Lower Bounds for Covering Problems

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Abstract

In this paper we present an $\Omega(n \log n)$ lower bound proofs for a several covering problems including optimal line bipartition problem, min-max covering by two axis parallel rectangles, discrete and continuous two-center problems, two-line center problem, etc. Our proofs are based on using "rotational reduce technique" and well known lower bound for the maximal gap problem.

Keywords: Lower bounds, reductions, Max-gap problem

1 Introduction

In this paper we present an $\Omega(n \log n)$ lower bound proofs in the algebraic decision-tree model for a several covering problems. In particular, we consider the case when we are given a set S of n points in the plane and

Optimal line bipartition. We wish to partition S by a line l into two subsets S_1 and S_2 such that $\max(f(S_1), f(S_2)\}$ is minimal, where f is some real-valued monotone function that is defined over the collection of subsets of S. We consider the case when $f(S')$ defines an area, perimeter of the convex hull of S' or diameter of S'.

Euclidean two-center. We wish to find a pair of two disks d_1 and d_2 of smallest possible common radius that cover S.

Two covering. We wish to find a pair of two axis-parallel rectangles r_1 and r_2 that together cover S and minimize the maximum of their perimeters.

L_∞ **discrete two-center.** We wish to find a pair of *discrete* squares b_1 and b_2, i.e. squares that are centered at points of some given set F, that cover S and minimize the size of the largest square.

Two-line center We wish to find a pair of strips w_1 and w_2 that cover S such that the width of the widest strip is minimized.

Smallest rotated enclosing square. We wish to find a minimal square s centered at some point q that covers S. The square s may rotate.

Largest rotated empty square. We wish to find a largest square s centered at some point q that does not contain any point from S. The square s may rotate.

Rotated squares. We wish to find a pair of side-parallel squares s_1 and s_2 whose union contains S, so as to to minimize the size (perimeter or area) of the largest square.

All these problems belong to the field of geometric optimization and covering. There are many results considering all the problems described above. Mitchell and Wynters [17] present solutions for the two instances of the optimal line bipartition problem, in which $f(S')$ is either the perimeter or area of the convex hull of S'. Their solutions require $O(n^3)$ and $O(n^2)$ time, respectively. Asano et al. [3] present an $O(n \log n)$ time algorithm for the diameter case. Later, Devillers and Katz [5] considered the restricted version of the problem where points of S are in convex position and gave $O(n \log n)$ time algorithm for all three cases. The Euclidean two-center problem received a lot of attention in the papers [2, 11, 14]. Megiddo [16] gave a linear time algorithm for the case of one disk, and since then only near quadratic results were achieved for two disks. In a major breakthrough, Sharir [19] showed that this problem can be solved in $O(n \log^9 n)$ time. Since then some improvements were made by using randomized techniques (see, e.g. [7]) Hershberger and Suri [9] considered two covering problem and solved in $O(n \log n)$ runtime the decision version of the problem where the value of perimeter $\mathcal{A}$ is given and we want to find whether exist two axis-parallel rectangles of a given perimeter $\mathcal{A}$ that cover S. Glozman et al. [8] succeeded to obtain an $O(n \log n)$ running time solution to the original two covering problem. Katz et al. [13] proposed an $O(n \log^2 n)$ time solution for L_∞ discrete two-center problem that was improved by Bespamyatnikh and Segal [4] to $O(n \log n)$ running time. We mention here that the case of the squares with arbitrary centers can be solved in linear time [20]. The best result for the two-line center problem was given by Jaromczyk and Kowaluk [12] and runs in time $O(n^2 \log^2 n)$ (see also [8, 14]). The smallest rotated enclosing square problem has been solved in Katz et al. [13] with $O(n \log n)$ time solution. The largest rotated empty square problem can be considered as some variant of a classical "largest empty circle" problem [18]. In our case, however, we are given already the center of the square which may rotate. Finally, Jaromczyk and Kowaluk [10] showed how to solve the rotated squares problem. They gave $O(n^2)$ runtime algorithm and showed how to generalize it to the case of rectangles.

Little has been done in proving lower bounds for geometric optimization and covering problems. In the extended version of paper [20] Sharir and Welzl showed a $\Omega(n \log n)$ lower bound for the L_∞ 4-center problem. Woeginger [21] proved a $\Omega(n \log n)$ lower bound for the restricted Euclidean 1-center problem. Drysdale and Jaromczyk [6] provided a $\Omega(n \log n)$ lower bound for the maximum area and maximum perimeter k-gon problems. The same result was obtained by Avis et al. [1] for determining the existence of a line stabber for a family of n line segments in the plane.

The main idea of our proofs is to show the reductions from the well known problem (MAX GAP). In order to find an appropriate reduction we will show in Section 2 how to use the "rotational" technique that transforms the instances of the MAX-GAP problem to the instances of our problems. This technique can be viewed as a “double cycle rotation”, “square rotation” and “quadruple cycle rotation”. In our case, the first two problems will be proven using “double cycle rotation’. The proofs for the two covering and L_∞ discrete two center problems are based on “square rotation” and the result for the rest of the problems can be obtained by using “four cycle rotation”. We conclude in Section 3.

