact, the adversary can introduce such a nested sequence consisting of at least $\lfloor \log_2 n \rfloor + 1$ many hyperedges. Thus, $|C_i| = i$ if $i \le \lfloor \log_2 n \rfloor + 1$. However, $r_1 \supset r_2 \supset \cdots$ is a decreasing chain, and hence, $|\textsc{opt}(\sigma_i)| = 1$, and the lower bound follows.

The upper bound $\rho(H) \le \lfloor \log_2 n \rfloor + 1$ is proved as follows. Let $c(x)$ denote a vertex ranking of the graph P_n that uses $\lfloor \log_2 n \rfloor + 1$ colors (see Proposition 5). Consider the deterministic hitting-set algorithm $\textsc{alg}_c$ defined as follows. Upon arrival of an unstabbed interval $[i, j]$, stab it by the point x in the interval $[i, j]$ with the highest color. Namely $x \triangleq \arg\max\{c(k) : i \le k \le j\}$.

We claim that $\rho_H(\textsc{alg}_c) \le 1 + \lfloor \log_2 n \rfloor$. The proof is based on the following observation. Consider a color γ and the subsequence of intervals $\sigma(\gamma)$ that consists of the intervals r_i in σ that satisfy the following two properties: (i) Upon arrival r_i is unstabbed. (ii) Upon arrival of r_i, $\textsc{alg}_c$ stabs r_i by a point colored γ. We claim that the intervals in $\sigma(\gamma)$ are pairwise disjoint. Indeed, if two intervals $r_1 \ne r_2$ in $\sigma(\gamma)$ intersect, then the maximum color in $r_1 \cup r_2$ is also γ, and it appears twice in $r_1 \cup r_2$. This contradicts the definition of a vertex ranking because $r_1 \cup r_2$ is also an interval. Thus, the optimum hitting set satisfies $|\textsc{opt}(\sigma)| \ge \max_\gamma |\sigma(\gamma)|$. But $|\textsc{alg}_c(\sigma)| = \sum_\gamma |\sigma(\gamma)| \le (1 + \lfloor \log_2 n \rfloor) \cdot \max_\gamma |\sigma(\gamma)|$, and hence $\rho_H(\textsc{alg}_c) \le 1 + \lfloor \log_2 n \rfloor$, as required. □

4. The main result

Our main result generalizes Propositions 5 and 6 regarding the equality of $\rho(H)$ and $\chi_{um}(H)$ when H is the hypergraph of intervals over n collinear points.

Theorem 7. *If a hypergraph $H = (X, R)$ is union-closed, then*

$$\chi_{um}(H) - 1 \leq \rho(H) \leq \chi_{um}(H).$$

The proof of Theorem 7 is by black-box reductions. The first reduction uses the unique-max coloring to obtain an online algorithm (simply stab a hyperedge with the vertex with the highest color). The second reduction uses a deterministic online hitting set algorithm to obtain a unique-max coloring.

We say that a hypergraph $H = (X, E)$ is *separable* if $\{x\} \in R$, for every $x \in X$. The proof of the following corollary appears in Section 4.3.

Corollary 8. *If a hypergraph $H = (X, R)$ is union-closed and separable, then $\rho(H) = \chi_{um}(H)$.*

4.1. Proof of $\rho(H) \leq \chi_{um}(H)$

The proof follows the reduction in the proof of Proposition 6. Let $k = \chi_{um}(H)$ and let $c : X \to [1, k]$ denote a unique-max coloring of $H = (X, R)$. Consider the deterministic hitting-set algorithm ALG_c defined as follows. Upon arrival of an unstabbed hyperedge $r \in R$, stab it by the vertex $x \in r$ colored $c_{\max}(r)$.

We claim that $\rho_H(\text{ALG}_c) \leq k$. Fix a sequence $\sigma = \{r_i\}_i$ of hyperedges input by the adversary. For a color γ, let $\sigma(\gamma)$ denote the subsequence of σ that consists of the hyperedges r_i in σ that satisfy the following properties: (i) r_i is unstabbed when it arrives. (ii) The first vertex that ALG_c uses to stab r_i is colored γ. The hyperedges in $\sigma(\gamma)$ are pairwise disjoint. Indeed, if two hyperedges $r_1 \neq r_2$ in $\sigma(\gamma)$ intersect, then $r_1 \cup r_2 \in R$. Moreover, the maximum color in $r_1 \cup r_2$ is also γ. But the color γ appears twice in the hyperedge $r_1 \cup r_2$; one vertex that stabs r_1 and another vertex that stabs r_2, a contradiction. Thus the optimum hitting set satisfies $\text{OPT}(\sigma) \geq \max_\gamma |\sigma(\gamma)|$. But

$$\text{ALG}_c(\sigma) = \sum_{\gamma=1}^{k} |\sigma(\gamma)| \leq k \cdot \max_\gamma |\sigma(\gamma)|$$

and hence $\rho_H(\text{ALG}_c) \leq k$, as required.

