

# LOWER AND UPPER BOUNDS FOR TRACKING MOBILE USERS

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**Abstract** In this paper we prove improved lower and upper bounds for the location of mobile facilities (in  $L_\infty$  and  $L_2$  metrics) under the motion of clients when facility moves faster than clients. This paper continues the research started in our joint paper where we present lower bounds and efficient algorithms for exact and approximate maintenance of the 1-center for a set of moving points in the plane. Our algorithms are based on the *kinetic* framework introduced by Basch, Guibas and Hershberger.

## 1. Introduction

The goal of this paper is to study problems related to the location of mobile facilities serving a set of customers. The notion of mobile servers has been well studied in connection with stationary but transitory customer sets [24]. Here we deal with the problems related to the location of mobile facilities, where customers are modelled by points moving continuously in  $d$ -dimensional space,  $d \geq 1$ . Specific example of such problems is the maintenance of the 1-center for moving points under the  $L_p$  metric. This has applications, for example in mobile wireless communication networks when the broadcast range should contain all the customers so as to provide service to the cellular phones. There are likely to be many applications of mobile facilities, particularly with respect to tactical mobile packet-radio networks. Servers such as battlefield map servers, or even servers whose sole purpose is to provide support for network control (such as those that maintain entity-to-address information) could be considered to be mobile facilities. Another application of these problems is locating a welding robot in an automobile manufacturing plant. Both problems are interesting from both theoretical and practical points of view.

Facility location is a classical problem of operations research that has also been examined in the computational geometry community. Most of the problems described in the facility location literature are concerned with finding a “desirable” facility location: the goal is to *minimize* a distance function between the facility (e.g., a service) and the sites (e.g., the customers). As was pointed out in [6], the data structures and algorithms that have been developed for the *static* problems (i.e., customers are not moving) are not directly applicable to the setting of moving points when the motion of the facilities must satisfy natural constraints. In this paper we continue the study of mobile version of the classical facility location problem:

- **$k$ -center:** Given a set  $S$  of  $n$  demand points in  $d$ -dimensional space ( $d \geq 1$ ), find a set  $P$  of  $k$  supply points so that the *radius* defined as the maximum  $L_p$  distance between a demand point and its nearest supply point in  $P$  is minimized. If  $P$  is a subset of  $S$  then  $P$  called a set of *discrete* centers. Note that for some metrics such a set  $P$  is not necessarily unique even for  $k = 1$  and  $d = 2$ .

This problem has been well studied in both the exact [2, 7, 8, 9, 10, 11, 13, 21, 25] and approximate [10, 12, 18] versions. In approximate versions a set  $P$  provides  $(1 + \varepsilon)$ -approximate  $k$ -center if the associated radius is at most  $(1 + \varepsilon)$  times the optimal radius, for any  $\varepsilon > 0$ . Facility location problems for time varying networks (when edge distances satisfy the triangle inequality) also have been studied, see [16, 19]. Let us define the *mobile  $k$ -center* problem as follows. Given a set  $S = \{p_1, p_2, \dots, p_n\}$  of  $n$  continuously moving points specified by a piecewise differentiable functions  $\{g_1, g_2, \dots, g_n\}$ , where  $g_i, 1 \leq i \leq n$  maps time interval  $[0, T]$  to  $\mathbb{R}^d$ , we want to determine whether there exist  $k$  continuous functions  $f_1, \dots, f_k$ ;  $f_i : [0, T] \rightarrow \mathbb{R}^d$  such that at any given moment  $t \in [0, T]$ , the points  $f_1(t), \dots, f_k(t)$  form a  $k$ -center for the points at locations  $g_1(t), \dots, g_n(t)$ , and, if so, find  $f_1, \dots, f_k$ . In this paper we concentrate mainly on the instances of the mobile 1-center problem, when  $p = \{1, 2, \infty\}$  and assume that  $d = 2$  if the dimension is not specified. We present improved lower and upper bounds for mobile rectilinear and Euclidean 1-center when facility is allowed to move faster than the clients.