2 Lower Bounds

In what follows we mention the MAX-GAP problem and then present our proofs.

- MAX-GAP problem: Given real numbers $0 \leq x_0, x_2, \ldots, x_{n-1} \leq 1$ we want to compute the maximum gap between two successive numbers in the sorted sequence.

In the algebraic decision-tree model, the MAX-GAP problem has a lower bound $\Omega(n \log n)$ [18, 15].

Remark 1. As was pointed out in the extended version of paper [20] there is a $\Omega(n \log n)$ lower bound for the GAP-EXISTENCE problem: Given real numbers $0 \leq x_0, x_2, \ldots, x_{n-1} \leq 1$ and a number $\delta > 0$, find whether exist two successive numbers in the sorted sequence with a difference at least δ. Using this result one can show that the decision variants of the problems with a $\Omega(n \log n)$ lower bound based on the reduction from MAX-GAP problem have also a $\Omega(n \log n)$ lower bound.

2.1 Double Cycle Rotation

Let $0 \leq x_0, x_2, \ldots, x_{n-1} \leq 1$ be an instance of the MAX-GAP problem and let ρ be a sorted permutation of these numbers. We compute the minimal element $x_{\rho(0)}$ and the maximal element $x_{\rho(n-1)}$ of this set of numbers. If $x_{\rho(0)} = x_{\rho(n-1)}$ then, obviously, MAX-GAP is 0. Thus, we assume that there are at least two distinct numbers. The main idea is to map each number to the two points onto the unit disk centered at the origin (one point will lie on the upper arc and the other on the lower arc). We subtract $x_{\rho(0)}$ from each number $x_i, 0 \leq i \leq n-1$. Let $x'_{\rho(i)} = x_{\rho(i)} - x_{\rho(0)}$.

Define the following mapping: $x_{\rho(i)}$ corresponds to the two points $p_{\rho(i)}$ and $p_{\rho(i)+n}$ with coordinates $(\cos(\pi - x'_{\rho(i)}), \sin(\pi - x'_{\rho(i)}))$ and $(\cos(x'_{\rho(i)}), -\sin(x'_{\rho(i)}))$, respectively. Note, that points $p_{\rho(0)}, p_{\rho(1)}, \ldots, p_{\rho(n-1)}, p_{\rho(0)+n}, \ldots, p_{\rho(n-1)+n}$ are listed in their clockwise order on the unit disk starting at the point $(-1, 0)$. We also want to avoid the case when the distance measured along the upper (lower) arc of the disk between the point $p_{\rho(n-1)}$ ($p_{\rho(n-1)+n}$) and the point $(1, 0)$ ($(-1, 0)$) is equal or larger than the MAX-GAP solution. In order to prevent this we shift all the points (except the points $(1, 0)$ and $(-1, 0)$) on the upper arc (lower arc) to the right (to the left) along this arc. But for how much? We need to find any two successive numbers in the sorted sequence of x_i numbers, $0 \leq i \leq n-1$, compute the gap $\gamma > 0$ between them and then shift the points along the upper (lower) arc by α, such that $\alpha < \pi - (x_{\rho(n-1)} - x_{\rho(0)})$ and $\gamma + \alpha > \pi - (x_{\rho(n-1)} - x_{\rho(0)}) - \alpha$. The number γ can be found by simple scanning the numbers $x_i, 1 \leq i \leq n$ and looking for the smallest element

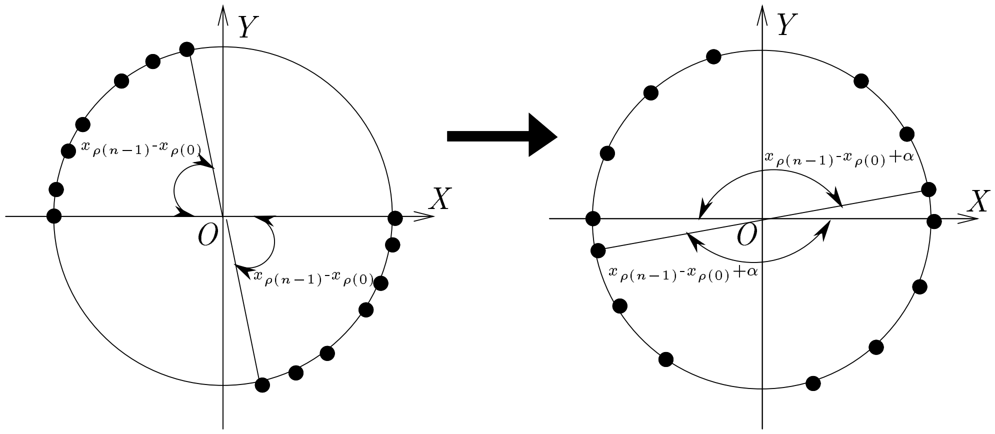

Figure 1: Mapping of MAX-GAP instance.

that is larger than $x_{\rho(0)}$ (by our assumption it exists). Thus, we have a simple inequality $(\pi - x_{\rho(n-1)} + x_{\rho(0)} - \gamma)/2 < \alpha < \pi - (x_{\rho(n-1)} - x_{\rho(0)})$. Shifting of points guarantee us existence of at least one index i, $0 \leq i \leq n-2$, such that the distance between points $p_{\rho(i)}$ and $p_{\rho(i+1)+n}$ is smaller than the distance between points $p_{\rho(0)}$ and $p_{\rho(n-1)}$. The situation is illustrated in Figure 1.