4.2. Proof of $\chi_{um}(H) \leq 1 + \rho(H)$

Let ALG denote a deterministic online hitting set algorithm that satisfies $\rho_H(\text{ALG}) = \rho(H)$. We use ALG as a "black box" to compute a unique-min coloring $c : X \to [0, \rho(H)]$. Note that we compute a unique minimum coloring rather than a unique maximum coloring; this modification simplifies the presentation. (If $c(x)$ is a unique-min coloring, then $c'(x) \triangleq \rho(H) - c(x)$ is a unique-max coloring.)

Terminology. Let $S \subseteq X$ be a subset of vertices. We say that a hyperedge $r \subseteq S$ is *S-maximal* if no hyperedge contained in S strictly contains r. Formally, for every hyperedge $r' \in R$, $r \subseteq r' \subseteq S$ implies that $r' = r$. Given a node v in a rooted tree, let $\text{path}(v)$ denote the path from the root to v. Define $\text{depth}(v)$ to be the distance from the root to v. (The distance of the root to itself is zero.) The *least common ancestor* of two nodes u and v in a rooted tree is the node of highest depth in $\text{path}(u) \cap \text{path}(v)$.

4.2.1. The decomposition

We use ALG to construct a decomposition forest consisting of rooted trees. Each node v in the forest is labeled by a hyperedge $r_v \in R$ and a vertex $x_v \in r_v$. The decomposition forest is defined inductively as follows.

For each X-maximal hyperedge in R we associate a distinct root. The labels of each root v are defined as follows. The hyperedge r_v is the X-maximal hyperedge associated with v. The vertex $x_v \in r_v$ is the vertex that ALG uses to stab r_v when the input sequence consists only of r_v.

We now describe the induction step for defining the children of a node v and its labels r_v and x_v. Let $X(\text{path}(v)) \triangleq \{x_u \mid u \in \text{path}(v)\}$ denote the sequence of vertices that appear along the path from the root to v. Similarly, let $\sigma(\text{path}(v)) \triangleq \{r_u \mid u \in \text{path}(v)\}$ denote the sequence of hyperedges that appear along $\text{path}(v)$. Let $S \triangleq r_v \setminus X(\text{path}(v))$. For each nonempty S-maximal hyperedge r, we add a child v' of v that is labeled by the hyperedge $r_{v'} = r$. The vertex $x_{v'}$ is the vertex x that stabs $r_{v'}$ when ALG is input the sequence of hyperedges $\sigma(\text{path}(v'))$. We stop with a leaf v if there is no hyperedge contained in $r_v \setminus X(\text{path}(v))$.

Proposition 9. *For every node v, the sequence of hyperedges in $\sigma(\text{path}(v))$ is a strictly decreasing chain. Namely, if v is a child of u then $r_v \subsetneq r_u$. Moreover, when this sequence is input to ALG, then each hyperedge is unstabbed upon arrival. Hence, the vertices in $X(\text{path}(v))$ are distinct.*

Proposition 10. *If v_1 and v_2 are siblings, then the hyperedges r_{v_1} and r_{v_2} are disjoint.*

Proof. Otherwise, since H is union-closed, $r_{v_1} \cup r_{v_2}$ is a hyperedge. This hyperedge contradicts the S-maximality of r_{v_1} and r_{v_2} for $S \triangleq r_v \setminus X(\text{path}(v))$, where v is the parent of v_1 and v_2. □

Proposition 11. *If v and u are two nodes such that v is neither an ancestor or a descendant of u, then the hyperedges r_v and r_u are disjoint.*

Proof. For the sake of contradiction, assume that $x \in r_u \cap r_v$. It follows that u and v must belong to the same tree whose root is labeled by the X-maximal hyperedge that contains x. The least common ancestor w of u and v has two distinct children w_1 and w_2 such that $w_1 \in \text{path}(u)$ and $w_2 \in \text{path}(v)$. By Proposition 9, $r_u \subset r_{w_1}$ and $r_v \subset r_{w_2}$. By Proposition 10, $r_{w_1} \cap r_{w_2} = \emptyset$, and it follows that $r_u \cap r_v = \emptyset$, as required. □

The following proposition is an immediate consequence of Propositions 9 and 11.

Proposition 12. *All the labels x_v of the nodes in the forest are distinct.*

4.2.2. Mapping hyperedges to nodes in the decomposition

Let $\tilde{X}$ denote the set of nodes in the decomposition forest. We now define a mapping $f : R \to \tilde{X}$ from the set of hyperedges R to the set of nodes of the decomposition forest.

Define $f(r)$ to be the forest node v of minimum depth such that x_v stabs r. Formally,

$$T(r) \triangleq \{v \in \tilde{X} \mid x_v \in r\}$$
$$f(r) \triangleq \arg\min\{\text{depth}(v) \mid v \in T(r)\}.$$

Claim 13. *The mapping $f(r)$ is well defined.*

Proof. We need to prove that (1) $T(r)$ is not empty for every hyperedge r, and (2) there exists a unique forest node $v \in T(r)$ of minimum depth.

