# Mobile Facility Location*

Sergei Bespamyatnikh[†] Binay Bhattacharya[‡] David Kirkpatrick[§]

Michael Segal[¶]

**Abstract**

In this paper we investigate the location of mobile facilities (in $L_\infty$ and $L_2$ metrics) under the motion of clients. In particular, we present lower bounds and efficient algorithms for exact and approximate maintenance of the 1-center and 1-median for a set of moving points in the plane. Our algorithms are based on the *kinetic* framework introduced by Basch et al. [6].

## 1 Introduction

The goal of this paper is to study problems related to the location of mobile facilities serving a set of customers. The notion of mobile servers has been well studied in connection with stationary but transitory customer sets [26]. Here we introduce problems related to the location of mobile facilities, where customers are modeled by points moving continuously in $d$-dimensional space, $d \geq 1$. Specific examples of such problems are the maintenance of the $k$-center and $k$-median for moving points under the $L_p$ metric. This has applications, for example in mobile wireless communication networks when the broadcast range should contain all the customers so as to provide service to the cellular phones. There are likely to be many applications of mobile facilities, particularly with respect to tactical mobile packet-radio networks. Servers such as battlefield map servers, or even servers whose sole purpose is to provide support for network control (such as those that maintain entity-to-address information) could be considered to be mobile facilities. Another application of these problems is locating a welding robot in an automobile manufacturing plant. Both problems are interesting from both theoretical and practical points of view. In general we suspect that proximity problems related to systems of moving points will attract increasing interest from both theoretical and applied perspectives.

Facility location is a classical problem of operations research that has also been examined in the computational geometry community. Most of the problems described in the facility location literature are concerned with finding a "desirable" facility location: the goal is to *minimize* a

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[†]Department of Computer Science, Duke University, Box 90129, Durham, NC 27708, USA. Email: besp@cs.duke.edu.

[‡]School of Computing Science, Simon Fraser University, Burnaby, B.C. Canada, V5A 1S6. Email: binay@cs.sfu.ca.

[§]Department of Computer Science, University of British Columbia Vancouver, B.C. Canada V6T 1Z4. Email: kirk@cs.ubc.ca.

[¶]Communication Systems Engineering Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel. Email: segal@cse.bgu.ac.il.

distance function between the facility (e.g., a service) and the sites (e.g., the customers). To our best knowledge nothing has been done in the area of facility location for continuously moving points. Surprisingly, the data structures and algorithms that have been developed for the *static* problems (i.e., customers are not moving) are not directly applicable to the setting of moving points when the motion of the facilities must satisfy natural constraints. In this paper we begin the study of mobile versions of the two classical facility location problems:

1. **$k$-center:** Given a set $S$ of $n$ demand points in $d$-dimensional space ($d \geq 1$), find a set $P$ of $k$ supply points so that the *radius* defined as the maximum $L_p$ distance between a demand point and its nearest supply point in $P$ is minimized. Note that for some metrics such a set $P$ is not necessarily unique even for $k = 1$ and $d = 2$.

2. **$k$-median:** Given a set $S$ of $n$ demand points in $d$-dimensional space ($d \geq 1$), find a set $P$ of $k$ supply points so that the *dispersion* defined as the sum of the $L_p$ distances between demand points and their nearest supply points in $P$ is minimized. Again, it is possible that for some metrics the solution is not unique.

These problems have been well studied in both the exact [2, 7, 8, 9, 12, 13, 15, 16, 22, 27] and approximate [3, 10, 11, 12, 14, 19, 21, 24] versions. In approximate versions a set $P$ provides $(1+\varepsilon)$-approximate $k$-center ($k$-median) if the associated radius (dispersion) is at most $(1+\varepsilon)$ times the optimal radius (dispersion), for any $\varepsilon > 0$. Facility location problems for time varying networks (when edge distances satisfy the triangle inequality) also have been studied, see [18, 20]. We define the *mobile $k$-center* (*mobile $k$-median*) problem as follows. Given a set $S = \{p_1, p_2, \ldots, p_n\}$ of $n$ continuously moving points specified by a piecewise differentiable functions $\{g_1, g_2, \ldots, g_n\}$, where $g_i, 1 \leq i \leq n$ maps time interval $[0, T]$ to $\mathbb{R}^d$, we want to determine whether there exist $k$ continuous functions $f_1, \ldots f_k$; $f_i : [0, T] \to \mathbb{R}^d$ such that at any given moment $t \in [0, T]$, the points $f_1(t), \ldots, f_k(t)$ form a $k$-center ($k$-median) for the points at locations $g_1(t), \ldots, g_n(t)$, and, if so, find $f_1, \ldots, f_k$. The *mobile approximate $k$-center ($k$-median)* problem is defined similarly. In this paper we focus on the instances of these mobile facility location problems, when $k = 1, p = \{1, 2, \infty\}$. We also assume that $d = 2$ if the dimension is not specified. Even these simple instances pose several challenging algorithmic and geometric questions. We also consider *velocity restricted* versions of these instances, i.e., when the velocity of the facility may vary in some range.

The mobile approximate 1-center and related problems are studied by Agarwal and Har-Peled [1] very recently. For the Euclidean 1-center in the plane, they presented an approximation algorithm based on a notion of *extent*. Their algorithm aims to maintain an approximate 1-center with a few events only. However, the velocity of the facility can be arbitrary large (Lemmas 10 and 11). We designed strategies that guarantee both an approximation factor and a bounded velocity of the facility (Theorem 12 and Lemma 14).

Our algorithms for these problems are based on a new class of data structures — *kinetic data structures (KDS)* — aimed at keeping track of attributes of interest in systems of moving objects [6, 17]. In the kinetic setting, a set of points is assumed to be continuously changing, or moving. Each point follows a posted *flight plan*, but a plan can change at any moment through a *flight plan update*. A KDS maintains a *configuration function* of interest (which in our case will be the set $P$) by watching for critical events as the objects move. A KDS for computing a particular attribute for a set of points in motion maintains a set of *certificates*. A certificate based on a tuple of points is a continuous function that associates a real number with each configuration of these points. For example, the certificates for a convex hull KDS are a collection of the triple of points, each with a particular orientation. At any one time, the conjunction of all the certificates being maintained by the kinetic data structure proves the combinatorial correctness of its output. As the points move, some certificates may become invalid, e.g., a triple of points changes orientation. When a certificate fails, the proof structure needs to be modified and the combinatorial description of the configuration function may need to be updated. Certificates are stored in a priority queue, ordered by failure time, and processed in order as they fail. A good kinetic data structure will take advantage of the continuity of the point motions to select certificate structures that are easy to update at these critical events; a structure satisfying this condition is called *responsive*. Other criteria for a KDS are *efficiency*, i.e., the number of events processed by KDS is not much greater than the number of combinatorial changes in the configuration function itself, *compactness*, i.e., the number of active certificates at any one time is roughly linear in the complexity of the moving points, and *locality*, i.e., a flight plan update for any one point affects only a small number of certificates.

This paper is organized as follows. In the next section we show several results regarding the mobile rectilinear and Euclidean 1-center problems. Section 3 deals with the mobile 1-median problem. Section 4 provides some concluding remarks and questions for future investigation.

## 2 Mobile 1-center problems

In this section we investigate the complexity of mobile 1-center problems under the $L_\infty$ and $L_2$ metrics. We first show how to deal with the case of the $L_\infty$ metric and then extend our result to the Euclidean case.

### 2.1 Rectilinear 1-center

**Exact algorithm.** We are given a set $S$ of $n$ moving points (customers) in $\mathbb{R}^d, d \geq 1$ and we want to maintain a 1-center of these points (under the $L_\infty$ metric), i.e., a point $c$ with the property that the maximum distance between a point of $S$ and $c$ is minimized. We assume, for simplicity, that the motion of each point of $S$ (i.e., its flight plan) is determined by a linear function. We assume that the velocities of points are bounded by 1. During a flight plan update the velocity of the point can change but still not exceed 1.

**Observation 1** *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be fixed numbers such that $a_i \geq 0$ for all $i$ and $\sum_{i=1}^n \alpha_i = 1$. The point $p$ defined as the linear combination of the customer points in $\mathbb{R}^d, d \geq 1$, i.e., $p = \sum_{i=1}^n \alpha_i p_i$, $p_i \in S$ moves with velocity at most 1.*

Next, we observe that the exact 1-center $c$ can move faster than all the points. See Figure 1.