We are now ready to prove the lower bound for the optimal line bipartition and the Euclidean two-center problems.

Theorem 1 *In the algebraic decision-tree model every solution algorithm for the optimal line bipartition problem (even if all the points are known in advance to lie on a convex hull) has time complexity* $\Omega(n \log n)$.

Proof. Let $0 \leq x_0, x_2, \ldots, x_{n-1} \leq 1$ be an input of the MAX-GAP problem and let $S = \{p_{\rho(0)}, p_{\rho(1)}, \ldots, p_{\rho(n-1)}, p_{\rho(0)+n}, \ldots, p_{\rho(n-1)+n}\}$ be the corresponding sequence of $2n$ shifted points, as described above. Consider Figure 2(a). We will solve the optimal line bipartition problem for S and derive the solution for the MAX-GAP problem for $x_0, \ldots, x_{n-1}$. Note, that by symmetry, in any optimal bipartition (S_1, S_2) for S the size of each set is n; otherwise, we can always improve the solution by decreasing the size of one of the sets and increasing the size of the other. Consequently, in the optimal solution the partitioning line l divides S between points $p_{\rho(i)}, p_{\rho(i+1)}$ and $p_{\rho(i)+n}, p_{\rho(i+1)+n}$, for some i, $0 \leq i \leq n-2$. (The remaining case when l divides S between points p_{2n-1}, p_0 and p_{n-1}, p_n is not optimal, since the points of S are shifted.) By the construction of S, the diameter of $S_1 = \{p_{\rho(i+1)+n}, \ldots, p_{\rho(0)}, \ldots, p_{\rho(i)}\}$, identical with the diameter of S_2, is equal to the distance between $p_{\rho(i)}$ and $p_{\rho(i+1)+n}$, which is

$$\sqrt{(\sin(x'_{\rho(i)} + \alpha) + \sin(x'_{\rho(i+1)} + \alpha))^2 + (\cos(x'_{\rho(i)} + \alpha) + \cos(x'_{\rho(i+1)} + \alpha))^2}$$

for some $0 \leq i \leq n-2$. After algebraic manipulation we obtain that the diameter is equal to $\sqrt{(2 + 2\cos(x'_{\rho(i+1)} - x'_{\rho(i)})}$ and attains its minimum when $\phi_i = x'_{\rho(i+1)} - x'_{\rho(i)}$ is the largest. Therefore, to solve the MAX-GAP problem we first solve the corresponding

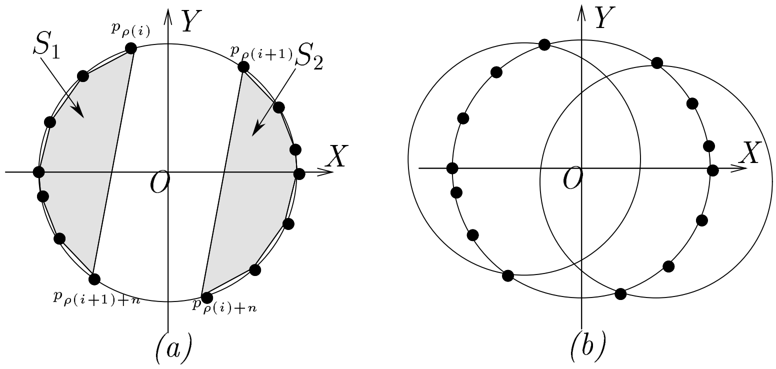

Figure 2: Two cases: (a) The optimal bipartition with respect to the area, perimeter or diameter of the convex hull is defined by MAX-GAP and (b) The optimal pair of the disks with respect to the radius is defined by the MAX-GAP.

bipartition problem for S that results in finding points $x'_{\rho(i)}$ and $x'_{\rho(i+1)}$. The absolute value of its difference is the solution to the MAX-GAP problem.

With one extra effort we will show how to prove two remaining cases. In fact, we show the area case; the perimeter case can be dealt similarly. Note that the optimal bipartition (S_1, S_2) defined by a line l with respect to the perimeter or area function is same as the the optimal bipartition with respect to the diameter function. (As before, the case when $S_1 = \{p_{\rho(0)}, \ldots, p_{\rho(n-1)}\}$, $S_2 = S - S_1$ is not optimal by definition of S.) The area of the convex hull of set S_1 (S_2) is equal one half of the area of the convex hull of S minus the area of the rectangle with corners at $p_{\rho(i)}, p_{\rho(i+1)}, p_{\rho(i)+n}, p_{\rho(i+1)+n}$, $0 \leq i \leq n-2$. This area is, therefore, minimized when the area of the rectangle is maximized, which is equal to

$$\sqrt{2 - 2\cos\phi_i} * \sqrt{2 + 2\cos\phi_i} = 2\sin\phi_i.$$

In the given interval, this value attains maximum when $\phi_i = x'_{\rho(i+1)} - x'_{\rho(i)}$ is maximal. Therefore, to solve the MAX-GAP problem we can solve the minimum area problem for the line bipartition problem for S. As a byproduct, this solution produces values $x'_{\rho(i)}$ and $x'_{\rho(i+1)}$. The absolute value of its difference is the solution to the MAX-GAP problem. This ends the proof. ∎

Theorem 2 *In the algebraic decision-tree model every solution algorithm for the Euclidean two-center problem (even if all the points are known in advance to lie on a convex hull) has time complexity* $\Omega(n \log n)$.

