Parameterized Algorithms

Lecture 5: DP + Color Coding
Steiner Tree. Given a graph $G$ and a terminal set $W \subseteq V(G)$, what is the minimum size of a subtree $T$ of $G$ such that $W \subseteq V(T)$?

Parameter. $k = |W|$.
**Steiner Tree.** Given a graph $G$ and a terminal set $W \subseteq V(G)$, what is the minimum size of a subtree $T$ of $G$ such that $W \subseteq V(T)$?

Parameter. $k = |W|$.

(Size of subtree $T = |E(T)|$.)
Notation.

\[ \text{dist}(u,v) = \text{length of a shortest path in } G \text{ between } u \text{ and } v. \]
DP: Steiner Tree

Assumptions.

1. $|W| > 1$. 
Assumptions.

1. $|W| > 1$. (Else the problem is trivial.)
DP: Steiner Tree

Assumptions.

1. $|W| > 1$. 
2. $G$ is connected.
DP: Steiner Tree

Assumptions.

1. $|W| > 1$.

2. $G$ is connected. (If there is no connected component that contains all terminals, then the answer is No.)
DP: Steiner Tree

Assumptions.

1. $|W| > 1$.
2. $G$ is connected.
3. $W$ is an independent set, and each vertex in $W$ is a leaf in $G$. 
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DP: Steiner Tree
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Assumptions.

1. $|W| > 1$.
2. $G$ is connected.
3. $W$ is an independent set, and each vertex in $W$ is a leaf in $G$. 
DP Table. Entry $M[v, W']$ for all $v \in V(G) \setminus W$ and $W' \subseteq W, \ W' \neq \emptyset$. 
**DP: Steiner Tree**

**DP Table.** Entry $M[v, W']$ for all $v \in V(G) \setminus W$ and $W' \subseteq W$, $W' \neq \emptyset$.

**Meaning.** $M[v, W']$: What is the minimum size of a subtree $T$ of $G$ such that $W' \cup \{v\} \subseteq V(T)$?
DP: Steiner Tree

DP Table. Entry $M[v, W']$ for all $v \in V(G) \setminus W$ and $W' \subseteq W$, $W' \neq \emptyset$.

Meaning. $M[v, W']$: What is the minimum size of a subtree $T$ of $G$ such that $W' \cup \{v\} \subseteq V(T)$?

Basis. $M[v, \{w\}] = \text{dist}(v, w)$. 
**Recursion.** $M[v, W'] = \min_{u \in V(G) \setminus W} \{ M[u, X] + M[u, W \setminus X] + \text{dist}(v,u) \}$. 

$X \subseteq W'$, $X \neq \emptyset$
**Proof.** $M[v, W'] \leq \min\{M[u, X] + M[u, W \setminus X] + \text{dist}(v, u)\}$.

$u \in V(G) \setminus W$

$X \subseteq W'$, $X \neq \emptyset$
**Proof.** \( M[v, W'] \leq \min\{M[u, X] + M[u, W \setminus X] + \text{dist}(v, u)\} \).

\[
\begin{align*}
&\text{for } u \in V(G) \setminus W \\
&X \subseteq W', X \neq \emptyset
\end{align*}
\]

For \( u \) and \( X \) that realize \( \min \), by the inductive hypothesis:
Proof. \( M[v,W'] \leq \min\{ M[u,X] + M[u,W\setminus X] + \text{dist}(v,u) \} \).

\[ u \in V(G) \setminus W \]
\[ X \subseteq W', X \neq \emptyset \]

For \( u \) and \( X \) that realize min, by the inductive hypothesis:

Subgraph \( S \) with at most \(*\) edges such that \( W' \cup \{v\} \subseteq V(S) \).
Proof. $M[v,W'] \geq \min\{M[u,X] + M[u,W' \setminus X] + \text{dist}(v,u)\}.
\begin{align*}
&\quad u \in V(G) \setminus W \\
&\quad X \subseteq W', X \neq \emptyset
\end{align*}
**DP: Steiner Tree**

**Proof.** \( M[v, W^*] \geq \min\{M[u, X] + M[u, W \setminus X] + \text{dist}(v, u)\}. \)

\[
\begin{align*}
u &\in V(G) \setminus W \\
X &\subseteq W^*, X \neq \emptyset
\end{align*}
\]

For an optimal Steiner tree w.r.t. \( W' \cup \{v\} \):
DP: Steiner Tree

Proof. \( M[v, W'] \geq \min\{M[u, X] + M[u, W \setminus X] + \text{dist}(v, u)\} \).

\[
\begin{align*}
u \in & V(G) \setminus W \\
X \subseteq & W', \ X \neq \emptyset
\end{align*}
\]

For an optimal Steiner tree w.r.t. \( W' \cup \{v\} \):

- \( |W'| > 1 \),
- \( W \) is an independent set of leaves
Proof. $M[v, W'] \geq \min \{M[u, X] + M[u, W \setminus X] + \text{dist}(v, u) \}$.