**Observation 2** *For any instance of the mobile 1-center problem in $\mathbb{R}^d, d \geq 1$ there is a rectilinear 1-center whose velocity is bounded by $\sqrt{d}$. Furthermore, there is an instance of the problem with a unique solution moving with velocity $\sqrt{d}$.*

*Proof.* The bounding box $\Pi_{i=1}^d [a_i, b_i]$ of the point set $S$ is defined by $2d$ points (some of them may coincide). The center of the bounding box has coordinates $((a_1 + b_1)/2, \ldots, (a_d + b_d)/2)$. The center of the bounding box can serve as a rectilinear 1-center. The $i$-th coordinate of the center of bounding box cannot move faster than velocity 1. Hence, the center of bounding box can move with velocity at most $\sqrt{d}$. On the other hand, Figure 1 shows an example when the rectilinear 1-center moves with velocity $\sqrt{2}$ in the plane. This can be extented to $\mathbb{R}^d, d \geq 1$ directly. ∎

The center of the bounding box and, thus, the rectilinear 1-center can be maintained using elementary data structures. The idea is to maintain points $p_l$,$p_r$, $p_t$, $p_b$ under motion of points.

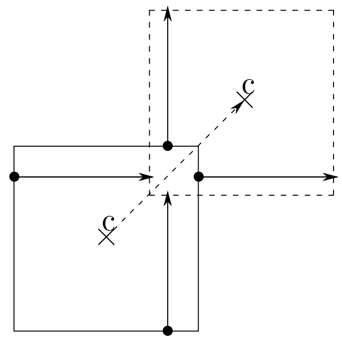


Figure 1: Rectilinear 1-center moves with velocity $\sqrt{2}$.

The problem is equivalent to the maximum maintenance problem [6]. Basch et al. [6] present several data structures to solve different variants of the maximum maintenance problem. For our purposes it suffices to use the kinetic swapping heap with $O(\log n)$ time responsiveness, $O(1)$ locality and $O(n \log^3 n)$ efficiency.

**Lemma 3** *A rectilinear* 1*-center can be maintained under the motion of points in an efficient KDS.*

Recall that the approximation factor at moment $t$ is defined as a relation between the radius determined by the facility $f$ and the radius determined by the optimal solution at this moment. Our goal is to minimize the approximation factor over all time. A natural question now is: What approximation factor can be achieved if we restrict the velocity of facility $f$ to some constant between 1 and $\sqrt{d}$? First, we consider the case of unit velocity. Notice that with restricted velocity we are not able to track the exact location of the rectilinear 1-center and, thus, kinetic versions of static 1-center algorithms do not suffice. Moreover, the approximation factor now depends on the initial position of the facility. In what follows we provide simple algorithms with guaranteed approximation factor 2 and then present a lower bound for the approximation factor when the facility moves with velocity at most 1.

**2-approximation factor.** If at the beginning of the motion of points we are allowed to put our facility $f$ on any point in the plane, the simple way to achieve 2-approximation factor is to put our facility on any demand point and follow its flight plan. Obviously, our facility will always be inside the bounding box of $S$, thus, providing a 2-approximation factor since the exact radius is equal to half the length of the largest side of the bounding box. In fact the same approximation factor can be achieved even in the case when the starting location of the facility is restricted to be any point inside the convex hull of $S$.

**Lemma 4** *Starting with the facility at an arbitrary location inside the convex hull of $S$ in $\mathbb{R}^d, d \geq 1$ there is a flight plan for a single facility that guarantees a* 2*-approximation of the rectilinear* 1*-center of $S$ and can be maintained efficiently.*

*Proof.* As a preprocessing step we compute the convex hull and a triangulation $T$ of $S$. We find the simplex $\Delta$ in $T$ that contains the facility $f$. Let $p_1, \ldots, p_d$ be the vertices of $\Delta$. See Figure 2.

Our goal is to keep the facility inside $\Delta$. A simple way is to find the convex combination $f = \alpha_1 p_1 + \ldots + \alpha_d p_d$, $\sum_{i=1}^{d} \alpha_i = 1$ and maintain $f$ according to this formula. By Observation 1, $f$ moves with velocity at most 1. ∎

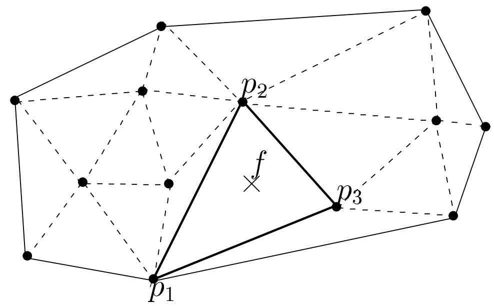


Figure 2: Facility lies inside triangle $\Delta$ with vertices $p_1$, $p_2$, $p_3$.

Being inside the convex hull (initially) is crucial for achieving a guaranteed approximation factor.

**Observation 5** *No approximation factor can be guaranteed if the initial location of the facility is outside the convex hull of $S$ in $\mathbb{R}^d, d \geq 1$.*

*Proof.* The points of $S$ can move according to the following strategy. They will move toward some point $r$ in the plane such that the distance from $f$ to $r$ is larger than any distance from $r$ to points in $S$. Such a point always exists since $f$ is outside convex hull. Thus, at point $r$ the radius determined by the optimal 1-center is equal to 0 and the radius determined by $f$ is greater than 0 which violates any approximation factor. ∎

Following Observation 5 we note that if the facility is restricted to move with velocity less that 1 then no approximation factor is achievable.

**Improvement of the approximation factor.** Now we present a new algorithm with a slightly improved approximation factor for a facility moving with unit velocity. The key idea is to use the notion of the center of mass. Given a set $S$ of $n$ points we can define mass in various ways:

- **Point mass.** The weight $\frac{1}{n}$ is assigned to each point.
- **Region mass.** The mass is uniformly distributed in the convex hull.
- **Boundary mass.** The mass is uniformly distributed on the boundary of the convex hull.

By Observation 1, the center of mass of the points moves with velocity at most 1. In contrast, the center of mass of the convex hull and the center of mass of the boundary of convex hull can move with velocity exceeding the maximum velocity of customers (See Figure 3(a)).

Figure 3(a) illustrates an example of four points $p_1, p_2, p_3, p_4$ moving with unit velocity over one unit of time. Initially, the center of mass of the convex hull (or its boundary) is on the segment $(p_2, l)$ and its final location is on the segment $(p_3, m)$. Any segment that connects interiors of $(p_2, l)$ and $(p_3, m)$ has a length larger than 1. Hence, the center of mass of the convex hull (or its boundary) moves with the velocity exceeding 1.

We have shown already an example (Figure 1) where the rectilinear 1-center moves with velocity $\sqrt{2}$ while the points move with unit velocity. When we allow the facility to move with unit velocity the best approximation factor that can be achieved in this example is $2-1/\sqrt{2}$. This approximation factor is defined by the facilities $f$ and $f'$ on the segment $cc'$ in Figure 3(b) such that the length of the segment connecting $f$ and $f'$ is 1. It is interesting to note that the center of mass of the

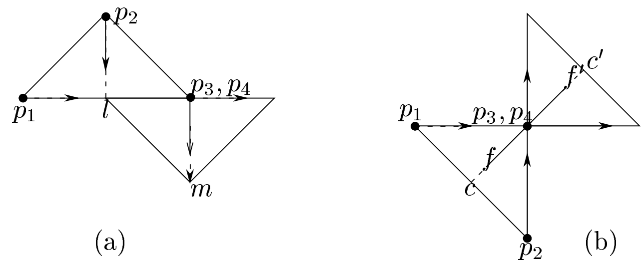


Figure 3: (a) Center of mass of the convex hull (or the boundary of convex hull) moves with velocity exceeding 1. (b) Center of mass of the boundary of convex hull is optimal strategy for given example.

boundary of the convex hull achieves this approximation factor. Clearly, that the center of mass the boundary of the convex hull is always on the segment $cc'$. Using the definition of the center of mass one can show that the $x$-coordinate of the center of mass the curve $f(x)$ defined on interval $[a, b]$ is $\frac{\int_a^b x\sqrt{1+(f'(x))^2}dx}{l}$, where $l$ is the length of the curve. It can be verified that the initial position of the center of mass of the triangle $\Delta p_1p_2p_3$ coincides with the position of facility $f$ and finally with the position of $f'$.