We prove that $T(r) \neq \emptyset$ by contradiction. Let $x \in r$ be any vertex in r. Consider the X-maximal hyperedge r_1 that contains x. Let v_1 be the root that is associated with r_1 (i.e., $r_{v_1} = r_1$). Clearly $r \subset r_{v_1}$ because $x \in r$ and r_{v_1} is X-maximal. By the assumption, $x_{v_1} \notin r$. Proceed along the tree rooted at v_1 to find a tree path $v_1, v_2, \ldots, v_k$ such that $r \subset r_{v_i}$ and $x_{v_i} \notin r$ for $1 \leq i \leq k$. To obtain a contradiction, we claim that one can find such an infinite path since $r \subseteq r_{v_k} \setminus \{x_{v_k}\}$. Indeed, r is contained in one of the S-maximal hyperedges for $S \triangleq r_{v_k} \setminus X(\text{path}(v_k))$. So we can define v_{k+1} to be the child of v_k such that $r \subseteq r_{v_{i+k}}$. Since $T(r)$ is empty $x_{v_{k+1}} \notin r$, the node v_{k+1} meets the requirement from the next node in the path. However, by Proposition 9, each tree in the forest is finite, a contradiction.

We prove that there exists a unique forest node $v \in T(r)$ of minimum depth. Assume that there are two forest nodes u and v of minimum depth such that both u and v are in $T(r)$. By the definition of $T(r)$, $x_u \in r$. By the fact that depth(u) is minimum it follows that $r \cap X(\text{path}(u)) = \{x_u\}$. Hence, by the maximality of r_u, it follows that $r \subseteq r_u$. Analogously, $r \subseteq r_v$. Hence, r_u and r_v are not disjoint. By Proposition 11, u is an ancestor of v, or vice-versa. This implies that $\text{depth}(u) \neq \text{depth}(v)$, a contradiction. □

4.2.3. The coloring

Define the coloring $c : X \to \mathbb{N}$ as follows. For each $x \in X$, if $x = x_v$ for some forest node v, then define $c(x) \triangleq \text{depth}(v)$. If x does not appear as a label x_v of any node in the forest, then $c(x) \triangleq \rho_H(\text{ALG})$. Note that Proposition 12 ensures that the coloring c is well defined.

Lemma 14. *The depth of every forest node is less than $\rho_H(\text{ALG})$.*

Proof. Consider a node v in the decomposition forest. By Proposition 9, the sequence $\sigma(\text{path}(v))$ of hyperedges is a decreasing chain, and when input to ALG, each hyperedge is unstabbed upon arrival. Therefore the cardinality of the hitting set that $\text{ALG}(\sigma(\text{path}(v)))$ returns equals $1 + \text{depth}(v)$. On the other hand, x_v stabs all these hyperedges. Since the competitive ratio of ALG with respect to H is $\rho_H(\text{ALG})$, it follows that the length of this sequence is not greater than $\rho_H(\text{ALG})$. The length of this sequence equals $1 + \text{depth}(v)$, and the lemma follows. □

Lemma 14 implies that the maximum color assigned by $c(x)$ is $\rho_H(\text{ALG})$. The following lemma implies that $\chi_{um}(H) \leq \rho_H(\text{ALG}) + 1$.

Lemma 15. *The coloring $c : X \to [0, \rho_H(\text{ALG})]$ is a unique-min coloring.*

Proof. Fix a hyperedge r. By Claim 13, $f(r)$ is well defined. Thus r contains only one vertex colored $c_{\min}(r)$, and all the other vertices in r are colored by higher colors. □

We remark that the proof of Lemma 15 uses the color $\rho_H(\text{ALG})$ as a "neutral" color that is never used as the minimum color in a hyperedge.

4.3. Proof of Corollary 8

Lemma 16. *If H is separable, then every vertex appears as a label x_v in the decomposition.*

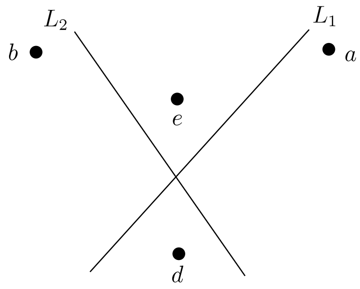

Fig. 1. Two intersecting half-planes, the union of which is not induced by a half-plane. The half-planes $r_1 = \{a, d\}$ and $r_2 = \{b, d\}$ intersect. By convexity, every half-plane that contains a, b, and d must contain e.

Proof. If H is separable, then the stopping condition in the construction of the decomposition trees is equivalent to $r_v = \{x_v\}$. Otherwise, for each vertex x in $r_v \setminus \{x_v\}$, the hyperedge $\{x\}$ would exclude the possibility that v is a leaf. This implies that every vertex appears as a label of a node in the decomposition forest, as required. □

Proof of Corollary 8. A vertex x is colored $\rho_H(\text{ALG})$ iff no node is labeled by x in the decomposition forest. By Lemma 16, if H is separable, then every vertex appears as a label x_v in the decomposition. Thus, the color $\rho_H(\text{ALG})$ is never used by $c(x)$. Hence the hyperedge of the coloring $c(x)$ is $[0, \rho_H(\text{ALG}) - 1]$ and the number of colors used by $c(x)$ is only $\rho_H(\text{ALG})$, as required. □

5. Online hitting-set for connected subgraphs

We consider the following setting of a hypergraph induced by connected subgraphs of a given graph. Formally, let $G = (V, E)$ be a graph. Let $H(G) = (V, R)$ denote the hypergraph over the same set of vertices V. A subset $r \subseteq V$ is a hyperedge in R if and only if the subgraph $G[r]$ induced by r is connected.