The mobile approximate 1-center and related problems are studied by Agarwal and Har-Peled [1] very recently. For the Euclidean 1-center in the plane, they presented

an approximation algorithm based on a notion of *extent*. Their algorithm aims to maintain an approximate 1-center with a few events only. However, the velocity of the facility can be arbitrary large (Lemma 4). We describe improved strategies that guarantee both an approximation factor and a bounded velocity of the facility (Theorem 5 and Lemma 7).

Gao et al. [14] proposed a randomized algorithm for maintaining a set of clusters among moving points in the plane. They presented algorithms with expected approximation factor (on the optimal number of centers) of  $c \log n$  for intervals and of  $c\sqrt{n} \log n$  for squares. The probability that there are more than expected number of centers is  $O(1/n^{\Theta(c^2)})$  for the case of intervals and  $O(1/n^{c^2 \log n})$  for the case of squares. An extension of this basic algorithm led to a hierarchical algorithm, also presented in [14], based on kinetic data structures [5] with expected constant approximation factor on the number of discrete centers, where the approximation factor also depends linearly on the constant  $c$ . The dependency of the approximation factor and the probability that the algorithm chooses more than a constant times the optimal number of centers is similar to that of the non-hierarchical algorithm for the squares case. The constant  $c$ , which has not been explicitly determined in [14], can be shown to be very large. Their algorithm has an expected update time of  $O(\log^{3.6} n)$ , the number of levels used in the obtained hierarchy is  $O(\log \log n)$  with  $O(n \log n \log \log n)$  space. Har-Peled [17] found a scheme of determining centers in advance, i.e. if in the optimal solution the number of centers is  $k$  and  $r$  is the optimal radius for the points moving with the degree of motion  $l$ , then his scheme guarantees  $2^{l+1}$ -approximation of the radius with  $k^{l+1}$  centers chosen from the set of input points before the points start to move. Huang et al. [20] present fully distributed (decentralized) constant-factor approximation algorithms for  $k$ -center problem in the  $L_1$ - and  $L_\infty$  norms for any  $d$ -dimensional space  $\mathbb{R}^d$ ,  $d \geq 1$ , and for the  $L_2$  norm in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , which either match or improve the best known constant approximation factors of a centralized algorithm, or have (asymptotically) optimal update costs improving [14].

Our algorithms for mobile 1-center problem are based on a relatively new class of data structures (used also in [6, 14]) — *kinetic data structures (KDS)* — aimed at keeping track of attributes of interest in systems of moving objects [5, 15]. In the kinetic setting, a set of points is assumed to be continuously changing, or moving. Each point follows a posted *flight plan*, but a plan can change at any moment through a *flight plan update*. A KDS maintains a *configuration function* of interest (which in our case will be the set  $P$ ) by watching for critical events as the objects move. A KDS for computing a particular attribute for a set of points in motion maintains a set of *certificates*. A certificate based on a tuple of points is a continuous function that associates a real number with each configuration of these points. For example, the certificates for a convex hull KDS are a collection of the triple of points, each with a particular orientation. At any one time, the conjunction of all the certificates being maintained by the kinetic data structure proves the combinatorial correctness of its output. As the points move, some certificates may become invalid, e.g., a triple of points changes orientation. When a certificate fails, the proof structure needs to be modified and the combinatorial description of the configuration function may need to be updated. Certificates are stored in a priority queue, ordered by failure time, and processed in order as they fail. A good kinetic data structure will take advantage of the continuity of the point motions to select certificate structures that are easy to update at these critical events; a structure satisfying this condition is called *responsive*. Other criteria for a KDS are *efficiency*, i.e., the number of events

processed by KDS is not much greater than the number of combinatorial changes in the configuration function itself, *compactness*, i.e., the number of active certificates at any one time is roughly linear in the complexity of the moving points, and *locality*, i.e., a flight plan update for any one point affects only a small number of certificates.

## 2. Mobile 1-center problems

In this section we present new lower and upper bounds for mobile 1-center problems under the  $L_\infty$  and  $L_2$  metrics for facility moving faster than the clients and also extend a lower bound proof for unit velocity of the facility presented in [6] to work in any metric and any dimension.