**Lemma 6 (Center of Mass)** *The center of mass of points guarantees a $(2-\frac{2}{n})$-approximation of the rectilinear 1-center of $S$ in $\mathbb{R}^d, d \geq 1$ and can be maintained efficiently under the motion of points in $S$.*

*Proof.* By Observation 1 the velocity of the center of mass of the points does not exceed the maximal velocity of the customers. Clearly, the worst possible case is when $n-1$ points coincide with the origin and the other point has coordinates $(1, 0, \ldots, 0)$. The center of mass of the points has coordinates $(\frac{1}{n}, 0, \ldots, 0)$. The relation between the radius determined by the center of mass and the radius determined by the optimal solution is $(1-\frac{1}{n})/\frac{1}{2} = 2-\frac{2}{n}$. In our kinetic model we maintain the queue of events: changing velocity of the points and changing flight plan of the points. We also maintain the location and velocity of the center of mass. Given a new event we can calculate in constant time the current location of the center of mass and its new velocity. ∎

Surprisingly, the approximation factor stated in the previous lemma is, in fact, asymptotically optimal. It is also interesting that the bound is optimal for *any* metric in *any* dimension.

**Theorem 7 (Lower Bound)** *Consider the mobile approximate 1-center problem in $\mathbb{R}^d, d \geq 2$ with any underlying metric. Any algorithm that starts with the facility inside the convex hull and moves the facility with at most unit velocity achieves an approximation factor of at least 2 (asymptotically) for the exact 1-center in the worst case.*

*Proof.* It suffices to prove the lower bound for $d = 2$ since two-dimensional worst case scenario can be embedded into $\mathbb{R}^d, d \geq 3$. We prove this theorem by an adversary argument. The adversary picks a set $S$ that contains points at vertices $A, B, C$ of an equilateral triangle in Figure 4. Without loss of generality we assume that there are arbitrary many points at each of the vertices $A$, $B$ and $C$ of triangle. The edge size of the triangle is 2. An algorithm starts with some initial location of facility $f$ inside triangle $\Delta ABC$. In order to prove a lower bound, the adversary will force an

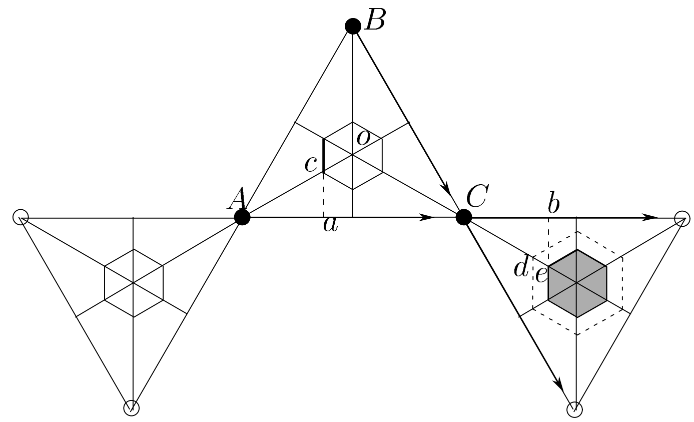


Figure 4: Adversary strategy for achieving lower bound for approximation factor.

algorithm to put a facility $f$ to the boundary of the bounding box of $S$. This goal can be achieved doing the following strategy. The adversary checks which one of the triangles $AoB, BoC, CoA$ contains the facility, where $o$ is the center of $\Delta ABC$. Depending on location of the facility the adversary moves points into the next equilateral triangle. For example, if the facility is located in triangle $AoB$ the adversary moves all the points to the right triangle in Figure 4. All points at vertices $A$ and $B$ move towards vertex $C$ and the points at vertex $C$ are split into two halves that move to the two remaining vertices of right triangle in Figure 4. Similarly, if the facility is located in triangle $BoC$ the adversary moves the points to the left triangle (the symmetric case of triangle $CoA$ is not shown in Figure 4). Notice that all points of $S$ move distance 2 from one triangle to another. The adversary continues to play this game until it forces the facility to be arbitrarily close to a vertex of the current triangle. To show this we define the distance of the facility to the center of the current triangle using a hexagon centered at the center of the triangle. Assume that the facility is located in $\Delta ABC$ on the left side of the hexagon centered at $o$. We show that the next location of the facility will be outside the hexagon of the same size, shaded in Figure 4. This follows easily from the fact that the distance between points $a$ and $b$ is 2. Consider the smallest hexagon (dashed in Figure 4) that can be reached by the facility. Let $c$ be the lowest point in the left side of the hexagon centered at $o$ and $d$ be the highest point on the left side of the dashed hexagon. The distance between $c$ and $d$ is 2. Let $x_n$ be the length of the segment $Ac$, i.e., $x_n = |Ac|$ and $x_{n+1} = |Cd|$. By Pythagoras's Theorem we have

$$((x_{n+1} - x_n)\sqrt{3}/2 + 1)^2 + \frac{(x_n + x_{n+1})^2}{4} = 1. \tag{1}$$

Solving this equation we obtain

$$x_{n+1} = \frac{x_n - \sqrt{3} + \sqrt{3 - 3x_n^2 + 2\sqrt{3}x_n}}{2} = \frac{x_n - \sqrt{3} + \sqrt{3}\sqrt{1 - x_n^2 + 2x_n/\sqrt{3}}}{2}$$

First we observe that $\sqrt{1 - x_n^2 + 2x_n/\sqrt{3}} < 1 + (1 - x_n)x_n/\sqrt{3}$ if $x_n < 1 - \sqrt{2\sqrt{3} - 3}$. Therefore, $x_{n+1} < x_n - \frac{x_n^2}{2}$ for $x_n < 1 - \sqrt{2\sqrt{3} - 3}$. Notice that $x_4 < 0.3 < 1 - \sqrt{2\sqrt{3} - 3}$ assuming that $x_1 = \frac{1}{\sqrt{3}} = |Ao|$. As we have shown above the sequence $\{x_n\}_{n=1}^{\infty}$ is monotonically decreasing. Hence $x_{n+1} < x_n - \frac{x_n^2}{2}$ for all $n \geq 4$. It can be shown by induction that $x_n < \frac{2}{n}$. Therefore, $\lim_{n\to\infty} x_n = 0$

and the facility is forced to be either outside the current triangle or arbitrarily close to one of the vertices of the current triangle. In the first case, the customers can run away from the facility since it is outside their convex hull.

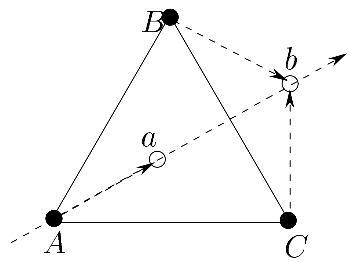


Figure 5: Final step of the adversary strategy.

In the second case, we continue the game according to the following rules, see Figure 5. Suppose that the facility is close to the vertex $A$. The points from the vertices $B$ and $C$ move toward any fixed point $b$ outside triangle $ABC$ that lies on the bisector of $\angle BAC$ and the points from the vertex $A$ move along the bisector to the point $a$ ($|Bb| = |Aa|$). When the points from the vertices $B$ and $C$ meet at $b$, the exact radius is equal to $\frac{|ab|}{2}$ and the facility is arbitrarily close to $a$ (to the right). Thus, the approximation factor is arbitrarily close to 2. ∎

**Motion with bounded velocity.** Suppose now, that the velocity of the facility is bounded by $v_{max} \in [1, \sqrt{2}]$. Recall that for values of $v_{max} = 1$ and $\sqrt{2}$ the best approximation factors are 2 and 1, respectively. We *mix* our two strategies — the center of mass of the points and the center of the bounding box — in the following way. Let $(f_1, v_1)$ be the location (vector in the plane) and velocity of the center of mass of the points. Similarly, $(f_2, v_2)$ are the location and velocity of the center of the bounding box. The mixing strategy is to maintain the *mixing center* $(f, v)$ defined as $(\alpha f_1 + (1-\alpha) f_2, \alpha v_1 + (1-\alpha) v_2)$, where $\alpha = (\sqrt{2} - v_{max})(\sqrt{2} - 1)$. We analyze the mixing strategy and show how it can be improved in the following theorem.