Proposition 17. *A coloring $c : V \to \mathbb{N}$ is a vertex ranking of G iff it is a unique-max coloring of $H(G)$. Hence, $\chi_{um}(H(G)) = \chi_{\text{vr}}(G)$.*

In particular, every vertex ranking of the path P_n is a unique-max coloring of the vertices with respect to intervals.

The following corollary characterizes the competitive ratio for the online hitting set problem for $H(G)$ in terms of the vertex ranking number of G. In fact, Propositions 5 and 6 imply the following corollary for the special case of the path P_n.

Corollary 18. $\rho(H(G)) = \chi_{\text{vr}}(G)$.

Proof. Follows from Corollary 8 and the Proposition 17. □

Corollary 18 implies optimal competitive ratios of online hitting set algorithms for a wide class of graphs that admit (hereditary) the so-called small balanced separators. For example, consider the online hitting set problem for connected subgraphs of a given planar graph. Let G be a planar graph on n vertices. It was proved in [16] that $\chi_{\text{vr}}(G) = O(\sqrt{n})$ and a vertex ranking with $O(\sqrt{n})$ colors can be computed efficiently. Therefore, Corollary 18 implies that the competitive ratio of our algorithm for connected subgraphs of planar graphs is $O(\sqrt{n})$. Corollary 18 also implies that this bound is optimal. Indeed, it was proved in [16] that for the $l \times l$ grid graph $G_{l\times l}$ (with l^2 vertices), $\chi_{\text{vr}}(G_{l\times l}) \geq l$. Hence, for $G_{l\times l}$, any deterministic online hitting set algorithm must have a competitive ratio at least l. In Table 1 we list several important classes of such graphs. (See the Appendix for a lower bound on the vertex ranking number for graphs of bounded tree-width.)

We note that in the case of a star (i.e., a vertex v with $n - 1$ neighbors), the number of subsets of vertices that induce a connected graph is 2^{n-1}. However, the star has a vertex ranking that uses just two colors, hence, the competitive ratio of our algorithm in this case is 2. This is an improvement over the analysis of the algorithm of Alon et al. [1] which only proves a competitive ratio of $O(n)$.

6. Points and half-planes

In this section we consider a special instance of the online hitting set problem for a finite set of points in the plane and hyperedges induced by half-planes.

We prove the following results for hypergraphs in which the ground set X is a finite set of n points in $\mathbb{R}^2$ and the hyperedges are all subsets of X that can be cut off by a half-plane. Namely, a subset of points that lie above (respectively, below) a given line ℓ.

We note that the hypergraph of points and half-planes is not union-closed. See Fig. 1 for an example. Thus, Theorem 7 is not immediately applicable.

Theorem 19. *The competitive ratio of every online hitting set algorithm for points and half-planes is $\Omega(\log n)$.*

Theorem 20. *There exists an online hitting set algorithm for points and half-planes that achieves a competitive ratio of $O(\log n)$.*

In the proofs we consider only hyperedges of points that are below a line; the case of points above a line is dealt with separately. This increases the competitive ratio by at most a factor of two.

Notation. Given a finite planar set of points X, let $V \subseteq X$ denote the subset of extreme points of X. That is, V consists of all points $p \in X$ such that there exists a half-plane h with $h \cap X = \{p\}$. Let $\{p_i\}_{i=1}^{|V|}$ denote an ordering of V in ascending x-coordinate order. Let $P = (V, E_P)$ denote the path graph over V where p_i is a neighbor of p_{i+1} for $i = 1, \ldots, |V| - 1$. The intersection of every half-plane with V is a subpath of P. Namely, the intersection of a nonempty hyperedge r_i with V is a set of the form $\{p_j \mid j \in [a_i, b_i]\}$. We refer to such an intersection as a discrete interval (or simply an interval, if the context is clear). We often abuse this notation and refer to a point $p_i \in V$ simply by its index i. Thus, the interval of points in the intersection of r_i and V is denoted by $I_i \triangleq [a_i, b_i]$.

6.1. Proof of Theorem 19

We reduce the instance of intervals on a line (or equivalently, the path P_n and its induced connected subgraphs) to an instance of points and half-planes. Simply place the n points on the parabola $y = x^2$. Namely, point i is mapped to the point (i, i^2). An interval $[i, j]$ of vertices is obtained by points below the line passing through the images of i and j. Hence, the problem of online hitting hyperedges induced by half-planes is not easier than the problem of online hitting intervals of P_n. The theorem follows from Proposition 6.

6.2. Proof of Theorem 20

Algorithm description. The algorithm reduces the minimum hitting set problem for points and half-planes to a minimum hitting set of intervals in a path. The reduction is to the path graph P over the extreme points V of X. To apply Algorithm ALG_c (see Section 4.1), a vertex ranking c for P is computed, and each half-plane r_i is reduced to the interval I_i. A listing of Algorithm HS_p appears as Algorithm 1. Note that the algorithm HS_p uses only the subset $V \subset X$ of extreme points of X.