### 2.1. Rectilinear 1-center

We consider the case when the velocity of the facility belongs to the range  $[1, \sqrt{2}]$ , define the *mixing strategy* and analyze it. Then we show the lower bound on the approximation factor for this case. At the end of this section we show an extended lower bound presented in [6] for facility moving with velocity 1 that works for any dimension and any metric.

**Motion with bounded velocity.** Suppose now, that the velocity of the facility is bounded by  $v_{max} \in [1, \sqrt{2}]$ . We notice that for values of  $v_{max} = 1$  and  $\sqrt{2}$  the best approximation factors are 2 and 1, respectively (see [6]). We *mix* our two strategies — the center of mass of the points and the center of the bounding box — in the following way. Let  $(f_1, v_1)$  be the location (vector in the plane) and velocity of the center of mass of the points. Similarly,  $(f_2, v_2)$  are the location and velocity of the center of the bounding box. (Bespamyatnikh et al. [6] explain how to maintain the center of bounding box and the center of mass of points using kinetic data structures.) The mixing strategy is to maintain the *mixing center*  $(f, v)$  defined as  $(\alpha f_1 + (1 - \alpha)f_2, \alpha v_1 + (1 - \alpha)v_2)$ , where  $\alpha = (\sqrt{2} - v_{max})(\sqrt{2} - 1)$ . We analyze the mixing strategy and show how it can be improved in the following theorem.

**Theorem 1 (Bounded Velocity)** *Suppose that the facility is allowed to move with velocity at most  $v_{max} \in [1, \sqrt{2}]$ . Let  $a_1(v)$  be the linear function defined by  $a_1(1) = 2$  and  $a_1(v') = a'$  where  $v' = \sqrt{2} \cos(\pi/8)$  and  $a' = 1.25$ . Let  $a_2(v)$  be the linear function defined by  $a_2(v') = a'$  and  $a_2(\sqrt{2}) = 1$ . There is a strategy of the facility that guarantees an approximation factor  $\max(a_1(v_{max}), a_2(v_{max}))$ .*

*Proof.* First we show that the velocity of the mixing center is bounded by  $v_{max}$ . We want to prove the inequality  $\alpha v_1 + (1 - \alpha)v_2 \leq v_{max}$ . In order to do this, we replace  $v_1$  by 1 and  $v_2$  by  $\sqrt{2}$ . It can easily be seen that the value of the expression  $\alpha + (1 - \alpha)\sqrt{2}$  when  $\alpha = (\sqrt{2} - v_{max})(\sqrt{2} - 1)$  is equal to  $v_{max}$ .

Regarding the approximation factor we consider the case in which the center of mass is at largest  $L_\infty$  distance from the center of the bounding box. This distance is  $1 - 2/n$  if the radius that is determined by the center of the bounding box is 1. Then the  $L_\infty$  distance between the mixing center and the center of the bounding box is  $\frac{\sqrt{2} - v_{max}}{\sqrt{2} - 1} (1 - \frac{2}{n})$ . The approximation factor bound is

$$1 + \frac{\sqrt{2} - v_{max}}{\sqrt{2} - 1} \left(1 - \frac{2}{n}\right).$$

The smallest bound that suits all  $n$  is the linear function  $1 + (\sqrt{2} - v_{max})(\sqrt{2} - 1)$ .

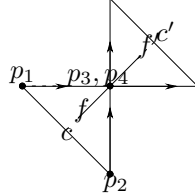


Figure 1. Movement of the center.

It turns out that the mixing strategy is not easy to improve. The reason is that the worst case velocities of two strategies, the center of mass (of all the points or a linear combination of a few points from  $S$ ) and the center of bounding box, can be in the same direction as shown in Figure 1.

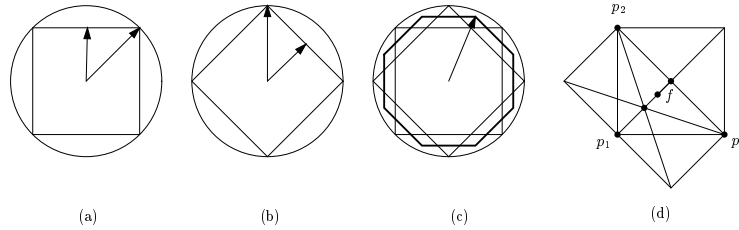


Figure 2. 8-gon strategy.