**Theorem 8 (Bounded Velocity)** *Suppose that the facility is allowed to move with velocity at most $v_{max} \in [1, \sqrt{2}]$. Let $a_1(v)$ be the linear function defined by $a_1(1) = 2$ and $a_1(v') = a'$ where $v' = \sqrt{2}\cos(\pi/8)$ and $a' = 1.25$. Let $a_2(v)$ be the linear function defined by $a_2(v') = a'$ and $a_2(\sqrt{2}) = 1$. There is a strategy of the facility that guarantees an approximation factor $\max(a_1(v_{max}), a_2(v_{max}))$.*

*Proof.* First we show that the velocity of the mixing center is bounded by $v_{max}$. We want to prove the inequality $\alpha v_1 + (1-\alpha) v_2 \leq v_{max}$. In order to do this, we replace $v_1$ by 1 and $v_2$ by $\sqrt{2}$. It can easily be seen that the value of the expression $\alpha + (1-\alpha)\sqrt{2}$ when $\alpha = (\sqrt{2} - v_{max})(\sqrt{2} - 1)$ is equal to $v_{max}$.

Regarding the approximation factor we consider the case in which the center of mass is at largest $L_\infty$ distance from the center of the bounding box. This distance is $1 - 2/n$ if the radius that is determined by the center of the bounding box is 1. Then the $L_\infty$ distance between the mixing center and the center of the bounding box is $\frac{\sqrt{2}-v_{max}}{\sqrt{2}-1}(1 - \frac{2}{n})$. The approximation factor bound is

$$1 + \frac{\sqrt{2} - v_{max}}{\sqrt{2} - 1}\left(1 - \frac{2}{n}\right).$$

The smallest bound that suits all $n$ is the linear function $1 + (\sqrt{2} - v_{max})(\sqrt{2} - 1)$.

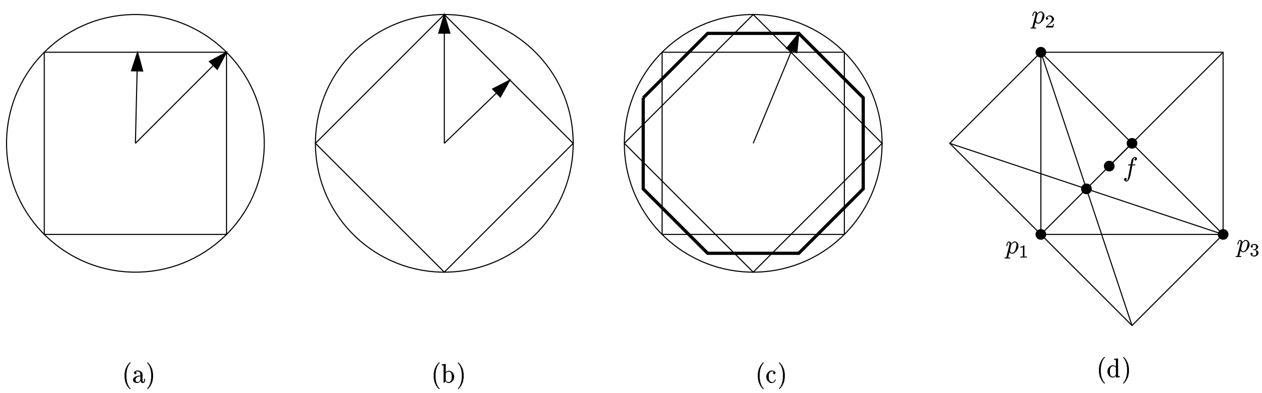


Figure 6: 8-gon strategy.

It turns out that the mixing strategy is not easy to improve. The reason is that the worst case velocities of two strategies, the center of mass (of all the points or a linear combination of a few points from $S$) and the center of bounding box, can be in the same direction as in Fig. 3(b). In this case the best motion of the facility is to move in this direction withthe maximum allowed velocity which corresponds to mixing strategy. The idea now is to interpolate two strategies that have the worst case velocities in different directions. Note that the maximum velocity of the center of bounding box depends on the direction and varies from 1 to $\sqrt{2}$, see Fig. 6(a). A good candidate for the interpolation is the strategy of the center of bounding box aligned to the coordinate axis rotated by 45°. The range of the maximum velocity is depicted in Fig. 6(b). Consider the midpoint of two centers. We call this *8-gon strategy* since the range of its maximum velocity forms the regular 8-gon, see Fig. 6(c). The maximum velocity of the facility following the 8-gon strategy is $v' = \sqrt{2}\cos(\pi/8)$ and the approximation in worst case is $a' = 1.25$ for three points $S = \{p_1, p_2, p_3\}$, see Fig. 6(d). One can show that the approximation factor of the 8-gon strategy is smaller than one of the mixing strategy for $v_{max} = v'$. In order to improve the approximation for other values of $v_{max}$ we

(i) interpolate (or mix) the strategy of the center of mass and the 8-gon strategy for $v_{max}$ in the range $[1, v']$, and

(ii) interpolate the strategy of the center of the bounding box and the 8-gon strategy for $v_{max} \in [v', \sqrt{2}]$.

The theorem follows. ∎

**Variants of mixing strategy.** We consider two different ways to improve the mixing strategy. First we notice that the mixing strategy described above constrains the initial location of the facility to some particular point. We avoid this by replacing the center of mass of the points in the mixing strategy by the algorithm described in Lemma 4. In other words, the initial location of the facility is a linear combination of any point $p$ inside the convex hull and the center $c$ of the bounding box. The resulting approximation factor, however, grows slightly to $1 + \frac{\sqrt{2}-v_{max}}{\sqrt{2}-1}$.

The second improvement concerns the approximation factor. The mixing center in the previous strategy sometimes may move with velocity less than $v_{max}$. In this case, we may use the excess of the velocity to improve the approximation factor by moving the mixing center $f$ towards the center of the bounding box. When the mixing center $f$ moves, the coefficient $\alpha$ in the linear combination of $p$ and $c$ becomes invalid. We would like to preserve the initial value of $\alpha$. It can be viewed as a new motion of point $p$. However, the point $p$, which initially lies in some triangle $\Delta p_i p_j p_k$

inside the convex hull of $S$ (since $c$ is inside the convex hull), may move outside this triangle. We can calculate this event and put it into the queue of events. When this event happens the point $p$ is still inside the convex hull and we need to compute a new triangle that contains $p$. It can be done as follows. We may assume that only points on the convex hull form a triangulation $T$. The mobile convex hull can be maintained kinetically [5] with $O(\log n)$ locality and $O(\log^2 n)$ time responsiveness. We introduce three new types of events:

1. one of the vertices of $\Delta p_i p_j p_k$ no longer serves as a vertex of the convex hull,
2. a new vertex appears on the convex hull,
3. the facility reaches the boundary of $\Delta p_i p_j p_k$.

In the first case we can update $T$ in $O(1)$ time and find a new triangle that contains $p$ and calculate the three new certificates providing the proof of $p$ inside this triangle. The second case is reverse to the first one and can be maintained similarly. In the third case $p$ moves from one triangle $\Delta$ in $T$ to an adjacent triangle $\Delta_1$, the vertices of $\Delta_1$ can be found in constant time using the maintained convex hull. From the practical point of view this strategy is more useful than the strategies described above.

**Remark.** It is well known that the metrics $L_1$ and $L_\infty$ are dual in the plane, in the sense that nearest neighbors under $L_1$ in a given coordinate system are also nearest neighbors under $L_\infty$ in a 45 degrees rotated coordinate system (and vice versa). The distances, however, are different by a multiplicative factor of $\sqrt{2}$. Thus, the results for $L_\infty$ carry over to the $L_1$ metric and vice versa.

**Lower bound for an approximation factor when** $v_{max} \in [1, \sqrt{2}]$. In fact we can show two different lower bound estimates. The first bound is based on the example depicted in Figure 3(b) when $|ff'| = v_{max}$ and $|p_1p_4| = |p_2p_4| = 1$. In this case the exact rectilinear radius is $\frac{1}{2}$ and the approximate radius is $1 - \frac{v_{max}}{2\sqrt{2}}$. Thus, the approximation factor $\beta(v_{max}) = 2 - \frac{v_{max}}{\sqrt{2}}$.