Proof. Let $0 \leq x_0, x_2, \ldots, x_{n-1} \leq 1$ be an input of the MAX-GAP problem and let $S = \{p_{\rho(0)}, p_{\rho(1)}, \ldots, p_{\rho(n-1)}, p_{\rho(0)+n}, \ldots, p_{\rho(n-1)+n}\}$ be the corresponding sequence of $2n$ shifted points, as described above. By symmetry, each of the two disks d_1 and d_2 in the optimal solution will contain exactly n points of S and disks will be of the same radius (see Figure 2(b)). The disk d_1 (d_2) will be defined by two points p^1 and p^2 of S (three points define the unit circle with all S points). Thus, these two points will define the diameter of each one of disks. In that case, the number of points between p^1 and p^2 lying on the unit disk (in clockwise direction) is exactly n; otherwise disk d_1 (d_2) will cover less than n points. Consequently, these points are $p_{\rho(i+1)}$ and $p_{\rho(i)+n}$, for some i, $0 \leq i \leq n-2$. (By construction and shifting of S, the points $p_{\rho(0)}$ and $p_{\rho(n-1)}$ can not produce an optimal solution.) The diameter of d_1 (d_2) is equal to $\sqrt{(2 + 2\cos(x'_{\rho(i+1)} - x'_{\rho(i)})}$ for some i, $0 \leq i \leq n-2$ and the smallest value is achieved when $x'_{\rho(i+1)} - x'_{\rho(i)}$ is maximal. Thus, in order to solve the MAX-GAP problem we can solve the Euclidean two-center problem, which gives us as a part of the solution $x'_{\rho(i+1)}$ and $x'_{\rho(i)}$ values. The absolute value of its difference is the solution to the MAX-GAP problem. ∎

2.2 Square rotation

We will show linear reductions from the MAX-GAP problem to the two covering problem and from GAP-EXISTENCE problem to the decision variant of the L_∞ discrete two center problem.

Theorem 3 *In the algebraic decision-tree model every solution algorithm for the two covering problem (even if all the points are known in advance to lie on a convex hull) has time complexity* $\Omega(n \log n)$.

Proof. Given an instance $0 \leq x_0, x_2, \ldots, x_{n-1} \leq 1$ of the MAX-GAP problem, we map each $x_{\rho(i)}$ to four points:

$$a_{\rho(i)} = (x_{\rho(i)}, 1-x_{\rho(i)}), b_{\rho(i)} = (1-x_{\rho(i)}, -x_{\rho(i)}), c_{\rho(i)} = (-x_{\rho(i)}, -1+x_{\rho(i)}), d_{\rho(i)} = (-1+x_{\rho(i)}, x_{\rho(i)})$$

Let $A = \{a_{\rho(i)}\}_{i=0}^{n-1}, B = \{b_{\rho(i)}\}_{i=0}^{n-1}, C = \{c_{\rho(i)}\}_{i=0}^{n-1}$ and $D = \{d_{\rho(i)}\}_{i=0}^{n-1}$. All the points of the set $A \cup B \cup C \cup D$ lie on the edges of the tilted unit square M (see Figure 3). We assume that the differences between $x_{\rho(n-1)}$ and 1 and between $x_{\rho(0)}$ and 0 are less than MAX-GAP solution; otherwise we can perform the similar transformation as we did before.

Define S be a set $A \cup B \cup C \cup D$ plus four more points that correspond to the vertices of M. Note, that S can be covered by two axis-parallel rectangles r_1 and r_2 of perimeter 6 each one. By symmetry, each of two axis-parallel rectangles r_1 and r_2 in the optimal solution will contain exactly $2n+2$ points; otherwise we will improve the solution by extending one of the rectangles and shrinking the other. The vertices of M are the *extreme* points of S and they serve as *determinators* of the smallest axis-parallel rectangle that covers S, that is, the edges of this rectangle must contain these points. If some rectangle $r_j, j = 1,2$ contains three determinators then the perimeter of r_j is at least 6. It remains to show what happens when each rectangle has to contain 2 determinators. We might notice, though, that if there is an index i such that any three of the points $a_{\rho(i)}$, $b_{\rho(i)}$, $c_{\rho(i)}$, $d_{\rho(i)}$ belong to the same covering axis-parallel rectangle, then the perimeter of this rectangle is greater or equal to 6. In order to improve the solution, we have to deal with two different cases (see Figure 3(a,b)).

Case (a). Each rectangle takes two adjacent determinators. Then, for each index i, $0 \leq i \leq n-1$ exactly two of $a_{\rho(i)}$, $b_{\rho(i)}$, $c_{\rho(i)}$, $d_{\rho(i)}$ points are covered by each rectangle and these two points must lie on the adjacent edges. In this case the perimeter of each rectangle equals $6 - 2(x_{\rho(i+1)} - x_{\rho(i)})$ for some i, $0 \leq i \leq n-2$.