In this case the best motion of the facility is to move in this direction within the maximum allowed velocity which corresponds to mixing strategy. The idea now is to interpolate two strategies that have the worst case velocities in different directions. Note that the maximum velocity of the center of bounding box depends on the direction and varies from 1 to  $\sqrt{2}$ , see Fig. 2(a). A good candidate for the interpolation is the strategy of the center of bounding box aligned to the coordinate axis rotated by  $45^\circ$ . The range of the maximum velocity is depicted in Fig. 2(b). Consider the midpoint of two centers. We call this *8-gon strategy* since the range of its maximum velocity forms the regular 8-gon, see Fig. 2(c). The maximum velocity of the facility following the 8-gon strategy is  $v' = \sqrt{2} \cos(\pi/8)$  and the approximation in worst case is  $a' = 1.25$  for three points  $S = \{p_1, p_2, p_3\}$ , see Fig. 2(d). One can show that the approximation factor of the 8-gon strategy is smaller than one of the mixing strategy for  $v_{max} = v'$ . In order to improve the approximation for other values of  $v_{max}$  we

- (i) interpolate (or mix) the strategy of the center of mass and the 8-gon strategy for  $v_{max}$  in the range  $[1, v']$ , and
- (ii) interpolate the strategy of the center of the bounding box and the 8-gon strategy for  $v_{max} \in [v', \sqrt{2}]$ .

The theorem follows. ■

**Lower bound for an approximation factor when  $v_{max} \in [1, \sqrt{2}]$ .** In fact we can show two different lower bound estimates. The first bound is based on the example depicted in Figure 1 when  $|ff'| = v_{max}$  and  $|p_1p_4| = |p_2p_4| = 1$ . In this case



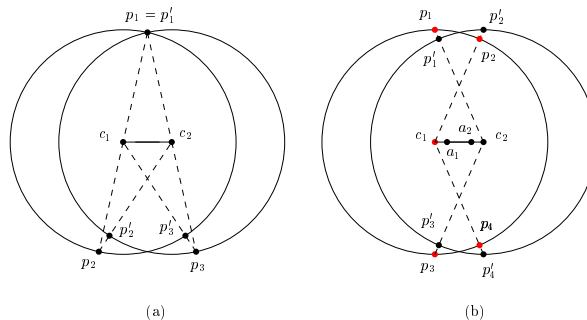


Figure 4. Euclidean 1-center may move with unbounded velocity.

To see this we observe that points  $p_1, \dots, p_4$  are located on the unit circle centered at  $c_1$ . A point  $p_i, i = 1, \dots, 4$  moves toward the point  $p'_i$ . The points  $p_i$  make the same length paths since  $c_1c_2p'_2p_1$  and  $c_1c_2p'_4p_3$  are rectangles. Let  $x = |c_1c_2|$  be the length of the path made by the 1-center and let  $y = |p_1p'_1|$  be the length of the path made by the point  $p_1$ . It suffices to show that  $y/x$  tends to 0 if  $x$  tends to 0. Indeed,  $1 + x^2 = (1 + y)^2$  since the triangle  $c_1c_2p_1$  is right. It implies  $x^2 = 2y + y^2$  and  $2y/x = x - y^2/x \leq x$ .

Next, we show that velocity of an approximate 1-center must be high.

**Lemma 4 (Lower Bound)** *For every  $\varepsilon > 0$ , any  $(1 + \varepsilon)$ -approximate mobile Euclidean 1-center has velocity at least  $1/4(\sqrt{2\varepsilon} + \varepsilon) = 1/(4\sqrt{2\varepsilon}) - O(1)$  in worst case.*

Due to lack of space the proof is omitted.

We show that the lower bound of the velocity needed to approximate mobile Euclidean 1-center established in Lemma 4 is optimal up to a constant factor.

