1) Finite Difference

Solve the Model Problem \( \frac{dy}{dx} = x + y; \ y(0) = 1 \) using Euler’s Method (EM) with \( h = 0.1 \) and the 2\textsuperscript{nd} order Runge-Kutta (RK2) with \( \lambda = 2/3, h = 0.1 \).

(a) Compare the solution with the exact solution \( y = 2e^x - x - 1 \) at \( x \) values between 0 and 1, with a step-size of 0.1 (i.e., \( x=0; x=0.1; x=0.2; \ldots; x=0.8; x=0.9; x=1.0 \)).
(b) Compare between RK2 and EM.
(c) Count the number of times \( f(x,y) \) was evaluated in both methods.
(d) How many time \( f(x,y) \) is called with Euler’s method with \( h = 0.05 \).
(e) Compare both methods for accuracy and efficiency: to be fair to both methods, adjust the value of \( h \) in EM so that both EM and RK2 use the same number of function evaluations. Now, compare the accuracy between the two methods.

2) Finite Element

Solve the Model Problem \( \frac{d^2y}{dx^2} - \left(1 - \frac{x}{5}\right)y = x \ ; \ y(1) = 2 \ ; \ y(3) = -1 \) using the Rayleigh-Ritz method. Approximate \( y(x) \) with \( u(x) = \sum_{i=1}^{N} c_i N_i(x) \) where \( N_i(x) \) are the triangular basis functions defined in class. Using the variational method find the coefficients \( C_1, C_2, C_3, C_4, C_5 \) (the first and last are given as the boundary conditions).

3) Numerical Differentiation

(a) Determine as accurately as possible approximations for each missing entry in the following table. (hint: employ the formulas for numerical differentiation)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3.0</td>
<td>9.3678</td>
<td></td>
</tr>
<tr>
<td>-2.8</td>
<td>8.2332</td>
<td></td>
</tr>
<tr>
<td>-2.6</td>
<td>7.1803</td>
<td></td>
</tr>
<tr>
<td>-2.4</td>
<td>6.2093</td>
<td></td>
</tr>
<tr>
<td>-2.2</td>
<td>5.3203</td>
<td></td>
</tr>
<tr>
<td>-2.0</td>
<td>4.5134</td>
<td></td>
</tr>
</tbody>
</table>

(b) Given that \( f(x) = e^{x/3} + x^2 \) find the bound for the error in each case.
4) **Numerical integration**

(a) Use the Composite Trapezoidal and Composite Simpson’s rules to approximate the following integral (the exact value is 19.9209…):

\[
\int_{-2}^{2} x^3 e^x \, dx \quad n = 4
\]

(b) Determine the values of \( n \) and \( h \) required to approximate:

\[
\int_{0}^{2} \frac{1}{x + 4} \, dx \quad \text{within } 10^{-5}
\]

*Good luck.*