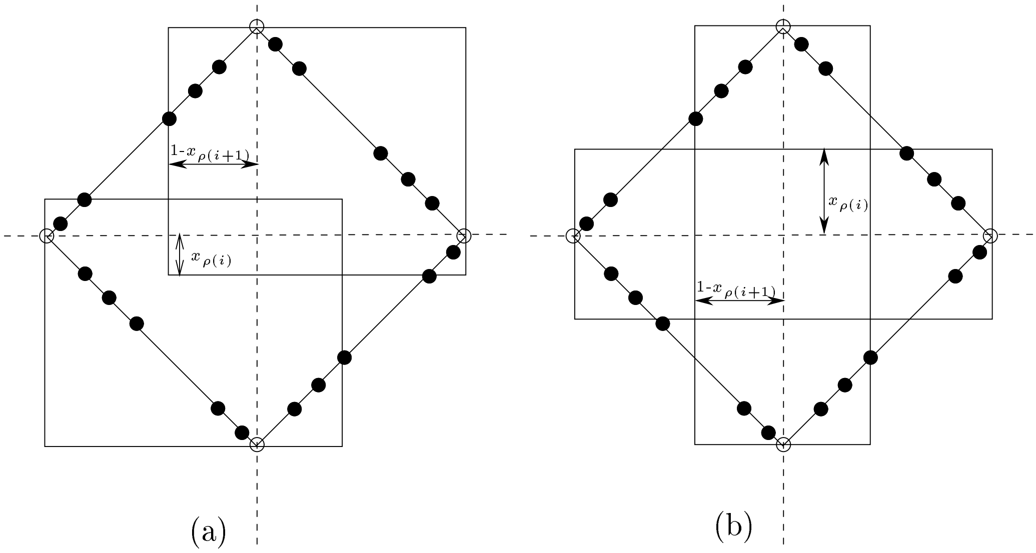

Figure 3: Min-Max Rectangle Problem: two cases.

Case (b). Each rectangle takes two opposite determinators. Then, for each index i, $0 \le i \le n-1$ exactly two of $a_{\rho(i)}$, $b_{\rho(i)}$, $c_{\rho(i)}$, $d_{\rho(i)}$ points are covered by each rectangle and these two points must lie on the opposite edges. In this case the perimeter of r_1 is equal to $8 - 4x_{\rho(i+1)}$ and the perimeter of r_2 is equal to $4 + 4x_{\rho(i)}$ for some i, $0 \le i \le n-2$.

Consider the sum of two perimeters in Case (b). The sum is equal to $12 - 4(x_{\rho(i+1)} - x_{\rho(i)})$. Thus, either each perimeter is equal to $6 - 2(x_{\rho(i+1)} - x_{\rho(i)})$ (which is the same result as in Case (a)), or one of the perimeters is smaller than $6 - 2(x_{\rho(i+1)} - x_{\rho(i)})$ and the other greater than $6 - 2(x_{\rho(i+1)} - x_{\rho(i)})$. It follows, that the solution obtained in Case (a) is optimal and attains its minimum when the difference $x_{\rho(i+1)} - x_{\rho(i)}$ is largest. This difference is the solution to the MAX-GAP problem. ∎

Theorem 4 *In the algebraic decision-tree model every solution algorithm for the L_∞ discrete two-center problem has time complexity $\Omega(n \log n)$.*

Proof. Given an instance $0 \le x_0, x_2, \ldots, x_{n-1} \le 1$, δ of the GAP-EXISTENCE problem, we map each $x_{\rho(i)}$ to four points:

$$a_{\rho(i)} = (x_{\rho(i)} + \delta, 1 - x_{\rho(i)}), b_{\rho(i)} = (1 - x_{\rho(i)}, -x_{\rho(i)})$$

$$c_{\rho(i)} = (-x_{\rho(i)}, -1 + x_{\rho(i)} - \delta), d_i = (-1 + x_{\rho(i)}, x_{\rho(i)})$$

Let $A = \{a_{\rho(i)}\}_{i=0}^{n-2}, B = \{b_{\rho(i)}\}_{i=0}^{n-1}, C = \{c_{\rho(i)}\}_{i=0}^{n-2}$ and $D = \{d_{\rho(i)}\}_{i=0}^{n-1}$. Define S be a set $B \cup D$ and put $F = A \cup C$ (as a matter of fact the proof holds for $S = A \cup B \cup C \cup D$). We are asking the question whether there are two discrete squares of radius 1 that centered at F can cover all the points of S? See Figure 4 above. Let b_1 be the square of radius 1 centered at $a_{\rho(i)} \in A$, for some i, $0 \le i \le n-2$. Then b_1 covers all the points $b_{\rho(0)}, b_{\rho(1)}, \ldots, b_{\rho(i)}$.