The second lower bound is based on the proof of Theorem 7. In fact, a similar proof can be applied for $v_{max} \in [1, \frac{2}{\sqrt{3}}]$. If $v_{max} \geq \frac{2}{\sqrt{3}}$ the facility can follow the center $o$ of the triangle. For $v_{max} \in [1, \frac{2}{\sqrt{3}}]$ the facility can always be inside a hexagon of some fixed size. Let $x = |Aa|$ and $|ce| = 2v$ on Figure 4. One can show that $x = \sqrt{3v_{max}^2 - 3}$. The rectilinear distance from $c$ to $B$ is $\sqrt{3} - \frac{x}{\sqrt{3}}$ and the distance from $c$ to $C$ is $2 - x$. The exact radius is equal to 1. Therefore, the approximation factor is $\gamma(v_{max}) = \max(2 - x, \sqrt{3} - \frac{x}{\sqrt{3}})$, for $x = \sqrt{3v_{max}^2 - 3}$. One can show that $\beta(v_{max}) > \gamma(v_{max})$ if $v_{max} > 2/\sqrt{3}$. Combining all the results together we obtain

**Lemma 9** *If the facility's maximal allowed velocity is* $v_{max} \in [1, \sqrt{2}]$*, then the worst-case approximation factor of* any *algorithm is at least* $\max(\beta(v_{max}), \gamma(v_{max}))$.

## 2.2 Euclidean 1-center

Although the static Euclidean 1-center problem, like the static rectilinear 1-center problem, can be solved in linear time [25], the mobile versions of these problems are quite different. In contrast to the rectilinear 1-center problem (see Observation 2) we show an example providing that the Euclidean 1-center may move with unbounded velocity. See Figure 5.

**Lemma 10 (Unbounded Velocity)** *For any velocity* $V \geq 0$ *there is a motion of three points* $p_1, p_2, p_3$ *in* $\mathbb{R}^d, d \geq 2$ *with unit velocity whose Euclidean* 1*-center moves instantaneously with velocity faster than* $V$.

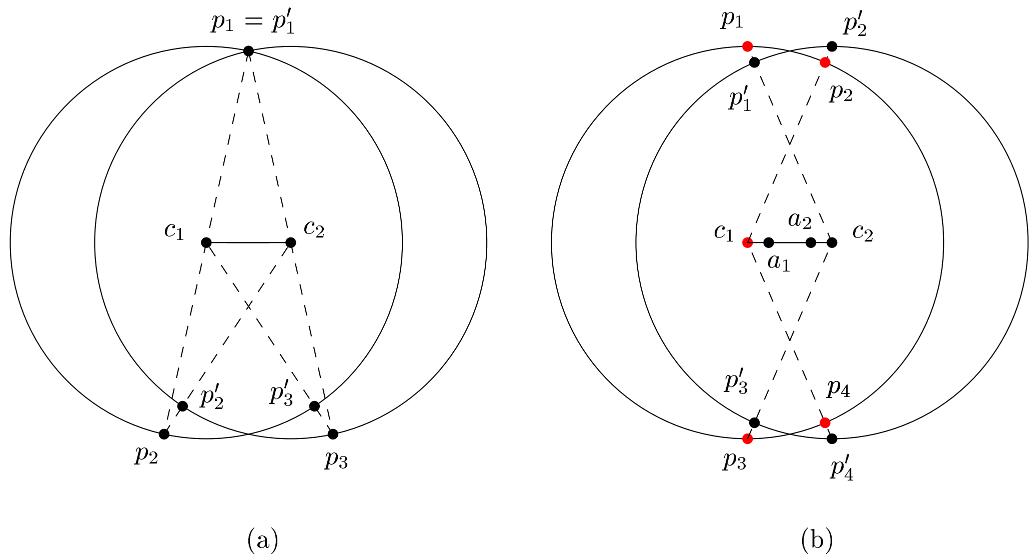


Figure 7: Euclidean 1-center may move with unbounded velocity.

*Proof.* The Lemma can be shown using an example of three points $p_1, p_2, p_3$ lying in any plane of $\mathbb{R}^d$ and moving to points $p_1', p_2', p_3'$, respectively, see Figure 5 (a). We use another example with four points $p_1, \ldots, p_4$ which provides a better bound of the velocity of the exact Euclidean 1-center, see Figure 7 (b). The points $p_1, \ldots, p_4$ are located on the unit circle centered at $c_1$. A point $p_i, i = 1, \ldots, 4$ moves toward the point $p_i'$. The points $p_i$ make the same length paths since $c_1 c_2 p_2' p_1$ and $c_1 c_2 p_4' p_3$ are rectangles.

Let $x = |c_1 c_2|$ be the length of the path made by the 1-center and let $y = |p_1 p_1'|$ be the length of the path made by the point $p_1$. It suffices to show that $y/x$ tends to 0 if $x$ tends to 0. Indeed, $1 + x^2 = (1+y)^2$ since the triangle $c_1 c_2 p_1$ is right. It implies $x^2 = 2y + y^2$ and $2y/x = x - y^2/x \leq x$. The Lemma follows. ∎

We show that velocity of an approximate 1-center must be high.

**Lemma 11 (Lower Bound)** *For every $\varepsilon > 0$, any $(1+\varepsilon)$-approximate mobile Euclidean 1-center has velocity at least $1/4(\sqrt{2\varepsilon} + \varepsilon) = 1/(4\sqrt{2\varepsilon}) - O(1)$ in worst case.*

*Proof.* Let $x = |c_1 c_2|$ and $y = |p_1 p_1'|$. The value of $y$ depends on $x$ that will be specified later. The exact Euclidean 1-center moves from $c_1$ to $c_2$. Consider the points $a_1$ and $a_2$ defined by $|c_1 a_1| = |a_2 c_2| = x/4$, see Figure 7 (b). Suppose that an approximate 1-center of points $p_1, \ldots, p_4$ is at $a_1$. Then the smallest radius of a disk covering the points is $|p_1 a_1|$. We assume that $|p_1 a_1| = 1+\varepsilon$. Then any $(1+\varepsilon)$-approximate 1-center must traverse the distance at least $|a_1 a_2| = x/2$. Its velocity is at least $|a_1 a_2|/|p_1 p_1'| = x/2y$.

We obtain $(1+\varepsilon)^2 = 1 + (x/4)^2$ from the triangle $c_1 a_1 p_1$. Hence

$$x = 4\sqrt{2\varepsilon + \varepsilon^2}.$$

We can bound $x \leq 4\sqrt{2\varepsilon} + 4\varepsilon$ since $16(2\varepsilon + \varepsilon^2) \leq 16(2\varepsilon + \varepsilon^2 + 4\varepsilon\sqrt{\varepsilon})$. Note that $y \leq x^2/2$ since $x^2 = 2y + y^2$. Therefore the velocity of an approximate 1-center is at least

$$V \geq \frac{x}{2y} \geq \frac{1}{x} \geq \frac{1}{4\sqrt{2\varepsilon} + 4\varepsilon}.$$

It remains to show that $1/(4\sqrt{2\varepsilon}+4\varepsilon) = 1/(4\sqrt{2\varepsilon}) - O(1/\varepsilon)$. Indeed,

$$\frac{1}{4\sqrt{2\varepsilon}} - \frac{1}{4\sqrt{2\varepsilon}+4\varepsilon} = \frac{1}{4\sqrt{2}(\sqrt{2}+\sqrt{\varepsilon})} \to \frac{1}{8} \text{ as } \varepsilon \to 0.$$

∎

We show that the lower bound of the velocity needed to approximate mobile Euclidean 1-center established in Lemma 11 is optimal up to a constant factor.

**Theorem 12 (Upper Bound)** *For any $\varepsilon > 0$ there is a strategy for mobile approximate Euclidean 1-center that guarantees the approximation factor $1+\varepsilon$ using velocity of the facility $4/\sqrt{\varepsilon}+o(1/\sqrt{\varepsilon})$ in the worst case.*

*Proof.* We apply the following strategy for the mobile facility. Let $V$ be the maximum velocity of the facility that will be specified later. Consider the initial configuration. We assume that in the beginning the facility $f$ is located so that the radius of the smallest circle enclosing all the customer points is at most $(1+\delta)$ times the optimal radius where $\delta \le \varepsilon$ and will be specified later. The goal of the facility now is to reach the position of the exact Euclidean 1-center at this moment. The facility heads to the target using the velocity $V$. We can compute the time $t$ needed for this. After reaching the target we repeat the procedure. It is interesting that the motion of the facility is independent on the motion of customers during the time when facility moves from one position to another. It is also interesting that the exact 1-center can achieve any velocity sometimes and the approximate facility just ignores any acceleration of the exact 1-center.