**Theorem 5 (Upper Bound)** *For any  $\varepsilon > 0$  there is a strategy for mobile approximate Euclidean 1-center that guarantees the approximation factor  $1 + \varepsilon$  using velocity of the facility  $4/\sqrt{\varepsilon} + o(1/\sqrt{\varepsilon})$  in the worst case.*

*Proof.* We apply the following strategy for the mobile facility. Let  $V$  be the maximum velocity of the facility that will be specified later. Consider the initial configuration. We assume that in the beginning the facility  $f$  is located so that the radius of the smallest circle enclosing all the customer points is at most  $(1 + \delta)$  times the optimal radius where  $\delta \leq \varepsilon$  and will be specified later. The goal of the facility now is to reach the position of the exact Euclidean 1-center at this moment. The facility heads to the target using the velocity  $V$ . We can compute the time  $t$  needed for this. After reaching the target we repeat the procedure. It is interesting that the motion of the facility is independent on the motion of customers during the time when facility moves from one position to another. It is also interesting that the exact 1-center can achieve any velocity sometimes and the approximate facility just ignores any acceleration of the exact 1-center.

Let  $c$  be the position of the exact Euclidean 1-center at the initial time  $t_0$  and let  $a$  be the position of the facility (the approximate 1-center) at the same time. Let  $t_1$  be the time when the facility reaches  $c$ . Let  $r$  denote the radius associated with  $c$  at the time  $t_0$ . Without loss of generality we can assume that  $r = 1$ .

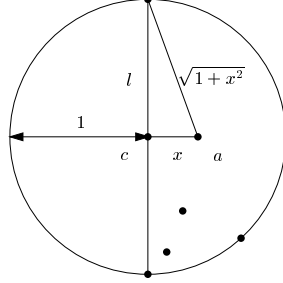


Figure 5.  $(1 + \varepsilon)$ -approximation of Euclidean 1-center.

First we estimate the distance between the exact and approximate centers at the time  $t_0$ . Denote  $x = |ac|$ . The line  $l$  orthogonal to the line  $ac$  passing through  $c$  partition the unit circle centered at  $c$  into 2 arcs, see Fig. 5. Each arc contains at least one customer point since the exact radius of the 1-center disk is 1. The minimum distance from  $a$  to the arc beyond  $l$  is  $\sqrt{1+x^2}$ . Clearly,  $\sqrt{1+x^2} \leq 1 + \delta$ . Therefore  $x = |ac| \leq \sqrt{2\delta + \delta^2}$ .

Let  $t$  be the time the facility need to get the exact center  $c$ , i.e.,  $t \leq |ac|/V$ . Consider the moment  $t_1$ . The radius associated with the approximate 1-center  $c$  is at most  $1+t$  since the velocities of the customers are bounded by 1. Let  $c'$  be the location of the exact 1-center. We show that the associated radius is at least  $1-t$ . Construct the line passing through  $c$  and orthogonal to  $cc'$ . The arc of the unit circle centered at  $c$  from which the exact center moves toward  $c'$  contains at least one customer point at the moment  $t_0$ . This customer is at distance at least  $1-t$  from  $c'$  at the moment  $t_1$ . Thus, the approximation ratio at the moment  $t_1$  is at least

$$\frac{1+t}{1-t} = 1 + \frac{2t}{1-t}.$$

We want to show that this is at most  $1 + \delta$ . It follows from

$$\frac{2t}{1-t} \leq \delta \text{ or, equivalently, } t \leq \frac{\delta}{2+\delta}. \quad (1)$$

We also want the approximation factor to be at most  $1 + \varepsilon$  for any moment  $t_0 + \Delta \in [t_0, t_1]$ . The radius associated with the approximate 1-center is at most  $1 + \delta + \Delta$ . The radius is at least  $1 - \Delta$ . Their ratio is at most

$$\frac{1 + \delta + \Delta}{1 - \Delta} = 1 + \frac{2\Delta + \delta}{1 - \Delta}.$$

The largest value of the upper bound is achieved when  $\delta = t$ . It suffices to have

$$\frac{2t + \delta}{1 - \delta} \leq \varepsilon \text{ or, equivalently, } t \leq \frac{\varepsilon - \delta}{2 + \varepsilon}. \quad (2)$$

Making the right sides of the equations 1 and 2 we obtain  $\delta^2 + 4\delta - 2\varepsilon = 0$  and  $\delta = \sqrt{4 + 2\varepsilon} - 2 \approx \varepsilon/2$ . By the equation 1 we can set  $t = \delta/(2 + \delta) \approx \varepsilon/4$ . Therefore, the velocity of the facility can be  $V = \sqrt{2\delta + \delta^2}/t \approx 4/\sqrt{\varepsilon}$ . The theorem follows. ■



We also can achieve reasonably good approximation factor for Euclidean 1-center when we trace the center of the bounding box of the points.