Algorithm 1 $\text{HS}_p(\{r_i\})$ - an online hitting set for points and half-planes

Require: $X \subset \mathbb{R}^2$ is a set of n points, and each r_i is an intersection of X with a half-plane.
1: $V \leftarrow$ the extreme points of X (i.e., lower envelope of the convex hull).
2: $\{p_i\}_{i=1}^{|V|} \leftarrow$ ordering of V in ascending x-coordinate order.
3: Let $P = (V, E_P)$ denote the path graph over V, where $E_P \triangleq \{(p_i, p_{i+1})\}_{i=1}^{|V|-1}$.
4: $c \leftarrow$ a vertex ranking of P (with $\lfloor \log_2 |V| \rfloor + 1$ colors).
5: Upon arrival of hyperedge r_i, reduce it to the interval $I_i = r_i \cap V$.
6: Run ALG_c with the sequence of hyperedges $\{I_i\}_i$.

Analysis of the competitive ratio. The analysis follows the proof of Proposition 6. Recall that $\sigma(a)$ denotes the subsequence of σ consisting of hyperedges r_i that are unstabbed upon arrival and stabbed initially by a point colored a.

Lemma 21. *The hyperedges in $\sigma(a)$ are pairwise disjoint.*

Proof. Assume for the sake of contradiction that $r_i, r_j \in \sigma(a)$ and $z \in r_i \cap r_j$. Let $[a_i, b_i]$ denote the endpoints of the interval $I_i = r_i \cap V$, and define $[a_j, b_j]$ and I_j similarly. Since a is the maximum color in $I_i \cup I_j$ and appears twice in $I_i \cup I_j$, it follows that $I_i \cup I_j$ is not an interval. Hence, $I_i \cap I_j = \emptyset$ and $z \notin V$. Consider the minimum interval $I \subseteq V$ that contains $I_i \cup I_j$. Let t denote the vertex in I with the highest color. Clearly, t is between I_i and I_j.

Let $(p)_x$ denote the x-coordinate of a point p. Without loss of generality, $(b_i)_x < (t)_x < (a_j)_x$. Assume that $(z)_x \leq (t)_x$ (the other case is handled similarly). See Fig. 2 for an illustration. Let L_j denote a line that induces the hyperedge r_j, i.e., the set of points below L_j is r_j. Let L_t denote a line that separates t from $X \setminus \{t\}$, i.e., t is the only point below L_t. Then, L_t passes below z, above t, and below a_j. On the other hand, L_j passes above z, below t, and above a_j. Since $(z)_x \leq (t)_x < (a_j)_x$, it follows that the lines L_t and L_j intersect twice, a contradiction, and the lemma follows. □

Lemma 21 implies that $|\text{OPT}(\sigma)| \geq \max_a |\sigma(a)|$. On the other hand $|\text{HS}_p(\sigma)| = \sum_a |\sigma(a)| \leq (1 + \log n) \cdot \max_a |\sigma(a)|$, and Theorem 20 follows.

7. Points and unit discs

In this section we consider a special instance of the online hitting set problem in which the ground set X is a finite set of n points in $\mathbb{R}^2$. The set of hyperedges consists of all the subsets of points that are contained in a unit disc. Formally, a unit disc d centered at o is the set $d \triangleq \{x \in \mathbb{R}^2 : \|x - o\|_2 \leq 1\}$. The hyperedge $r(d)$ induced by a disc d is the set $r(d) \triangleq \{x \in X : x \in d\}$. The circle ∂d is defined by $\partial d \triangleq \{x \in \mathbb{R}^2 : \|x - o\|_2 = 1\}$.

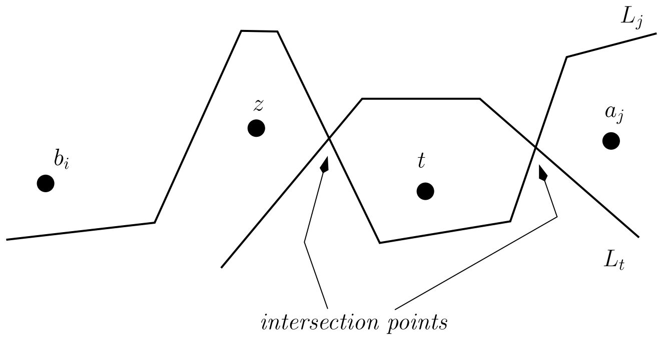

Fig. 2. Proof of Lemma 21. The lines L_j and L_t are depicted as polylines only for the purpose of depicting their above/below relations with the points.

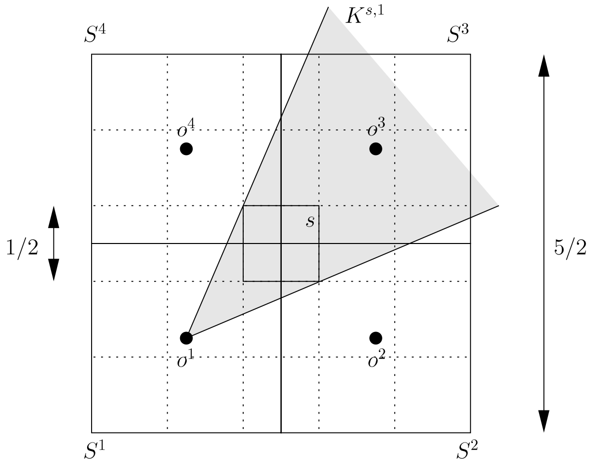

Fig. 3. A partitioning of the plane from Chen et al. [6].