Let $c$ be the position of the exact Euclidean 1-center at the initial time $t_0$ and let $a$ be the position of the facility (the approximate 1-center) at the same time. Let $t_1$ be the time when the facility reaches $c$. Let $r$ denote the radius associated with $c$ at the time $t_0$. Without loss of generality we can assume that $r = 1$.

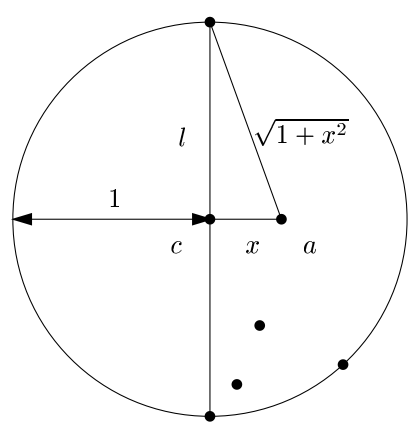


Figure 8: $(1+\varepsilon)$-approximation of Euclidean 1-center.

First we estimate the distance between the exact and approximate centers at the time $t_0$. Denote $x = |ac|$. The line $l$ orthogonal to the line $ac$ passing through $c$ partition the unit circle centered at $c$ into 2 arcs, see Fig. 8. Each arc contains at least one customer point since the exact radius of the 1-center disk is 1. The minimum distance from $a$ to the arc beyond $l$ is $\sqrt{1+x^2}$. Clearly, $\sqrt{1+x^2} \le 1+\delta$. Therefore $x = |ac| \le \sqrt{2\delta+\delta^2}$.

Let $t$ be the time the facility need to get the exact center $c$, i.e., $t \leq |ac|/V$. Consider the moment $t_1$. The radius associated with the approximate 1-center $c$ is at most $1 + t$ since the velocities of the customers are bounded by 1. Let $c'$ be the location of the exact 1-center. We show that the associated radius is at least $1 - t$. Construct the line passing through $c$ and orthogonal to $cc'$. The arc of the unit circle centered at $c$ from which the exact center moves toward $c'$ contains at least one customer point at the moment $t_0$. This customer is at distance at least $1 - t$ from $c'$ at the moment $t_1$. Thus, the approximation ratio at the moment $t_1$ is at least

$$\frac{1+t}{1-t} = 1 + \frac{2t}{1-t}.$$

We want to show that this is at most $1 + \delta$. It follows from

$$\frac{2t}{1-t} \leq \delta \quad \text{or, equivalently,} \quad t \leq \frac{\delta}{2+\delta}. \tag{2}$$

We also want the approximation factor to be at most $1 + \varepsilon$ for any moment $t_0 + \Delta \in [t_0, t_1]$. The radius associated with the approximate 1-center is at most $1 + \delta + \Delta$. The radius is at least $1 - \Delta$. Their ratio is at most

$$\frac{1+\delta+\Delta}{1-\Delta} = 1 + \frac{2\Delta+\delta}{1-\Delta}.$$

The largest value of the upper bound is achieved when $\delta = t$. It suffices to have

$$\frac{2t+\delta}{1-\delta} \leq \varepsilon \quad \text{or, equivalently,} \quad t \leq \frac{\varepsilon-\delta}{2+\varepsilon}. \tag{3}$$

Making the right sides of the equations 2 and 3 we obtain $\delta^2 + 4\delta - 2\varepsilon = 0$ and $\delta = \sqrt{4+2\varepsilon} - 2 \approx \varepsilon/2$. By the equation 2 we can set $t = \delta/(2+\delta) \approx \varepsilon/4$. Therefore, the velocity of the facility can be $V = \sqrt{2\delta + \delta^2}/t \approx 4/\sqrt{\varepsilon}$. The theorem follows. ■

Lemmas 10 and 11 demonstrate negative results. Fortunately, results analogous to Lemma 4 and Lemma 6 for the Euclidean 1-center can easily be obtained.

**Theorem 13 (Bounding Box Strategy)** *The strategy of tracing the center of the bounding box of $n \geq 3$ points has an approximation factor of $(1+\sqrt{2})/2$ for the mobile Euclidean 1-center and this bound is tight.*

*Proof.* Let $s = (1+\sqrt{2})/2$. Without less of generality we can assume that the bounding box is $B = [0,2] \times [0,2t]$ for some $t \geq 1$, see Fig. 9 (a). Let $S$ be any configuration of $n \geq 3$ points in $B$ such that each side of $B$ contains at least one point of $S$. Let $r$ be the approximate radius, i.e., $r_a = \max_{p \in S}\{|po|\}$ where $c_a = (1,t)$ is the center of $B$. Let $r$ be the exact radius of 1-center. We want to prove that $r_a \leq sr$.

We first consider the case where one of the points of $S$ is located at a vertex of $B$, say $p_1 = (2, 2t)$. It implies $r_a = \sqrt{1+t^2}$. Two sides of $B$ contain $p_1$, and there are at least two points $p_2$ and $p_3$ lying on the sides of $B$ along $x$- and $y$-axis, respectively. We can assume that $p_2 \neq p_3$, otherwise $p_2 = p_3 = (0,0)$ and $r = r_a$. If $t > 2$ then $r \geq |p_1p_2| \geq t$ and one can check that

$$\frac{r_a}{r} \leq \frac{\sqrt{1+t^2}}{t} = \sqrt{\frac{1}{t^2} + 1} \leq \sqrt{1\frac{1}{4}} < s.$$

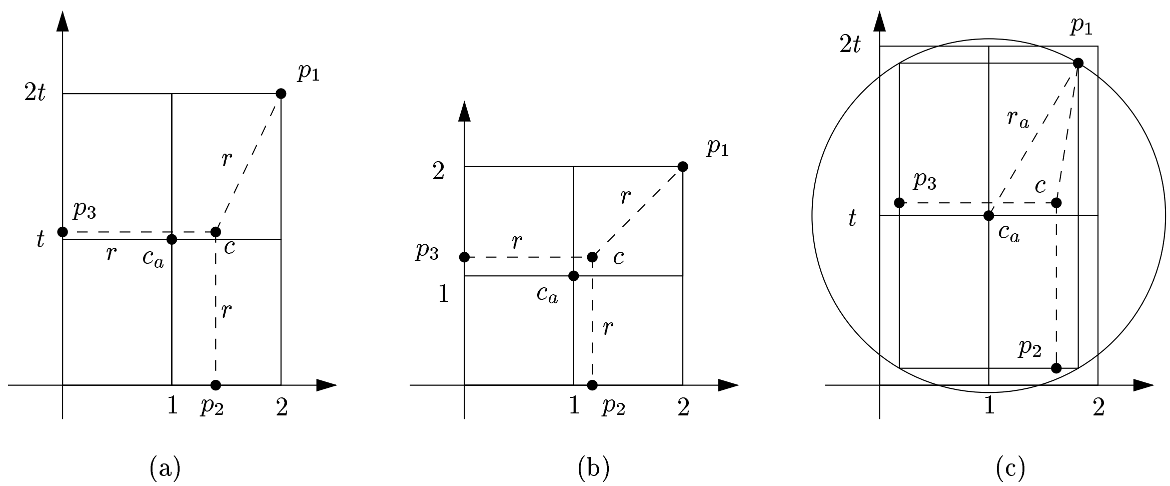


Figure 9: Bounding box strategy for approximating Euclidean 1-center.

We assume that $t \in [1, 2]$. The smallest value of $r$ is achieved when $p_2$ is strictly below the 1-center and $p_3$ is to the left of it, see Fig. 9 (a). Thus the exact 1-center has coordinates $c = (r, r)$. The 1-center is at distance $r$ from $p_1$,

$$(2 - r)^2 + (2t - r)^2 = r^2.$$

This can be transformed to the quadratic equation

$$r^2 - 4(t + 1)r + 4(t^2 + 1) = 0.$$

$r = 2(t + 1 - \sqrt{2t})$ since the root with "+" is greater than $2t$ which is impossible.