**Theorem 6 (Bounding Box Strategy)** *The strategy of tracing the center of the bounding box of  $n \geq 3$  points has an approximation factor of  $(1+\sqrt{2})/2$  for the mobile Euclidean 1-center and this bound is tight.*

*Proof.* Let  $s = (1 + \sqrt{2})/2$ . Without loss of generality we can assume that the bounding box is  $B = [0, 2] \times [0, 2t]$  for some  $t \geq 1$ , see Fig. 6 (a). Let  $S$  be any configuration of  $n \geq 3$  points in  $B$  such that each side of  $B$  contains at least one point of  $S$ . Let  $r$  be the approximate radius, i.e.,  $r_a = \max_{p \in S} \{ |p_0| \}$  where  $c_a = (1, t)$  is the center of  $B$ . Let  $r$  be the exact radius of 1-center. We want to prove that  $r_a \leq sr$ .

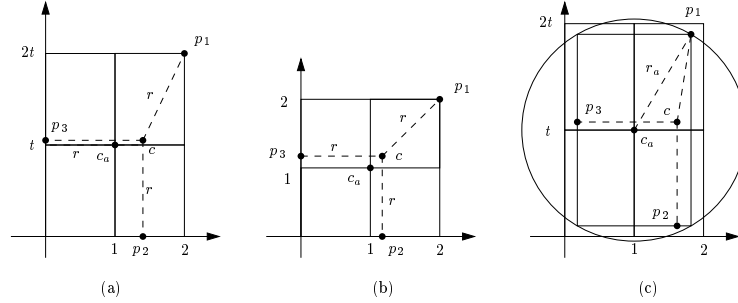


Figure 6. Bounding box strategy for approximating Euclidean 1-center.

We first consider the case where one of the points of  $S$  is located at a vertex of  $B$ , say  $p_1 = (2, 2t)$ . It implies  $r_a = \sqrt{1+t^2}$ . Two sides of  $B$  contain  $p_1$ , and there are at least two points  $p_2$  and  $p_3$  lying on the sides of  $B$  along  $x$ - and  $y$ -axis, respectively. We can assume that  $p_2 \neq p_3$ , otherwise  $p_2 = p_3 = (0, 0)$  and  $r = r_a$ . If  $t > 2$  then  $r \geq |p_1 p_2| \geq t$  and one can check that

$$\frac{r_a}{r} \leq \frac{\sqrt{1+t^2}}{t} = \sqrt{\frac{1}{t^2} + 1} \leq \sqrt{1 + \frac{1}{4}} < s.$$

We assume that  $t \in [1, 2]$ . The smallest value of  $r$  is achieved when  $p_2$  is strictly below the 1-center and  $p_3$  is to the left of it, see Fig. 6 (a). Thus the exact 1-center has coordinates  $c = (r, r)$ . The 1-center is at distance  $r$  from  $p_1$ ,

$$(2-r)^2 + (2t-r)^2 = r^2.$$

This can be transformed to the quadratic equation

$$r^2 - 4(t+1)r + 4(t^2+1) = 0.$$

$r = 2(t+1 + \sqrt{2t})$  since the root with “+” is greater than  $2t$  which is impossible.

$r_a$  and  $r$  can be viewed as functions of  $t$ . In order to prove that  $r_a/r \leq s$  for  $t \in [1, 2]$  we show that (i)  $r_a(1)/r(1) = s$ , and (ii)  $(r_a/r)' \leq 0$  for any  $t \in [1, 2]$ . If  $t = 1$  then  $r_a = \sqrt{2}$  and  $r = 2(2 - \sqrt{2})$ . One can check that  $r_a/r = s$ . Note that this gives an example with tight bound, see Fig. 6 (b).