$\forall u \in V(G) \setminus W$

$X \subseteq W$, $X \neq \emptyset$

For an optimal Steiner tree w.r.t. $W' \cup \{v\}$:
Proof. $M[v,W'] \geq \min\{M[u,X] + M[u,W'\backslash X] + \text{dist}(v,u)\}$.

$u \in V(G) \backslash W$

$X \subseteq W', X \neq \emptyset$

For an optimal Steiner tree w.r.t. $W' \cup \{v\}$:
Recursion. $M[v,W'] = \min \{M[u,X] + M[u,W \setminus X] + \text{dist}(v,u)\}.$

$u \in V(G) \setminus W$
$X \subseteq W', X \neq \emptyset$

Running Time.
**Recursion.** $M[v, W'] = \min \{M[u, X] + M[u, W \setminus X] + \text{dist}(v, u)\}$.

\[
\begin{align*}
u & \in V(G) \setminus W, \\
X & \subset W', \ X \neq \emptyset
\end{align*}
\]

**Running Time.**

\[
O(\sum_{W' \subset W} \sum_{v \in V(G)} \sum_{X \subset W'} \sum_{u \in V(G) \setminus W} 1) = O(n^2 \sum_{W' \subset W} \sum_{X \subset W'} 1) = O(3^k n^2)
\]
Color Coding

Highlight a small pattern in a large input so that it will be easy to find it and then solve the problem.
The pattern is usually the solution or part of it.
Color Coding: \( k \)-Path

\( k \)-Path.

**Input:** Graph \( G \); parameter \( k \).

**Question:** Does \( G \) have a path on at least \( k \) vertices?

\( k = 5 \)
**Color Coding: $k$-Path**

**$k$-Path.**

**Input:** Graph $G$; parameter $k$.

**Question:** Does $G$ have a path on at least $k$ vertices?
Color Coding: $k$-Path

Color-set: \{1,2,...,k\}.  

To each vertex, randomly assign a color.  

Highlight a solution $\rightarrow$ colorful solution.
Color Coding: $k$-Path

Color-set: $\{1,2,...,k\}$.  
To each vertex, randomly assign a color.
Highlight a solution $\rightarrow$ colorful solution.

Unlike Lecture 1, now we ignore the order!
Color Coding: $k$-Path

Color-set: $\{1,2,...,k\}$.  
To each vertex, randomly assign a color.  
Highlight a solution $\rightarrow$ colorful solution.  
We are also happy with...
Color Coding: $k$-Path

Color-set: $\{1,2,...,k\}$. 

To each vertex, randomly assign a color. Highlight a solution $\rightarrow$ colorful solution. 

And with...
Color Coding: $k$-Path

Color-set: $\{1,2,\ldots,k\}$.  
To each vertex, randomly assign a color.  
Highlight a solution $\rightarrow$ colorful solution.  
But not with...
Color Coding: $k$-Path

The probability of highlighting a solution: $k!/k^k$
The probability of highlighting a solution:

\[
k!/k^k > (k^k/e^k)/k^k = 1/e^k. \quad \text{(Stirling’s approx.)}
\]
Probability of highlighting a solution: $(1/e)^k$. \( \rightarrow O(e^k) \) iterations.

$k = 5$
Color Coding: $k$-Path

Probability that a single iteration fails: $1 - (1/e^k)$.

Prob. that $100e^k$ iterations fail: $[1 - (1/e^k)]^{100e^k} < e^{-100}$

Prob. of success $> 1 - e^{-100}$. 
Color Coding: $k$-Path

Probability of highlighting a solution: $(1/e)^k$. \(\rightarrow O(e^k)\) iterations.

$k=5$
Color Coding: $k$-Path

One-sided error:
- If we say Yes, then there exists a solution.
- If there exists a solution, then we say Yes with high probability.
Running time:
Let $T$ be the time in which we can decide whether there is a colorful $k$-path. Then, the running time is $O(T \cdot e^k)$. 
Running time:
Let $T$ be the time in which we can decide whether there is a colorful $k$-path. Then, the running time is $O(T \cdot e^k)$. We will get $T = O^*(2^k)$. 

