A Unified Approach for Registration and Depth in Depth from Defocus

Rami Ben-Ari

Appendices

A Euler-Lagrange Derivation

This appendix includes 3 sections. The first two address the derivation of the Euler-Lagrange (EL) equations (21) and (23) in the manuscript. The last section elaborates on the relative blur model in Section 2.3.

A.1 Variation with respect to $V_r$

Let us begin our derivation with the definition of the defocus operation:

$$h \ast f := \int_{\Gamma} h(u, V_r(x)) f(x - u) du$$

(A.1)

with $x \in \Omega \subset \mathbb{R}^2$, $f : \Omega \rightarrow [0, \infty]$ representing the image, $h : \Gamma \subset \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow [0, 1)$ the spatially varying defocus kernel and $\ast$ denoting the defocus operator as defined in (3). We are interested in the functional derivative of the following energy functional:

$$F(V_r) = \int_{\Omega} h(x, V_r) \otimes f(x) dx$$

(A.2)

Following the standard derivation in calculus of variation, the variational derivative can be expressed by:

$$\frac{\delta F}{\delta V_r} = \frac{\partial}{\partial \lambda} F(V_r + \lambda \eta(x)) \bigg|_{\lambda=0}$$

(A.3)

where $\lambda \in \mathbb{R}$ and $\eta \in C^1(\Omega)$, prescribes the variation.

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Rami Ben-Ari
Orbotech Ltd., Yavne, Israel
E-mail: benari.rami@gmail.com
Substituting the expression for the defocus operator in (A.3) yields:

\[
\int_{\Omega} \int_{\Gamma} \frac{\partial}{\partial \lambda} \left[ h(u, V_r(x) + \lambda \eta) f(x - u) \right] \bigg|_{\lambda=0} \, du \, dx
\]  

(A.4)

Note that \( u \) and \( x \) are independent variables. Derivation by \( \lambda \) yields:

\[
\int_{\Omega} \int_{\Gamma} \frac{\partial h}{\partial V_r}(u, V_r(x) + \lambda \eta) \otimes f(x) \bigg|_{\lambda=0} \, du \eta(x) \, dx
\]  

(A.5)

Substituting \( \lambda = 0 \) and using fundamental lemma of calculus of variations, the EL term emerges as:

\[
\frac{\delta \mathcal{F}}{\delta V_r} = \frac{\partial h}{\partial V_r} \otimes f(x)
\]  

(A.6)

A.2 Variation with respect to transformation parameters

In this section we derive the expression for the variational derivative of type:

\[
\mathcal{F}(V_r, T) = \int_{\Omega} h(x, V_r) \otimes f(Tx) \, dx
\]  

(A.7)

corresponding to the Euler-Lagrange PDE derived in (23). According to definition of the defocus operator in (A.1), the variation by each transformation component \( \zeta_i \) is carried out as:

\[
\frac{\delta \mathcal{F}}{\delta \zeta_i} V_r, T = \frac{\partial}{\partial \zeta_i} \int_{\Omega} h(x, V_r(x)) f(T(x - u)) \, du \, dx
\]  

(A.8)

Considering the defocus as a smoothing operator, we can approximate the integrand by a continuously differentiable function. According to Leibnitz integral rule then, the derivative can be passed inside the integral:

\[
= \int \int \frac{\partial h(x, V_r(x)) f(T(x - u))}{\partial \zeta_i} \, du \, dx
\]  

(A.9)

\[
= h \otimes \frac{\partial f(Tx)}{\partial \zeta_i}
\]

where the equality is satisfied due to independence of \( h(\cdot, \cdot) \) to the geometric transformation (according to Propositions (1) and (2) in the paper). Note that the defocus kernel is defined w.r.t the reference frame and depends on the recovered shape \( z(x) \) corresponding to this frame and therefore is independent to \( T \).
A.3 Relative blur

To make the paper self contained, we hereby show briefly the relative blur model as often used in DFD. For sake of simplicity let us assume the images are aligned (for proof in case of misaligned images please see the paper). The image formation model for a Gaussian defocus kernel implies:

\[ I_i(x) = G(V_i) \ast r(x) \]  \hspace{1cm} (A.10)

where \( G \) denotes the Gaussian kernel with variance \( V_i(x) \) and \( r(x) \) stands for the radiance image. The index \( i \in \{1, 2\} \) corresponds to the appropriate observation and the sign \( \ast \) denotes the shift-variant defocus blur. Restricting ourselves to a local patch allows us to assume a constant blur. The defocus operation then reduces to a convolution. Considering \( V_2 > V_1 \) without loss of generality, we can decompose the defocus using the cascade property of Gaussian convolution:

\[
I_2 = G(V_2 - V_1) \ast \underbrace{G(V_1) \ast r(x)}_{I_1} = G(V_r) \ast I_1
\]  \hspace{1cm} (A.11)

where the last equality is based on the image formation (A.10) where \( V_r := V_2 - V_1 \) denotes the so called relative blur (scale).