To prove the second condition we differentiate  $r$  and  $r_a$

$$r' = 2(1 - 1/\sqrt{2t}) \quad \text{and} \quad r'_a = t/\sqrt{1+t^2}.$$

We want to prove that  $r'_a r - r_a r' \leq 0$  or, equivalently,

$$\frac{2t}{\sqrt{1+t^2}} (t + 1 - \sqrt{2t}) \leq 2\sqrt{1+t^2} \left(1 - \frac{1}{\sqrt{2t}}\right).$$

Let  $x = \sqrt{t/2}$ . Then  $t = 2x^2$  for  $x \in [\sqrt{2}/2, 1]$  and the inequality can be written as

$$2x^2(2x^2 + 1 - 2x) \leq (1 + 4x^4) \left(1 - \frac{1}{2x}\right).$$

Multiplying by  $2x$  we obtain

$$(12x^4 - 4x^3) + (2x - 1) \geq 0.$$

Clearly,  $12x^4 - 4x^3 \geq 0$  and  $2x - 1 \geq 0$ .

It remains to consider the case where there is no point of  $S$  at a vertex of  $B$ . Let  $p_1$  be a point of  $S$  at distance  $r_a$  from the center of  $B$ , see Fig. 6 (c). Note that  $r_a < \sqrt{1+t^2}$ . The idea is to shrink the bounding box  $B$  so that  $p_1$  is the vertex of new bounding box  $B'$ . To achieve this we move every point of  $S \setminus B'$  toward the exact 1-center  $c$  until it reaches the boundary of  $B'$  (note that  $c \in B$ ). The approximate radius does not change but the exact radius can be reduced only. The exact radius can be bounded from below as described above and the inequality  $r_a \leq sr$  holds. The theorem follows. ■

**Lemma 7 (Mixing Strategy)** *If the facility is allowed to move with velocity  $v_{max} \in [1, \sqrt{2}]$ , the approximation factor*

$$\alpha \frac{1 + \sqrt{2}}{2} + (1 - \alpha) \left(2 - \frac{2}{n}\right), \quad \text{where} \quad \alpha = \frac{\sqrt{2} - v_{max}}{\sqrt{2} - 1} \quad (3)$$

*is achievable.*

*Proof.* The same mixing strategy described for rectilinear 1-center can be applied to the Euclidean 1-center problem for the facility moving with velocity  $v_{max} \in [1, \sqrt{2}]$ . Let  $c_m$  be the center of mass and let  $c_b$  be the center of the bounding box. Let  $r$  be the exact radius of 1-center and let  $r_m, r_b$  be the radii determined by  $c_m$  and  $c_b$ , respectively. The mixing center is defined as  $c_{mix} = \alpha c_m + (1 - \alpha)c_b$ .

Using the result obtained in [6] which says that the center of mass of points guarantees a  $(2 - 2/n)$ -approximation of the rectilinear 1-center of  $S$  and using Theorem 6 we have  $r_m \leq (2 - 2/n)r$  and  $r_b \leq (\sqrt{2} + 1)r/2$ . Let  $p$  be any point in  $S$ . It suffices to prove that  $|pc_{mix}| \leq Ar$ , where  $A$  is the required approximation factor from (3). The mixing center has a property that  $pc_{mix} = \alpha pc_m + (1 - \alpha)pc_b$ . Therefore

$$\begin{aligned} |pc_{mix}|^2 &= \alpha^2 |pc_m|^2 + (1 - \alpha)^2 |pc_b|^2 + 2\alpha(1 - \alpha) pc_m \cdot pc_b \\ &\leq \alpha^2 |pc_m|^2 + (1 - \alpha)^2 |pc_b|^2 + 2\alpha(1 - \alpha) |pc_m| |pc_b| \\ &= (\alpha^2 |pc_m| + (1 - \alpha) |pc_b|)^2. \end{aligned}$$

Thus,  $|pc_{mix}| \leq \alpha |pc_m| + (1 - \alpha) |pc_b| \leq \alpha(2 - 2/n)r + (1 - \alpha)(\sqrt{2} + 1)r/2 = Ar$  and we are done. ■

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