Deriving the CPAB Derivative

Oren Freifeld
The Department of Computer Science, Ben-Gurion University

Abstract

This documents contains the details for deriving the CPAB derivative which we first introduced at [1]. We here also fix a small mistake in Equation (24) from [1]. All the notation used below was defined in [1].

1 Erratum: Fixing a Small Mistake from Equation (24) from [1]

Equation (24) in [1] appeared there as

$$y(x,t) = \int_0^t B_{c,j}\widetilde{\phi^\theta(x,\tau)} + A_{c,\theta}\widetilde{y(x,\tau)}d\tau\,. \tag{1}$$

Which, according to the notation from [1], stands for

$$y(x,t) = \int_0^t B_{c,j}\widetilde{\phi^\theta(x,\tau)} + A_{c,\theta}\begin{pmatrix} y(x,\tau) \\ 1 \end{pmatrix} d\tau\,, \tag{2}$$

since $\widetilde{y(x,\tau)} = \begin{pmatrix} y(x,\tau) \\ 1 \end{pmatrix}$. However, using $\begin{pmatrix} y(x,\tau) \\ 1 \end{pmatrix}$ in that equation was wrong. The correct equation is

$$\boxed{y(x,t) = \int_0^t B_{c,j}\widetilde{\phi^\theta(x,\tau)} + A_{c,\theta}\begin{pmatrix} y(x,\tau) \\ 0 \end{pmatrix} d\tau\,.} \tag{3}$$

Below we provide all the details for the derivation, as well as an explanation for how this mistake occurred.

2 Deriving Equation (24) from [1]

Recall, from the paper, that

$$\phi^\theta(x,t) = x + \int_0^t v^\theta(\phi^\theta(x,\tau))d\tau\,, \tag{4}$$

and define

$$y(x,t) \triangleq \frac{d}{d\theta_j}\phi^\theta(x,t)\,. \tag{5}$$

It follows that

$$y(x,t) = \frac{d}{d\theta_j}\phi^\theta(x,t) = \frac{d}{d\theta_j}\int_0^t v^\theta(\phi^\theta(x,\tau))d\tau \overset{t\neq\text{func}(\theta)}{=} \int_0^t \frac{d}{d\theta_j}v^\theta(\phi^\theta(x,\tau))d\tau\,. \tag{6}$$

Thus:

$$\begin{aligned} y(x,t) &= \int_0^t \frac{d}{d\theta_j}v^\theta(\phi^\theta(x,\tau))d\tau && (7)\\ &= \int_0^t \frac{d}{d\theta_j}\sum_{j'=1}^{d}\theta_{j'}v_{\text{vec}^{-1}(B_{j'})}\widetilde{\phi^\theta(x,\tau)}d\tau && (8)\\ &= \int_0^t \left(\frac{d}{d\theta_j}\sum_{j'=1}^{d}\theta_{j'}v_{\text{vec}^{-1}(B_{j'})}\right)\widetilde{\phi^\theta(x,\tau)} + \sum_{j'=1}^{d}\theta_{j'}v_{\text{vec}^{-1}(B_{j'})}\frac{d}{d\theta_j}\widetilde{\phi^\theta(x,\tau)}d\tau\,. && (9) \end{aligned}$$

Note that

$$\frac{d}{d\theta_j}\sum_{j'=1}^{d}\theta_{j'}v_{\text{vec}^{-1}(B_{j'})} = \frac{d}{d\theta_j}\theta_j v_{\text{vec}^{-1}(B_j)} \overset{B_j\neq\text{func}(\theta)}{=} v_{\text{vec}^{-1}(B_j)} \tag{10}$$

and that

$$\frac{d}{d\theta_j}\widetilde{\phi^\theta(x,\tau)} = \frac{d}{d\theta_j}\begin{pmatrix} \phi^\theta(x,\tau) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{d}{d\theta_j}\phi^\theta(x,\tau) \\ \frac{d}{d\theta_j}1 \end{pmatrix} = \begin{pmatrix} y(x,\tau) \\ 0 \end{pmatrix} \in \mathbb{R}^{n+1}\,. \tag{11}$$

Thus, the equation becomes

$$y(x,t) = \int_0^t v_{\text{vec}^{-1}(B_j)}\widetilde{\phi^\theta(x,\tau)} + \sum_{j'=1}^{d}\theta_{j'}v_{\text{vec}^{-1}(B_{j'})}\begin{pmatrix} y(x,\tau) \\ 0 \end{pmatrix} d\tau \tag{12}$$

$$= \int_0^t B_{c,j}\widetilde{\phi^\theta(x,\tau)} + A_{c,\theta}\begin{pmatrix} y(x,\tau) \\ 0 \end{pmatrix} d\tau \tag{13}$$

and we conclude that

$$y(x,t) = \int_0^t B_{c,j}\widetilde{\phi^\theta(x,\tau)} + A_{c,\theta}\begin{pmatrix} y(x,\tau) \\ 0 \end{pmatrix} d\tau\,. \tag{14}$$

2.1 Explaining the Aforementioned Mistake

As mentioned above, in the original paper we made a small mistake. Here is what went wrong: $\frac{d}{d\theta_j}\widetilde{\phi^\theta(x,\tau)}$ was taken as $\widetilde{y(x,\tau)} = \begin{pmatrix} y(x,\tau) \\ 1 \end{pmatrix}$ instead of $\begin{pmatrix} y(x,\tau) \\ 0 \end{pmatrix}$. Thus, instead of ending up with the **correct equation** above,

$$y(x,t) = \int_0^t B_{c,j}\widetilde{\phi^\theta(x,\tau)} + A_{c,\theta}\begin{pmatrix} y(x,\tau) \\ 0 \end{pmatrix} d\tau, \tag{15}$$

we mistakenly ended up with the **wrong equation**

$$y(x,t) = \int_0^t B_{c,j}\widetilde{\phi^\theta(x,\tau)} + A_{c,\theta}\begin{pmatrix} y(x,\tau) \\ 1 \end{pmatrix} d\tau \tag{16}$$

$$= \int_0^t B_{c,j}\widetilde{\phi^\theta(x,\tau)} + A_{c,\theta}\widetilde{y(x,\tau)}d\tau\,. \tag{17}$$

REFERENCES

[1] O. Freifeld, S. Hauberg, B. Kayhan, and J. W. Fisher III, “Transformations based on continuous piecewise-affine velocity fields,” 2017.