Deriving the CPAB Derivative

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Abstract
This document contains the details for deriving the CPAB derivative which we first introduced at [1]. We here also fix a small mistake in Equation (24) from [1]. All the notation used below was defined in [1].

1 ERRATUM: FIXING A SMALL MISTAKE FROM EQUATION (24) FROM [1]

Equation (24) in [1] appeared there as
\[ y(x, t) = \int_0^t B_{c,j} \phi^\theta(x, \tau) + A_{c,\theta} y(x, \tau) d\tau. \]  

Which, according to the notation from [1], stands for
\[ y(x, t) = \int_0^t B_{c,j} \phi^\theta(x, \tau) + A_{c,\theta} \left( y(x, \tau) \right) d\tau, \]  

since \( y(x, \tau) = \left( y(x, \tau) \right) \). However, using \( y(x, \tau) = \left( y(x, \tau) \right) \) in that equation was wrong. The correct equation is
\[ y(x, t) = \int_0^t B_{c,j} \phi^\theta(x, \tau) + A_{c,\theta} \left( y(x, \tau) \right) d\tau. \]  

Below we provide all the details for the derivation, as well as an explanation for how this mistake occurred.

2 DERIVING EQUATION (24) FROM [1]

Recall, from the paper, that
\[ \phi^\theta(x, t) = x + \int_0^t v^\theta(\phi^\theta(x, \tau)) d\tau, \]  
and define
\[ y(x, t) = \frac{d}{d\theta_j} \phi^\theta(x, t). \]  

It follows that
\[ y(x, t) = \frac{d}{d\theta_j} \phi^\theta(x, t) = \frac{d}{d\theta_j} \int_0^t v^\theta(\phi^\theta(x, \tau)) d\tau \neq \int_0^t \frac{d}{d\theta_j} v^\theta(\phi^\theta(x, \tau)) d\tau. \]  

Thus:
\[ y(x, t) = \int_0^t \frac{d}{d\theta_j} v^\theta(\phi^\theta(x, \tau)) d\tau \]
\[ = \int_0^t \left( \frac{d}{d\theta_j} \sum_{j'=1}^d \theta_{j'} v_{vec^{-1}(B_j)} \phi^\theta(x, \tau) \right) d\tau \]
\[ = \int_0^t \left( \sum_{j'=1}^d \theta_{j'} v_{vec^{-1}(B_j)} \frac{d}{d\theta_j} \phi^\theta(x, \tau) \right) + \sum_{j'=1}^d \theta_{j'} v_{vec^{-1}(B_j)} \frac{d}{d\theta_j} \phi^\theta(x, \tau) d\tau. \]  

Note that
\[ \frac{d}{d\theta_j} \sum_{j'=1}^d \theta_{j'} v_{vec^{-1}(B_j)} = \frac{d}{d\theta_j} \theta_{j'} v_{vec^{-1}(B_j)} = v_{vec^{-1}(B_j)} \]  

\[ \frac{d}{d\theta_j} \theta_{j'} v_{vec^{-1}(B_j)} \]  

and that
\[
\frac{d}{d\theta} \phi^\theta(x, \tau) = \frac{d}{d\theta} \left( \begin{array}{c} \phi^\theta(x, \tau) \\ 1 \end{array} \right) = \left( \begin{array}{c} \frac{d}{d\theta} \phi^\theta(x, \tau) \\ \frac{d}{d\theta} 1 \end{array} \right) = \left( \begin{array}{c} y(x, \tau) \\ 0 \end{array} \right) \in \mathbb{R}^{n+1}.
\] (11)

Thus, the equation becomes
\[
y(x, t) = \int_0^t v_{\text{vec}^{-1}(B_j)} \phi^\theta(x, \tau) + \sum_{j'=1}^d \theta_{j'} v_{\text{vec}^{-1}(B_{j'})} \left( \begin{array}{c} y(x, \tau) \\ 0 \end{array} \right) d\tau
\] (12)
\[
= \int_0^t B_{c,j} \phi^\theta(x, \tau) + A_{c,\theta} \left( \begin{array}{c} y(x, \tau) \\ 0 \end{array} \right) d\tau
\] (13)

and we conclude that
\[
y(x, t) = \int_0^t B_{c,j} \phi^\theta(x, \tau) + A_{c,\theta} \left( \begin{array}{c} y(x, \tau) \\ 0 \end{array} \right) d\tau.
\] (14)

2.1 Explaining the Aforementioned Mistake

As mentioned above, in the original paper we made a small mistake. Here is what went wrong: \( \frac{d}{d\theta} \phi^\theta(x, \tau) \) was taken as \( \widehat{y(x, \tau)} = \left( \begin{array}{c} y(x, \tau) \\ 1 \end{array} \right) \) instead of \( \left( \begin{array}{c} y(x, \tau) \\ 0 \end{array} \right) \). Thus, instead of ending up with the correct equation above, we mistakenly ended up with the wrong equation
\[
y(x, t) = \int_0^t B_{c,j} \phi^\theta(x, \tau) + A_{c,\theta} \left( \begin{array}{c} y(x, \tau) \\ 0 \end{array} \right) d\tau,
\] (15)

we mistakenly ended up with the wrong equation
\[
y(x, t) = \int_0^t B_{c,j} \phi^\theta(x, \tau) + A_{c,\theta} \left( \begin{array}{c} y(x, \tau) \\ 1 \end{array} \right) d\tau
\] (16)
\[
= \int_0^t B_{c,j} \phi^\theta(x, \tau) + A_{c,\theta} y(x, \tau) d\tau.
\] (17)

REFERENCES