As in the case of points and half-planes, the hypergraph of points and unit discs is not union-closed. To see this, assume that the distances between the four points in Fig. 1 are small. In this case, the lines L_1 and L_2 can be replaced by unit discs that induce the same hyperedges.

Theorem 22. *The competitive ratio of every online hitting set algorithm for points and unit discs is* $\Omega(\log n)$.

Proof. Reduce an instance of intervals on a line to points and unit-discs as follows. Position n points on a line such that the distance between the first and last point is less than one. For each interval, there exists a unit disc that intersects the points in exactly the same points as the interval. Thus, the lower bound for points and intervals (Proposition 6) holds also for unit discs. □

Theorem 23. *There exists an online hitting set algorithm for points and unit discs that achieves a competitive ratio of* $O(\log n)$.

7.1. Proof of Theorem 23

Partitioning. We follow Chen et al. [6] with the following partitioning of the plane (see Fig. 3). Partition the plane into square tiles with side-lengths 1/2. Consider a square s in this tiling. Let S denote a square concentric with s whose side length is 5/2. Partition S into four quadrants, each a square with side length 5/4. Let S^i denote a quadrant of S and let o^i denote its center, for $i \in \{1, 2, 3, 4\}$. Let D_s denote the set of unit discs d such that $d \cap s \neq \emptyset$.

Proposition 24. *If* $d \in D_s$, *then* $d \cap \{o^1, \ldots, o^4\} \neq \emptyset$.

For $d \in D_s$, let $\tau(s, d) \triangleq \min\{i : o^i \in d\}$. For $\tau \in \{1, \ldots, 4\}$, let $D_{s,\tau}$ denote the set $\{d \in D_s \mid \tau(s, d) = \tau, d \cap s \cap X \neq \emptyset\}$. The following lemma shows that circles bounding the discs in $D_{s,\tau}$ behave like pseudo-lines when restricted to a subregion of S.

Lemma 25 ([6])**.** *Let* $K^{s,\tau}$ *denote the convex cone with apex* o^τ *spanned by s. Then, for any pair of discs* $d, d' \in D_{s,\tau}$, *the circles* ∂d *and* $\partial d'$ *intersect at most once in* $K^{s,\tau}$.

Extreme points. For every square tile s and every $\tau \in \{1, \ldots, 4\}$, we define a set $V_{s,\tau}$ of *extreme points* as follows.

$$V_{s,\tau} \triangleq \{x \in X \mid \exists d \in D_{s,\tau} : d \cap s \cap X = \{x\}\}.$$

Note that if $d \in D_{s,\tau}$ and $d \cap s \cap X \neq \emptyset$, then $d \cap V_{s,\tau} \neq \emptyset$.

Let $\theta_{s,\tau} : V_{s,\tau} \to [0, 2\pi]$ denote an *angle function*, where $\theta_{s,\tau}(x)$ equals the slope of the line $o^\tau x$. Let $\{p_i\}_{i=1}^{|V_{s,\tau}|}$ denote an ordering of $V_{s,\tau}$ in increasing $\theta_{s,\tau}$ order. For a disc $d \in D_{s,\tau}$, we say that $d \cap V_{s,\tau}$ is an *interval* if there exist i, k such that $d \cap V_{s,\tau} = \{p_j \mid i \leq j \leq k\}$.

Proposition 26 ([5]). *The angle function $\theta_{s,\tau}$ is one-to-one, and $d \cap V_{s,\tau}$ is an interval, for every $d \in D_{s,\tau}$.*

Vertex ranking. Let $P_{s,\tau}$ denote the path graph over $V_{s,\tau}$ where p_i is a neighbor of p_{i+1} for $i = 1, \ldots, |V_{s,\tau}| - 1$. Let $c^{s,\tau} : V_{s,\tau} \to \mathbb{N}$ denote a vertex ranking with respect to $P_{s,\tau}$ that uses $\lfloor \log_2(2|V_{s,\tau}|) \rfloor$ colors.

Consider a disc $d \in D_{s,\tau}$. Let $r = r(d)$ denote the hyperedge $d \cap X$. Assume that $r \cap s \neq \emptyset$. Let $c^{s,\tau}_{\max}(r) \triangleq \max\{c^{s,\tau}(v) \mid v \in r \cap V_{s,\tau}\}$. Let $v^{s,\tau}_{\max}(r)$ denote the vertex $v \in r \cap V_{s,\tau}$ such that $c^{s,\tau}(v) = c^{s,\tau}_{\max}(r)$.

7.1.1. Algorithm description

A listing of the algorithm HS_d appears as Algorithm 2. The algorithm requires the following preprocessing: (i) Compute a tiling of the plane with $1/2 \times 1/2$ squares. Each point $x \in X$ must lie in the interior of a tile. This is easy to achieve since X is finite. (ii) For every tile s, compute the four types of extreme points $V_{s,\tau}$, and order each $V_{s,\tau}$ in increasing $\theta_{s,\tau}$ order. (iii) Compute a vertex ranking $c^{s,\tau}$ for each $V_{s,\tau}$. The algorithm maintains a hitting set C_i of the $i-1$ hyperedges $\{r_1, \ldots, r_{i-1}\}$ that have been input so far. Upon arrival of a hyperedge $r_i = r(d_i)$, if it is stabbed by C_{i-1}, then simply update $C_i \leftarrow C_{i-1}$. Otherwise, a vertex $v_{i,s}$ is selected from each square tile s such that $r_i \cap s \neq \emptyset$. These vertices are added to C_{i-1} to obtain C_i.