**Color Coding: $k$-Path**
- Compared to Lecture 1, now it was much easier to highlight a solution.
- But the highlighted problem we got is NP-hard. Nevertheless, it can be solved "efficiently".
**Color Coding: $k$-Path**

**$k$-Path. Input:** Graph $G$; parameter $k$.

**Question:** Does $G$ have a path $k$ vertices?

**DP Table.** $M[U,v]$ for every subset of vertices $U$ of $G$ and vertex $v$ in $U$.

**Meaning.** Is there a $(|U|)$-path whose set of vertices is exactly $U$ and it ends at $v$? (True or False.)
Color Coding: $k$-Path

**$k$-Path. Input:** Graph $G$; parameter $k$.

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Color Coding: \( k \)-Path

\textbf{\( k \)-Path. Input:} Graph \( G \); parameter \( k \).
\textbf{Question:} Does \( G \) have a path \( k \) vertices?

\textbf{DP Table.} \( M[U,v] \) for every subset of vertices \( U \) of \( G \) and vertex \( v \) in \( U \).
\textbf{Meaning.} Is there a \((|U|\text{-})\)path whose set of vertices is exactly \( U \) and it ends at \( v \)? (True or False.)

\textbf{Basis.} \(|U|=1\). \( M[U,v]=\text{True iff } U=\{v\} \).

\textbf{Recursion.} \(|U|\geq2\).

\[ M[U,v] = \text{OR}_{u \text{ neighbor of } v \text{ and in } U} (M[U\setminus\{v\},u]). \]
**Color Coding: k-Path**

**k-Path. Input:** Graph $G$; parameter $k$.
**Question:** Does $G$ have a path $k$ vertices?

**DP Table.** $M[U,v]$ for every subset of vertices $U$ of $G$ and vertex $v$ in $U$.

**Meaning.** Is there a $(|U|)$-path whose set of vertices is exactly $U$ and it ends at $v$? (True or False.)

**Basis.** $|U|=1$. $M[U,v]=\text{True}$ iff $U=\{v\}$.

**Recursion.** $|U|\geq2$.

$M[U,v] = \text{OR}_{u \text{ neighbor of } v \text{ and in } U}(M[U\setminus\{v\},u])$.

**Answer.** Yes iff $M[U,v]$ is True for some $U$ of size $k$ and $v$. 
**Color Coding: $k$-Path**

**$k$-Path. Input:** Graph $G$; parameter $k$.
**Question:** Does $G$ have a path $k$ vertices?

**DP Table.** $M[U,v]$ for every subset of vertices $U$ of $G$ and vertex $v$ in $U$.

**Meaning.** Is there a $(|U|)$-path whose set of vertices is exactly $U$ and it ends at $v$? (True or False.)


**Recursion.** $|U|\geq 2$.

$$M[U,v] = \text{OR}_{u \text{ neighbor of } v \text{ and in } U}(M[U\setminus\{v\},u]).$$

**Time.** $O(2^n\sum_v \deg(v))=O(2^nm)$. 
Color Coding: $k$-Path

$k$-Path. Input: Graph $G$; parameter $k$.

Question: Does $G$ have a path $k$ vertices?

DP Table. $M[U,v]$ for every subset of vertices $U$ of $G$ and vertex $v$ in $U$.

Meaning. Is there a ($|U|$-)path whose set of vertices is exactly $U$ and it ends at $v$? (True or False.)


Recursion. $|U|\geq2$.

$$M[U,v] = \bigvee_{u \text{ neighbor of } v \text{ and in } U}(M[U\setminus\{v\},u]).$$

Time. $O(2^nm)$. (Can prune to $O(\binom{n}{k}m)$).
**Color Coding: \(k\)-Path**

**Colorful \(k\)-Path. Input:** Graph \(G\); parameter \(k\).

**Question:** Does \(G\) have a colorful path \(k\) vertices?

**DP Table.** \(M[C,v]\) for every subset of colors \(C\) of \(\{1,...,k\}\) and vertex \(v\) in \(G\) whose color is in \(C\).

**Meaning.** Is there a colorful (\(|C|\)-)path whose set of colors is exactly \(C\) and it ends at \(v\)?
**Color Coding: \( k \)-Path**

**Colorful \( k \)-Path. Input:** Graph \( G \); parameter \( k \).

**Question:** Does \( G \) have a colorful path \( k \) vertices?

**DP Table.** \( M[C,v] \) for every subset of colors \( C \) of \( \{1,\ldots,k\} \) and vertex \( v \) in \( G \) whose color is in \( C \).