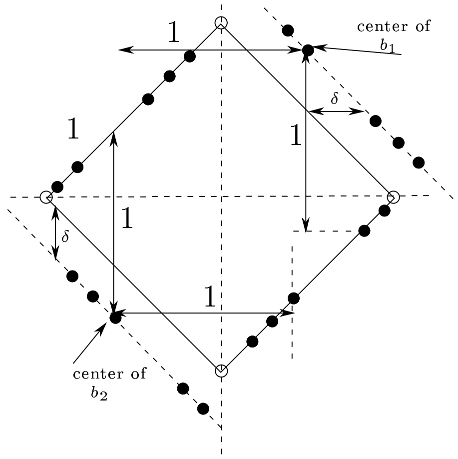

Figure 4: Centers of two discrete squares.

Assume that $x_{\rho(i+1)} - x_{\rho(i)} \geq \delta$. Then $1 - (x_{\rho(i+1)} - x_{\rho(i)}) + \delta \leq 1$. It turns out that b_1 also covers all the points $d_{\rho(i+1)}, d_{\rho(i+2)}, \ldots, d_{\rho(n-1)}$. Thus, the remaining uncovered points are $D' = \{d_{\rho(1)}, d_{\rho(2)}, \ldots, d_{\rho(i)}\}$ and $B' = \{b_{\rho(i+1)}, b_{\rho(i+2)}, \ldots, b_{\rho(n-1)}\}$. If we choose b_2 to be centered at point $c_{\rho(i+1)}$ then b_2 covers B' and D'. This is because the edge size of b_2 is 1 and $1 - (x_{i+1} - x_i) + \delta \leq 1$. If there is no index i, such that $x_{i+1} - x_i \geq \delta$ then points $b_{\rho(i+1)}$ and $d_{\rho(i+1)}$ remain uncovered by b_2 and there is no center in F for b_2 with the edge size 1 that will cover them. Thus, the answer to the GAP-EXISTENCE problem, is the same as the answer for the L_∞ discrete two-center problem for the sets S and F. ∎

2.3 Quadruple cycle rotation

Given an instance $0 \leq x_0, x_2, \ldots, x_{n-1} \leq 1$ of the MAX-GAP problem, we perform a similar transformation as for the first two problems considered in this paper, with the difference that now we map each number to four points, one point per quadrant (we assume that there are at least two distinct numbers in the MAX-GAP instance). Let $x'_{\rho(i)}, 0 \leq i \leq n-1$ be defined as before. Each $x_{\rho(i)}$ corresponds to four points $p_{\rho(i)}$,$p_{\rho(i)+n}$,$p_{\rho(i)+2n}$,$p_{\rho(i)+3n}$ with coordinates: $(\cos(\pi - x'_{\rho(i)}), \sin(\pi - x'_{\rho(i)}))$, $(\cos(\frac{\pi}{2} - x'_{\rho(i)}), \sin(\frac{\pi}{2} - x'_{\rho(i)}))$, $(\cos(x'_{\rho(i)}), -\sin(x'_{\rho(i)}))$ and $(\cos(\frac{3\pi}{2} - x'_{\rho(i)}), \sin(\frac{3\pi}{2} - x'_{\rho(i)}))$, respectively. In order to guarantee the existence of at least one index i, $0 \leq i \leq n-2$, such that the distance between points $p_{\rho(i)}$ and $p_{\rho(i+1)}$ is smaller than the distance between points $p_{\rho(n-1)}$ and $p_{\rho(0)+n}$, we shift all the points along the the arcs of the unit disk, similarly to what we did in the double cycle rotation transformation. Clearly, all these steps can be accomplished in linear time.

Theorem 5 *In the algebraic decision-tree model every solution algorithm for the two-line center problem (even if all the points are known in advance to lie on a convex hull) has time complexity* $\Omega(n \log n)$.

Proof. Given an instance $0 \leq x_0, x_2, \ldots, x_{n-1} \leq 1$ of the MAX-GAP problem, let S be a set of $4n$ points defined above. Definitely, S can be covered by two strips w_1 and w_2,

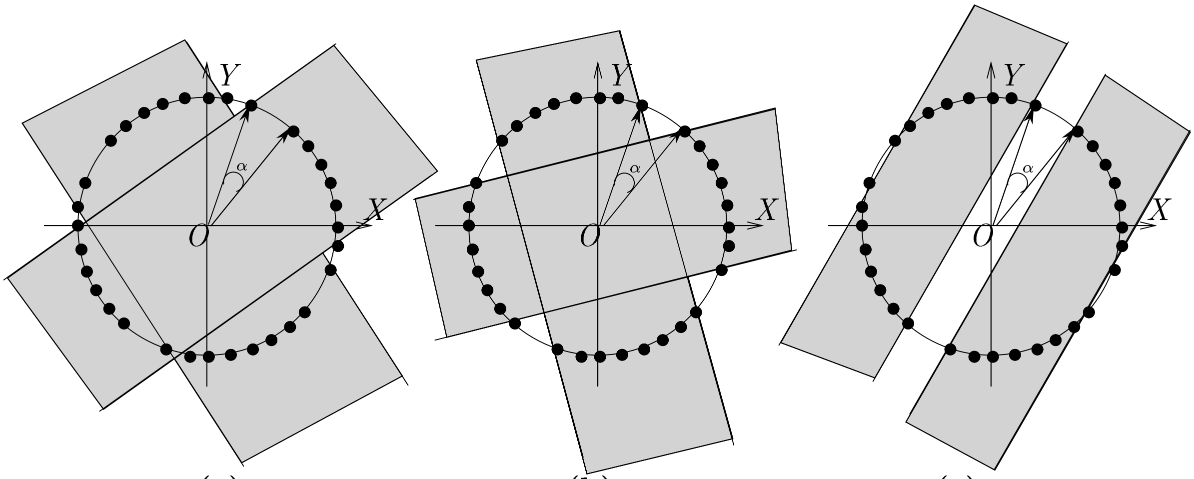

Figure 5: The optimal solution is achieved in the (c) case ($\alpha = x_{\rho(i+1)} - x_{\rho(i)}$).