$r_a$ and $r$ can be viewed as functions of $t$. In order to prove that $r_a/r \leq s$ for $t \in [1, 2]$ we show that (i) $r_a(1)/r(1) = s$, and (ii) $(r_a/r)' \leq 0$ for any $t \in [1, 2]$. If $t = 1$ then $r_a = \sqrt{2}$ and $r = 2(2 - \sqrt{2})$. One can check that $r_a/r = s$. Note that this gives an example with tight bound, see Fig. 9 (b).

To prove the second condition we differentiate $r$ and $r_a$

$$r' = 2(1 - 1/\sqrt{2t}) \quad \text{and} \quad r'_a = t/\sqrt{1 + t^2}.$$

We want to prove that $r'_a r - r_a r' \leq 0$ or, equivalently,

$$\frac{2t}{\sqrt{1 + t^2}} \left( t + 1 - \sqrt{2t} \right) \leq 2\sqrt{1 + t^2} \left( 1 - \frac{1}{\sqrt{2t}} \right).$$

Let $x = \sqrt{t/2}$. Then $t = 2x^2$ for $x \in [\sqrt{2}/2, 1]$ and the inequality can be written as

$$2x^2(2x^2 + 1 - 2x) \leq (1 + 4x^4) \left( 1 - \frac{1}{2x} \right).$$

Multiplying by $2x$ we obtain

$$(12x^4 - 4x^3) + (2x - 1) \geq 0.$$

Clearly, $12x^4 - 4x^3 \geq 0$ and $2x - 1 \geq 0$.

It remains to consider the case where there is no point of $S$ at a vertex of $B$. Let $p_1$ be a point of $S$ at distance $r_a$ from the center of $B$, see Fig. 9 (c). Note that $r_a < \sqrt{1+t^2}$. The idea is to shrink the bounding box $B$ so that $p_1$ is the vertex of new bounding box $B'$. To achieve this we move every point of $S \setminus B'$ toward the exact 1-center $c$ until it reaches the boundary of $B'$ (note that $c \in B$). The approximate radius does not change but the exact radius can be reduced only. The exact radius can be bounded from below as described above and the inequality $r_a \leq sr$ holds. The theorem follows. ∎

**Lemma 14 (Mixing Strategy)** *If the facility is allowed to move with velocity* $v_{max} \in [1, \sqrt{2}]$, *the approximation factor*

$$\alpha\frac{1+\sqrt{2}}{2} + (1-\alpha)\left(2 - \frac{2}{n}\right), \quad \textit{where} \quad \alpha = \frac{\sqrt{2} - v_{max}}{\sqrt{2}-1} \tag{4}$$

*is achievable.*

*Proof.* The same mixing strategy described in Lemma 8 can be applied to the Euclidean 1-center problem for the facility moving with velocity $v_{max} \in [1, \sqrt{2}]$. Let $c_m$ be the center of mass and let $c_b$ be the center of the bounding box. Let $r$ be the exact radius of 1-center and let $r_m$, $r_b$ be the radii determined by $c_m$ and $c_b$, respectively. The mixing center is defined as $c_{mix} = \alpha c_m + (1-\alpha)c_b$.

By Lemma 6 and Theorem 13 $r_m \leq (2 - 2/n)r$ and $r_b \leq (\sqrt{2}+1)r/2$. Let $p$ be any point in $S$. It suffices to prove that $|pc_{mix}| \leq Ar$, where $A$ is the required approximation factor from (4). The mixing center has a property that $pc_{mix} = \alpha pc_m + (1-\alpha)pc_b$. Therefore

$$\begin{aligned} |pc_{mix}|^2 &= \alpha^2|pc_m|^2 + (1-\alpha)^2|pc_b|^2 + 2\alpha(1-\alpha)pc_m \cdot pc_b \\ &\leq \alpha^2|pc_m|^2 + (1-\alpha)^2|pc_b|^2 + 2\alpha(1-\alpha)|pc_m||pc_b| \\ &= (\alpha^2|pc_m| + (1-\alpha)|pc_b|)^2|. \end{aligned}$$

Thus, $|pc_{mix}| \leq \alpha|pc_m| + (1-\alpha)|pc_b| \leq \alpha(2-2/n)r + (1-\alpha)(\sqrt{2}+1)r/2 = Ar$ and we are done. ∎

## 3 Mobile 1-median problems

Similarly to the 1-center problem, we consider the 1-median problem under the $L_\infty$, $L_1$ and $L_2$ metric. Intuitively, the maintenance of the 1-median seems harder than that of the 1-center because now all points are participating in definition of the median (while only four points define the center).

### 3.1 Rectilinear 1-median

We consider the problem in which the distances are measured by the $L_1$ metric. We create a grid $M$ by drawing a horizontal and a vertical line through each point of $S$. Assume the points of $S$ are sorted according to their $x$ coordinates and according to their $y$ coordinates. Denote by $M(i,j)$ the grid point that was generated by the $i^{th}$ horizontal line and the $j^{th}$ vertical line in the $y$ and $x$ orders of $S$ respectively. Bajaj [4] observed that the solution to the 1-median problem should be a grid point. As a matter of fact it has been shown that for this problem the point $M(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ is the required point. (For an even $n$ the solution is not unique and there is a whole grid rectangle whose points can be chosen as the solution.) We are interested in maintaining $M(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$

under the motion of points of $S$ with unit velocity. It can easily be observed that as in the case of the rectilinear 1-center (see Observation 2) the velocity of the 1-median can be as large as $\sqrt{2}$ times the maximum velocity of the points.

**Lemma 15** *The rectilinear* 1*-median can be maintained under the motion of points in an efficient kinetic data structure.*

*Proof.* We show how to maintain efficiently the $x$-coordinate of the median. The same approach works for the $y$-coordinate. Assume that the indices of the points of $S$ correspond to the sorted $x$ order of points. The idea is to split the points into two sets $S_1$ and $S_2$ such that $S_1 = \{p_1, \ldots, p_{\lfloor n/2 \rfloor}\}$ and $S_2 = \{p_{\lfloor n/2 \rfloor+1}, \ldots, p_n\}$. We maintain $S_1$ in the kinetic swapping heap $H_1$ (of the maximum maintenance problem) and $S_2$ in the kinetic swapping heap $H_2$ (of the minimum maintenance problem). We also need an additional certificate that states that $x(p_{\lfloor n/2 \rfloor}) \leq x(p_{\lfloor n/2 \rfloor+1})$. As we update the maximal value of $H_1$ and the minimal value of $H_2$ we check whether this certificate is valid, and if not, we exchange two points defining these two values between $H_1$ and $H_2$. Similarly to the maximum maintenance case the responsiveness is $O(\log n)$ and locality is $O(1)$. ∎

**Approximating median by the center of mass**.

**Lemma 16** *The center of mass of $n$ points provides an approximation factor of $2 - \frac{2}{n}$ for the rectilinear* 1*-median problem and this bound is tight for facilities with unit velocity.*

*Proof.* Let $(c_x, c_y)$ be the coordinates of the center c of mass of the points of $S$ and $(m_x, m_y)$ be the coordinates of the rectilinear 1-median $m$. We want to prove that

$$\Sigma_{i=1}^{n} |cp_i|_1 \leq (2 - \frac{2}{n}) \Sigma_{i=1}^{n} |mp_i|_1,$$

where $| * |_1$ stands for the $L_1$ distance. Following the definition of the $L_1$ distance it is sufficient to prove that $\Sigma_{i=1}^{n}|c_x - x(p_i)| \leq (2 - \frac{2}{n})\Sigma_{i=1}^{n}|m_x - x(p_i)|$ and $\Sigma_{i=1}^{n}|c_y - y(p_i)| \leq (2 - \frac{2}{n})\Sigma_{i=1}^{n}|m_y - y(p_i)|$. We will show the first inequality. The second inequality follows analogously. Notice that the first inequality can be viewed as the 1-dimensional case of the Lemma for the points $\{x(p_1), x(p_2), \ldots, x(p_n)\}$ because their center of mass is the point $c_x$ and their 1-median is the point $m_x$.