**Meaning.** Is there a colorful \((|C|-)\)path whose set of colors is exactly \( C \) and it ends at \( v \)?

**Note.** If every color is used only once, a walk is a path. We use colors (rather than vertices) to ensure that no vertex is visited more than once.
**Color Coding: $k$-Path**

**Colorful $k$-Path. Input:** Graph $G$; parameter $k$.  
**Question:** Does $G$ have a colorful path $k$ vertices?

**DP Table.** $M[C,v]$ for every subset of colors $C$ of $\{1,...,k\}$ and vertex $v$ in $G$ whose color is in $C$.  
**Meaning.** Is there a colorful $(|C|)$-path whose set of colors is exactly $C$ and it ends at $v$?  
**Basis.** $|C|=1$. $M[C,v]=\text{True}$ iff $C=\{\text{col}(v)\}$. 
**Color Coding: k-Path**

**Colorful k-Path. Input:** Graph $G$; parameter $k$.

**Question:** Does $G$ have a colorful path $k$ vertices?

**DP Table.** $M[C,v]$ for every subset of colors $C$ of $\{1,\ldots,k\}$ and vertex $v$ in $G$ whose color is in $C$.

**Meaning.** Is there a colorful ($|C|$-)path whose set of colors is exactly $C$ and it ends at $v$?

**Basis.** $|C|=1$. $M[C,v]=\text{True}$ iff $C=\{\text{col}(v)\}$.

**Recursion.** $|C|\geq 2$.

$M[C,v] = \text{OR}_{u \text{ neighbor of } v \text{ with color in } C\setminus\{\text{col}(v)\}} (M[C\setminus\{\text{col}(v)\},u])$. 
**Color Coding: k-Path**

**Colorful k-Path.** **Input:** Graph $G$; parameter $k$.

**Question:** Does $G$ have a colorful path $k$ vertices?

**DP Table.** $M[C,v]$ for every subset of colors $C$ of $\{1,...,k\}$ and vertex $v$ in $G$ whose color is in $C$.

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**Recursion.** $|C|\geq 2$.

$M[C,v] = \text{OR}_{u \text{ neighbor of } v \text{ with color in } C\{\text{col}(v)\}} (M[C\{\text{col}(v)\},u])$.

**Answer.** Yes iff $M[\{1,...,k\},v]$ is True for some $v$. 
Color Coding: \(k\)-Path

**Colorful \(k\)-Path.** Input: Graph \(G\); parameter \(k\).

**Question:** Does \(G\) have a colorful path \(k\) vertices?

**DP Table.** \(M[C,v]\) for every subset of colors \(C\) of \(\{1,...,k\}\) and vertex \(v\) in \(G\) whose color is in \(C\).

**Meaning.** Is there a colorful (\(|C|\)-)path whose set of colors is exactly \(C\) and it ends at \(v\)?

**Basis.** \(|C| = 1\). \(M[C,v] = \text{True} \iff C = \{\text{col}(v)\}\).

**Recursion.** \(|C| \geq 2\).

\(M[C,v] = \text{OR}_{u \text{ neighbor of } v \text{ with color in } C \setminus \{\text{col}(v)\}} (M[C \setminus \{\text{col}(v)\}, u])\).

**Time.** \(O(2^km)\).
**d-Set Packing.** Given a family $F=\{S_1,S_2,...,S_m\}$ of sets of size exactly $d$ over a universe $U$ and a parameter $k$, decide whether there is a subset $F'$ of $F$ of size at least $k$ whose sets are pairwise vertex disjoint.
**Color Coding: \(d\)-Set Packing**

**\(d\)-Set Packing.** Given a family \(F=\{S_1, S_2, \ldots, S_m\}\) of sets of size exactly \(d\) over a universe \(U\) and a parameter \(k\), decide whether there is a subset \(F'\) of \(F\) of size at least \(k\) whose sets are pairwise vertex disjoint.

**\(O^*((2e)^{dk})\)-Time Algorithm.** Use the method of color coding with \(dk\) colors. Here, a colorful solution refers to a solution \(F'\) such that all elements in \(UF'\) have distinct colors.
**Color Coding: $d$-Set Packing**

**$d$-Set Packing.** Given a family $F=\{S_1, S_2, \ldots, S_m\}$ of sets of size exactly $d$ over a universe $U$ and a parameter $k$, decide whether there is a subset $F'$ of $F$ of size at least $k$ whose sets are pairwise vertex disjoint.