each of width 1. Moreover, if one of the strips contains any three of four extreme points $(0,-1)$,$(-1,0)$,$(0,1)$,$(1,0)$ then it will have a width at least 1. Thus, we can assume that each strip contains only two extreme points. By symmetry, in any optimal solution each strip will cover exactly $2n$ points and strips will be of equal width; otherwise we increase the width of one of the strips and decrease the width of the other. Hence, according to our construction there are only two possible configurations. In the first configuration, each strip contains two opposite extreme points; in the second configuration, each strip contains two adjacent extreme points. It is easy to see that for the first configuration the optimal solution is achieved when there are exactly n points out of the left side of strip w_1 and there are exactly n points out of right side of w_1 (otherwise, one of the strips is wider than the other, see Figure 5(a,b)). It means that each side of strip w_1 (resp. w_2) in the first configuration is defined by two points $p_{\rho(i)}$ and $p_{\rho(i+1)+n}$ (resp. $p_{\rho(i)+n}$ and $p_{\rho(i+1)+2n}$) for some $i, 0 \leq i \leq n-2$. The value of the width of w_1 (w_2) can be obtained by computing the distance between these two points and is equal to $\sqrt{2 - 2\sin(x'_{\rho(i+1)} - x'_{\rho(i)})}$ for some $i, 0 \leq i \leq n-2$. For the second configuration, the optimal solution is defined by two strips such that each one of them contains $2n$ consecutive points along the arc of the unit disk (Figure 5(c)). In that case the width of each strip is equal to $\sqrt{1 - \sin(x'_{\rho(i+1)} - x'_{\rho(i)})}$, for some $i, 0 \leq i \leq n-2$. Thus, we obtain that the solution attains its minimum in the second configuration when $x'_{\rho(i+1)} - x'_{\rho(i)}$ is the largest. Each $x'_{\rho(i)}$ is between 0 and 1 and $\sin x$ is an increasing function in the given interval $[0, \frac{\pi}{2}]$. Hence, to solve the MAX-GAP problem we first solve the corresponding two-line center problem for S and then find a solution for the MAX-GAP problem in linear time. ∎

Theorem 6 *In the algebraic decision-tree model every solution algorithm for the smallest rotated enclosing square (even if all the points are known in advance to lie on a convex hull) has time complexity* $\Omega(n \log n)$.

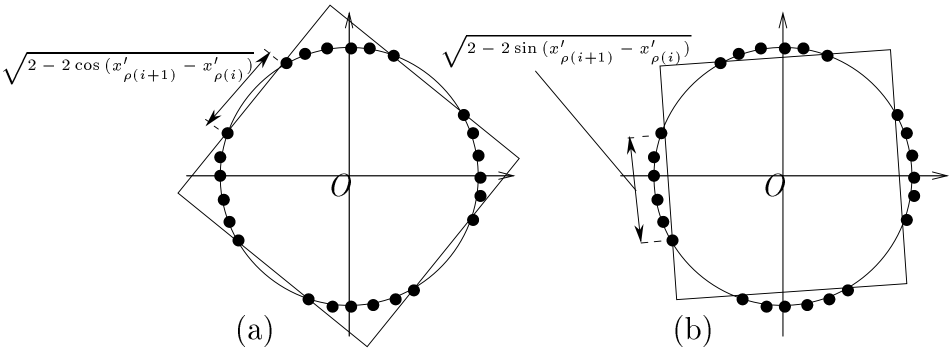

Figure 6: Smallest enclosing square (a) and largest empty inscribed square (b).

Proof. Let $x_0, \ldots, x_{n-1}$ be an input to the MAX-GAP problem and let S be a set of points as in the previous proof. We set $q = (0, 0)$. Obviously, set S can be covered by the enclosing square s of size 2. In order to do better we try to shrink s from all its sides till we hit some point of S, then rotate it, while trying to shrink it more. We repeat this process till some side of s will contain two points (otherwise we are able to continue rotate the square). See Figure 6(a). These two points should be consecutive on the disk (in a clockwise direction) because otherwise square s will miss some point of S. Therefore, these points are $p_{\rho(i)}$ and $p_{\rho(i+1)}$ for some i, $0 \leq i \leq n-2$ and the distance between them is equal to $\sqrt{2 - 2\cos(x'_{\rho(i+1)} - x'_{\rho(i)})}$ (according to cosines theorem). The total edge length of s is $2\sqrt{1 - \sin(x'_{\rho(i+1)} - x'_{\rho(i)})} + \sqrt{2 - 2\cos(x'_{\rho(i+1)} - x'_{\rho(i)})}$ for some $i, 0 \leq i \leq n-2$ which is a decreasing function of its argument $x'_{\rho(i+1)} - x'_{\rho(i)}$ in the interval $[0, \frac{\pi}{2}]$. Thus, in order to find a solution to the MAX-GAP problem we find the smallest enclosing square s centered at origin. This square is defined by points $p_{\rho(i+1)}$ and $p_{\rho(i)}$. The difference between values $x'_{\rho(i+1)}$ and $x'_{\rho(i)}$ is the solution to the MAX-GAP problem. ∎

Remark. If we allow s to be centered at any point in the plane, we obtain the same solution as described in the theorem.