Assume that $x(p_1) \leq x(p_2) \leq \ldots \leq x(p_n)$. For simplicity, we also assume that $n$ is even. Let $M = \sum_{i=1}^{n} |m_x - x(p_i)|$. Rewriting $M$ we obtain $M = \sum_{i=n/2+1}^{n} x(p_i) - \sum_{i=1}^{n/2} x(p_i)$. Let $C = \sum_{i=1}^{n} |c_x - x(p_i)|$. Our goal is to show that $C \leq (2 - 2/n)M$. Let $x_l$ and $x_r$ be the centers of masses of $\{x(p_1), \ldots, x(p_{n/2})\}$ and $\{x(p_{n/2+1}), \ldots, x(p_n)\}$, respectively, i.e., $x_l = 2(\Sigma_{i=1}^{n/2} x(p_i))/n$ and $x_r = 2(\Sigma_{i=n/2+1}^{n} x(p_i))/n$. Clearly, $c_x = (x_l + x_r)/2$ and $x_r - x_l = 2M/n$. Let $D = \sum_{i=n/2+1}^{n}(x(p_i) - x_l) + \sum_{i=1}^{n/2}(x_r - x(p_i))$. We can rewrite $D$ as follows:

$$D = \sum_{i=n/2+1}^{n} x(p_i) - \sum_{i=1}^{n/2} x(p_i) + n/2(x_r - x_l) = M + n/2(x_r - x_l) = 2M.$$

First we show that $C \leq D$. Let $x(p_i)$ and $x(p_j)$ be two arbitrary points such that $1 \leq i \leq n/2$ and $n/2 + 1 \leq j \leq n$. Obviously, $x(p_j)$ lies to the right of $x_l$. Without any loss of generality we assume that $c_x \geq x(p_{n/2})$. We show that the contribution of $x(p_i)$ and $x(p_j)$ in $D$ is greater than their contribution in $C$. We now consider the following two cases:

Case 1: $x(p_j) \geq c_x$. In this case we have $(x_r - x(p_i)) + (x(p_j) - x_l) = (c_x - x(p_i)) + (x(p_j) - c_x) + (x_r - x_l) \geq (c_x - x(p_i)) + (x(p_j) - c_x)$.

Case 2: $x(p_j) < c_x$. In this case $(x_r - x(p_i)) + (x(p_j) - x_l) = (x(p_j) - x_l) + (x_r - c_x) + (c_x - x(p_i)) \geq (c_x - x(p_j)) + (c_x - x(p_i))$, because $x_r - c_x = c_x - x_l \geq c_x - x(p_j)$.

From the above two cases we can conclude that $C \leq D$. However, we can make this inequality a bit tighter. We notice that if we have $k$ pairs of points satisfying Case 1, then actually $D = 2M \geq C + k(x_r - x_l)$. Since $x_r - x_l = 2M/n$, we have in this case $(2 - 2k/n)M \geq C$. The number $k$ is defined by the number of points lying in interval $[x_r, \infty)$. Obviously there is at least one such point, i.e., $k \geq 1$. Thus, $(2 - 2/n)M \geq C$.

The tightness of this bound follows from the Example in Lemma 6. It can be easily verified that in this case $M = 1$ and $C = \frac{n-1}{n} + 1 - \frac{1}{n} = 2 - \frac{2}{n}$. ∎

**Lower bound.** We prove the following lower bound.

**Lemma 17** *Any algorithm that moves the facility with at most unit velocity achieves an approximation factor of at least* $2 - \frac{1}{\sqrt{2}}$ *for the mobile rectilinear* 1*-median problem in the worst case.*

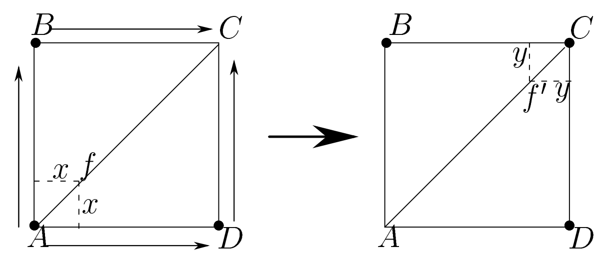


Figure 10: Lower bound for the rectilinear 1-median.

*Proof.* We prove this lemma using the following construction. We pick a set of 4 points such that there are 2 points at vertex $A$, 1 point at vertex $B$ and $D$ of the left square with unit side in Figure 10. Obviously, the rectilinear 1-median $m$ coincides with the vertex $A$ and the sum of distances defined by $m$ is 2. We will move points with unit velocity from $B$ and $D$ toward a vertex $C$, one point from $A$ towards $B$ and one point from $A$ towards $D$. This will guarantee that the new median $m'$ will coincide with the vertex $C$ in the right square and the sum of the distances defined by $m'$ is also 2. By the symmetry argument, an algorithm starts with some initial location of facility $f$ lying in the diagonal in the left square at distance $2x$ from the vertex $A$. The final position of the facility is $f'$ at distance $2y$ from the vertex $C$ in the right square. It is easy to see that $x = y$. The sum of the distances from $f$ to all the points in the left square is $2 + 4x$ and equals to the sum of the distances from $f'$ to all the points in the right square. Thus, the approximation factor is equal to $\frac{2+4x}{2} = 1 + 2x$. On the other hand, $2\sqrt{2}x + 1 = \sqrt{2}$. Therefore, the approximation factor is $1 + 2(\sqrt{2} - 1)/(2\sqrt{2}) = 2 - 1/\sqrt{2}$. ∎

## 3.2 Euclidean 1-median

This problem, also known as the Fermat-Weber problem, received a lot of attention during last century. Only iterative algorithms (see, for example, [23]) are known for this problem. The 1-median problem differs from the 1-center problem not only by its difficulty but also in terms of approximation. Unlike the 1-center case, we now approximate the sum of the distances and not the distance between the facility $f$ and the exact 1-median. Fortunately, the rectilinear 1-median provides a constant approximation for the Euclidean 1-median.

**Lemma 18** *The rectilinear* 1*-median provides a* $\sqrt{2}$*-approximation factor for the Euclidean* 1*-median.*

*Proof.* Let $m_1$ and $m_2$ be the locations of the exact $L_1$ and $L_2$ medians, respectively, for a given set of points $S$. Let $r_1$ be the sum of the $L_1$ distances from $m_1$ to points in $S$ and $r_2$ be the sum of the $L_2$ distances from $m_2$ to points in $S$, i.e., $r_1 = \Sigma_{i=1}^{n}|m_1 p_i|_1$ and $r_2 = \Sigma_{i=1}^{n}|m_2 p_i|_2$. Define $r' = \Sigma_{i=1}^{n}|m_1 p_i|_2$. Note that for any two points $a, b$ in the plane the following holds:

$$|ab|_2 \leq |ab|_1 \leq \sqrt{2}|ab|_2. \qquad (**)$$

We want to show that $r' \leq \sqrt{2} r_2$. Using (**) we obtain $r_2 = \Sigma_{i=1}^{n}|m_2 p_i|_2 \geq \frac{1}{\sqrt{2}}\Sigma_{i=1}^{n}|m_2 p_i|_1 \geq \frac{r_1}{\sqrt{2}}$. On the other hand, $r' = \Sigma_{i=1}^{n}|m_1 p_i|_2 \leq \Sigma_{i=1}^{n}|m_1 p_i|_1 = r_1 \leq \sqrt{2} r_2$. The last inequality follows from the bound on $r_2$. ■

Based on this result and Lemma 14 we immediately conclude

**Corollary 19** *The center of mass of the points gives a* $\sqrt{2}(2 - \frac{2}{n})$ *approximation factor for the Euclidean* 1*-median.*

We can again use the mixing strategy from Lemma 8 for the facility moving with velocity $v_{max} \in [1, \sqrt{2}]$ by mixing the strategies of the center of mass of the points and the rectilinear 1-median. It provides a better approximation factor since the function $s(x, y) = \Sigma_{i=1}^{n}\sqrt{(x - x_i)^2 + (y - y_i)^2}$ defining the sum of the distances from the point $(x, y)$ in the plane to points of $S$ is concave.

# 4 Conclusion

In this paper we introduced mobile versions of two classical facility location problems and investigated the complexity of the mobile 1-center and 1-median problems in the plane. Future directions for the research in this area include: providing tighter bounds for mobile 1-median problem, obtaining precise trade-offs between the velocities of the facilities and approximation factors and generalization of our results to the mobile $k$-center and $k$-median problems for $k \geq 2$ and in dimensions $d > 2$.

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