Lemma 25 provides an interpretation of Algorithm HS_d as a reduction to the case of hitting subsets of points below a pseudo-line (i.e., pseudo half-planes). Each square tile s and type $\tau \in \{1, \ldots, 4\}$ defines an instance of points and pseudo half-planes with respect to the set $X_s \triangleq X \cap s$ of points and the subsets $d \cap X_s$ for discs $d \in D_{s,\tau}$. The algorithm maintains a different invocation of HS_p for each square s and type τ. Upon arrival of an unstabbed disc d, the algorithm inputs the hyperedge $d \cap s \cap X$ to each instance of HS_p corresponding to a square s and a type τ such that $d \in D_{s,\tau}$.

Algorithm 2 $\text{HS}_d(X)$ - an online hitting set for unit discs.

Require: $X \subset \mathbb{R}^2$ is a set of n points. A tiling by $1/2 \times 1/2$ squares. Four types of extreme points $V_{s,\tau}$ per tile. A vertex ranking $c^{s,\tau}$ of $V_{s,\tau}$ with respect to the "angular" order.

```
C_0 ← ∅
for i = 1 to ∞ do {arrival of a hyperedge r_i = r(d_i)}
  if r_i not stabbed by C_{i-1} then
    for all square tiles s such that r_i ∩ s ≠ ∅ do
      τ ← τ(s, d_i) {find the type of d_i wrt s}
      v_{s,i} ← v^{s,τ}_max(r_i ∩ V_{s,τ}) {find the vertex with the max color}
      C_i ← C_{i-1} ∪ {v_{s,i}}
    end for
  else
    C_i ← C_{i-1}
  end if
end for
```

7.1.2. Analysis of the competitive ratio

Let $\sigma = \{r_i\}_i$ denote the input sequence. Let $\sigma^A \subseteq \sigma$ denote the subsequence of hyperedges r_i such that r_i is unstabbed upon arrival (i.e., r_i is not stabbed by C_{i-1}).

Proposition 27. $|\text{HS}_d(\sigma)| \leq 16 \cdot |\sigma^A|$.

Proof. Each disc intersects at most 16 square tiles. Upon arrival of an unstabbed disc, at most one point is added to the hitting set, for each intersected square. □

The following lemma shows that, if two discs contain a common point $x \in s$, are of the same type τ, and are unstabbed upon arrival, then they are stabbed by extreme points in $V_{s,\tau}$ of different colors.

Lemma 28. *If $x \in X \cap s$, $r_i, r_j \in \sigma^A \cap D_{s,\tau}$ and $x \in r_i \cap r_j$, then $c^{s,\tau}(v_{s,i}) \neq c^{s,\tau}(v_{s,j})$.*

Proof. To shorten notation let $c = c^{s,\tau}$, $V = V_{s,\tau}$, and $\theta = \theta_{s,\tau}$. Assume for the sake of contradiction that $c(v_{s,i}) = c(v_{s,j})$. By Proposition 26, $r_i \cap V$ is an interval, which we denote by $I_i = [a_i, b_i]$. Similarly, $I_j = [a_j, b_j]$ is the interval for $r_j \cap V$. Since $c^{s,\tau}$ is a unique-max coloring of the intervals in $V_{s,\tau}$, $I_i \cup I_j$ is not an interval, so there must be an extreme point in between the intervals. Denote this in-between point by t. Without loss of generality, $\theta(b_i) < \theta(t) < \theta(a_j)$. Assume that $\theta(x) \leq \theta(t)$. Consider a disc $d_j \in D_{s,\tau}$ such that $r_j = r(d_j)$. Consider a disc $d_t \in D_{s,\tau}$ such that $d_t \cap X_s = \{t\}$. We claim that the circles ∂d_j and ∂d_t intersect twice in the cone $K^{s,\tau}$, contradicting Lemma 25. Indeed, ∂d_t passes "below" x, "above" t, and "below" a_j. On the other hand, ∂d_j passes above x, below t, and above a_j. The case $\theta(x) > \theta(t)$ is proved similarly by considering the discs d_t and d_i. □

Let $\sigma(x)$ denote the subsequence of hyperedges r_i such that $x \in r_i$. The following lemma proves that the algorithm stabs a sequence of discs that share a common point by $O(\log n)$ points.

Lemma 29. *For every $x \in X$, $|\text{HS}_d(\sigma(x))| \leq 64 \cdot \lfloor \log_2(2n) \rfloor$.*

Proof. Fix a point $x \in X$, and let s denote the tile such that $x \in s$. Let $\sigma^A(x)$ denote the sequence of hyperedges in $\sigma(x)$ that were unstabbed upon arrival in an execution of $\text{ALG}(\sigma(x))$. By Proposition 27, $|\text{HS}_d(\sigma(x))| \leq 16 \cdot |\sigma^A(x)|$.