Theorem 7 *In the algebraic decision-tree model every solution algorithm for the largest rotated empty square problem (even if all the points are known in advance to lie on a convex hull) has time complexity* $\Omega(n \log n)$.

Proof. For a given instance $x_0, \ldots, x_{n-1}$ of the MAX-GAP problem, let S and q be as in the previous theorem. There is an inscribed empty square s with the edge size equals $\sqrt{2}$ and vertices at extreme points of S. We try to extend the boundary of s from all its sides till we hit some point of S, then rotate s, while trying to extend it more. This process is repeated similarly to the previous case till some edge of s contains two points. It follows, by symmetry, that there are exactly $n-2$ points between these two points (See Figure 6(b). Opposite to the previous theorem these two points are $p_{\rho(i)}$ and $p_{\rho(i+1)+3n}$ for some i, $0 \leq i \leq n-2$. The distance between them is equal to $\sqrt{2 - 2\sin(x'_{\rho(i+1)} - x'_{\rho(i)})}$. Thus, the total edge length of

s is $\sqrt{2 - 2\sin(x'_{\rho(i+1)} - x'_{\rho(i)})} + 2\sqrt{1 - \cos(x'_{\rho(i+1)} - x'_{\rho(i)})}$ for some $i, 0 \leq i \leq n-2$ which is increasing function of its argument $x'_{\rho(i+1)} - x'_{\rho(i)}$ argument in the interval $[0, \frac{\varrho}{2}]$. Hence, in order to find a solution to the MAX-GAP problem we find the largest empty inscribed square s centered at origin for a set S. This square is defined by points $p_{\rho(i)}$ and $p_{\rho(i+1)+3n}$. The difference between values $x'_{\rho(i+1)}$ and $x'_{\rho(i)}$ is the solution to the MAX-GAP problem. ∎

Theorem 8 *In the algebraic decision-tree model every solution algorithm for the rotated squares problem has time complexity* $\Omega(n \log n)$.

Proof. Let $x_0, \ldots, x_{n-1}$ be an instance of the MAX-GAP problem. Let S be a set of points as used in the last theorem. We duplicate set S, obtaining a set S' of $4n$ points. We translate all the points of S' in the plane by vector $\vec{a} = (5, 0)$, i.e. we add 5 to the x-coordinate of every point in S'. Let $S'' = S \cup S'$. We want to find two mutually parallel squares s_1 and s_2 that cover S'', so as to minimize the edge size of the largest square. Obviously, S'' can be covered by two axis-parallel squares s_1 and s_2 with edge size 2; square s_1 covers S and square s_2 covers S'. In the better solution, s_1 (s_2) cannot contain any point of S' (S) since it would lead to the increasing edge size. Thus, the only possibility to make s_1 and s_2 smaller is to rotate them around S and S', respectively. Since a set S' is a translated copy of a set S, the squares s_1 and s_2 will be of the same size and remain mutually parallel in the optimal solution. According to the remark after theorem 6, the best possible placement of s_1 (and s_2) is defined by two points are $p_{\rho(i)}$ and $p_{\rho(i+1)}$ for some i, $0 \leq i \leq n-2$. As was pointed out before, the edge length of s_1 is $2\sqrt{1 - \sin(x'_{\rho(i+1)} - x'_{\rho(i)})} + \sqrt{2 - 2\cos(x'_{\rho(i+1)} - x'_{\rho(i)})}$ for some $i, 0 \leq i \leq n-2$ which is decreasing function of its argument $x'_{\rho(i+1)} - x'_{\rho(i)}$ in the interval $[0, \frac{\pi}{2}]$. This function achieves its minimum when the difference $x'_{\rho(i+1)} - x'_{\rho(i)}$ is the largest. Then, after we find the solution to the rotated squares problem, we can find points $p_{\rho(i+1)}$ and $p_{\rho(i)}$ that define square s_1. The difference between values $x'_{\rho(i+1)}$ and $x'_{\rho(i)}$ is the solution to the MAX-GAP problem. ∎

3 Conclusions

In this paper we have presented lower bounds for a number of different covering and optimization problems. All the proofs are based on a linear time reductions from MAX-GAP problem. The crucial property of the reductions is that they can be obtained by mapping the numbers from the MAX-GAP instance to the points on the circle or tilted square and their subsequent rotations. The challenging questions remaining are how to narrow a gap between the existing algorithms for a several problems mentioned in this paper and their lower bounds.

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