**$O^*((2e)^{dk})$-Time Algorithm.** Use the method of color coding with $dk$ colors. Here, a colorful solution refers to a solution $F'$ such that all elements in $U \setminus F'$ have distinct colors. Coloring iterations and analysis almost identical to the case of $k$-Path.
**Color Coding: $d$-Set Packing**

**$d$-Set Packing.** Given a family $F=\{S_1, S_2, \ldots, S_m\}$ of sets of size exactly $d$ over a universe $U$ and a parameter $k$, decide whether there is a subset $F'$ of $F$ of size at least $k$ whose sets are pairwise vertex disjoint.

**$O^*((2e)^{dk})$-Time Algorithm.** Use the method of color coding with $dk$ colors. Here, a colorful solution refers to a solution $F'$ such that all elements in $U \cup F'$ have distinct colors. Coloring iterations and analysis almost identical to the case of $k$-Path.

Design a DP-based algorithm that works in time $O^*(2^{dk})$ (exercise or on board).
Perfect Hash Family. A family $F$ of functions $f : \{1,\ldots,n\} \rightarrow \{1,\ldots,k\}$ is an $(n,k)$-perfect hash family if for every subset $S$ of $\{1,\ldots,n\}$ of size $k$, there exists a function in $F$ that is injective when restricted to $S$. 

Color Coding: Derandomization
Perfect Hash Family. A family $F$ of functions $f : \{1,\ldots,n\} \rightarrow \{1,\ldots,k\}$ is an $(n,k)$-perfect hash family if for every subset $S$ of $\{1,\ldots,n\}$ of size $k$, there exists a function in $F$ that is injective when restricted to $S$.

Why does such a family exist?
Perfect Hash Family. A family $F$ of functions $f : \{1, \ldots, n\} \rightarrow \{1, \ldots, k\}$ is an $(n,k)$-perfect hash family if for every subset $S$ of $\{1, \ldots, n\}$ of size $k$, there exists a function in $F$ that is injective when restricted to $S$.

Why does such a family exist? Consider the family of all functions.
### Color Coding: Derandomization

**Perfect Hash Family.** A family $F$ of functions $f : \{1,\ldots,n\} \rightarrow \{1,\ldots,k\}$ is an *(n,k)*-perfect hash family if for every subset $S$ of $\{1,\ldots,n\}$ of size $k$, there exists a function in $F$ that is injective when restricted to $S$.

**Theorem.** An $(n,k)$-perfect hash family of size $e^{k+O(\log^2 k) \log n}$ can be constructed in time $e^{k+O(\log^2 k) n \log n}$. 
Perfect Hash Family. A family $F$ of functions $f : \{1,\ldots,n\} \rightarrow \{1,\ldots,k\}$ is an $(n,k)$-perfect hash family if for every subset $S$ of $\{1,\ldots,n\}$ of size $k$, there exists a function in $F$ that is injective when restricted to $S$.

Theorem. An $(n,k)$-perfect hash family of size $e^{k+O(\log^2 k)} \log n$ can be constructed in time $e^{k+O(\log^2 k)} n \log n$.

How to use this theorem?
Perfect Hash Family. A family $F$ of functions $f : \{1,\ldots,n\} \rightarrow \{1,\ldots,k\}$ is an $(n,k)$-perfect hash family if for every subset $S$ of $\{1,\ldots,n\}$ of size $k$, there exists a function in $F$ that is injective when restricted to $S$.

Theorem. An $(n,k)$-perfect hash family of size $e^{k+O(\log^2 k)} \log n$ can be constructed in time $e^{k+O(\log^2 k)} n \log n$.

How to use this theorem? Instead of random colorings, for each function $f$ in $F$, consider one coloring: Vertex number $i$ will be colored by color number $f(i)$. 
**Color Coding: Derandomization**

*Perfect Hash Family.* A family $F$ of functions $f : \{1, \ldots, n\} \rightarrow \{1, \ldots, k\}$ is an $(n,k)$-perfect hash family if for every subset $S$ of $\{1, \ldots, n\}$ of size $k$, there exists a function in $F$ that is injective when restricted to $S$.

*Theorem.* An $(n,k)$-perfect hash family of size $e^{k+O(\log^2 k)} \log n$ can be constructed in time $e^{k+O(\log^2 k)} n \log n$.

**How to use this theorem?** Instead of random colorings, for each function $f$ in $F$, consider one coloring: Vertex number $i$ will be colored by color number $f(i)$.

If there is a solution, then there is an iteration where it will be colorful.