The disc d_i of each hyperedge $r_i \in \sigma^A(x)$ belongs to one of four types $D_{s,\tau}$, for $1 \leq \tau \leq 4$ By Lemma 28, the hyperedges in $\sigma^A(x) \cap D_{s,\tau}$ are stabbed by extreme points in $V_{s,\tau}$, the colors of which are distinct. Each vertex ranking $c^{s,\tau}$ uses at most $\lfloor \log_2(2n) \rfloor$ colors. Thus, $|\sigma^A(x)| \leq \sum_{\tau=1}^{4} |\sigma^A(x) \cap V_{s,\tau}| \leq 4 \cdot \lfloor \log_2(2n) \rfloor$, and the lemma follows. □

Proof of Theorem 23. Consider an execution of $\text{HS}_d(\sigma)$ and independent executions of $\text{HS}_d(\sigma(x))$, for every $x \in \text{OPT}(\sigma)$. Every time $\text{HS}_d(\sigma)$ is input an unstabbed hyperedge r_i, at least one of the executions of $\text{HS}_d(\sigma(x))$ is also input r_i, and r_i is also unstabbed upon arrival. This implies that $|\text{HS}_d(\sigma)| \leq \sum_{x \in \text{OPT}(\sigma)} |\text{HS}_d(\sigma(x))|$.

By Lemma 29, $|\text{HS}_d(\sigma(x))| = O(\log n)$. This implies that $|\text{HS}_d(\sigma)| = O(\log n) \cdot |\text{OPT}(\sigma)|$, and the theorem follows. □

8. Discussion

We would like to suggest two open problems.

1. Design an online hitting set algorithm for points and arbitrary discs, the competitive ratio of which is $o(\log^2 n)$ or prove a lower bound of $\Omega(\log^2 n)$.
2. Design an online hitting set algorithm with a logarithmic competitive ratio for any hypergraph with bounded VC-dimension or obtain a lower bound as above. Using Sauer's lemma that states that the number of hyperedges in a hypergraph with n vertices having VC-dimension d is $O(n^d)$, Alon et al. obtain an $O(\log^2 n)$ competitive ratio, and the best known lower bound is $\Omega(\log n)$ as demonstrated here for hypergraphs induced by points w.r.p to intervals.

Acknowledgments

We would like to thank Nabil Mustafa and Saurabh Ray for discussions on the open problems suggested in the last section.

Appendix. A lower bound on the vertex ranking number of graphs with bounded tree-width

Consider the graph $G_{d,k} = (V, E)$ that is the Cartesian product of a complete graph K_d and a path P_k. Namely, $G_{d,k}$ is obtained by cascading k cliques each with d vertices. We denote these k cliques by $C_1, \ldots, C_k$. Between every two cliques C_i and C_{i+1} the edges form a complete bipartite graph. Clearly, the tree-width of $G_{d,k}$ is $2d - 1$.

Claim 30. $\chi_{\text{vr}}(G_{d,k}) = d \cdot (\lfloor \log k \rfloor + 1)$.

Proof. The upper bound follows by assigning $\lfloor \log k \rfloor + 1$ distinct palettes to the cliques along the path (as in a vertex ranking of P_k). Each palette contains d distinct colors used to color the vertices in the clique C_i assigned to the palette.

The lower bound is proved as follows. First, we need to prove that (without loss of generality) the highest d colors appear in the same clique. We say that $X \subseteq V$ is d-clique-free if none of the cliques $C_1, \ldots, C_k$ is contained in X. Observe that if X is d-clique-free, then $G \setminus \{X\}$ is still connected. Consider a vertex ranking c of G. Order the vertices in descending colors, and consider the longest prefix $A \subset V$ that is d-clique-free. The vertices in A must have distinct colors and vertices outside A cannot use these colors. Indeed, let $c(A)$ denote the set of colors used to color vertices in A. If two different vertices u, v satisfy $c(u) = c(v)$ and $c(u) \in C(A)$, then there is a path from u to v, all the interior vertices of which are in $G \setminus \{A\}$. Such a path contradicts the assumption that c is a vertex ranking. Let v denote the next vertex (in the descending color order), and let $A' \triangleq A \cup \{v\}$. By the maximality of A, it follows that A' contains a clique. Let C_i denote the clique in A'. Note also that C_i is the only clique among $C_1, \ldots, C_k$ that is contained in A'. Repeating a similar argument, it follows that no other vertex in V has the same color as v. This implies that every coloring c' that is obtained by permuting the colors of the vertices in A' is also a vertex ranking. Indeed, two vertices with the same color do not belong to A'. Moreover, any path between them that intersects A' will retain the property of being a vertex ranking even after the colors of A' are permuted. Hence, without loss of generality, the vertices of C_i are colored by the d highest colors, as required.

The proof now follows by considering the recurrence for $\chi_{\text{vr}}(G_{d,k})$ which follows by the fact that we may assume that one of the cliques is colored by the d highest colors. Namely,

$$\chi_{\text{vr}}(G_{d,k}) \geq \begin{cases} d & \text{if } k = 1 \\ d + G_{d,\lceil k/2 \rceil} & \text{if } k > 1. \end{cases} \quad \square$$

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