Curve fitting – 7. חתך 'א'

**Approximation via curve fitting**

**Problem:**

The measured data is given by the set of points $\{(x_i, y_i)\} = \{(2, 4), (3, 4.5), (6, 6), (8, 7)\}$.

The function to be fitted is $f_p(x) = \beta_1 x + \beta_2 + \beta_3 \sin(x)$.

We aim to find the parameters $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$ that minimize the sum of squared errors $S(\beta)$

$$S(\beta) = \sum_{i=1}^{n} (y_i - f_p(x_i))^2$$

**Solution:**

We need to find the parameter values that minimize $S(\beta)$, i.e.,

$$\beta = \arg \min_{\beta} S(\beta)$$

We have

$$\nabla S(\beta) = 0$$

And the gradient is given by

$$\frac{\partial S}{\partial \beta_1} = 0 \rightarrow \hat{\beta}_1 \Sigma_{i=1}^{4} x_i \Sigma_{i=1}^{4} x_i^2 + \hat{\beta}_2 \Sigma_{i=1}^{4} x_i + \hat{\beta}_3 \Sigma_{i=1}^{4} \sin(x_i) = \Sigma_{i=1}^{4} y_i x_i$$

$$\frac{\partial S}{\partial \beta_2} = 0 \rightarrow \hat{\beta}_2 \Sigma_{i=1}^{4} x_i + \hat{\beta}_3 \Sigma_{i=1}^{4} \sin(x_i) = \Sigma_{i=1}^{4} y_i$$

$$\frac{\partial S}{\partial \beta_3} = 0 \rightarrow \hat{\beta}_3 \Sigma_{i=1}^{4} \sin(x_i) = \Sigma_{i=1}^{4} y_i$$

These expressions are similar to the least squares method. The iterated function is:

$$\hat{\beta}_1 \Sigma_{i=1}^{4} x_i \Sigma_{i=1}^{4} x_i^2 + \hat{\beta}_2 \Sigma_{i=1}^{4} x_i + \hat{\beta}_3 \Sigma_{i=1}^{4} \sin(x_i) = \Sigma_{i=1}^{4} y_i x_i$$

To find the specific function, we solve the above equation.
$$\frac{\partial S}{\partial \beta_3} = 0 \rightarrow$$

III. \[ \beta_1 \sum_{i=1}^{4} x_i \sin(x_i) + \beta_2 \sum_{i=1}^{4} \sin(x_i) + \beta_3 \sum_{i=1}^{4} \sin(x_i)^2 = \sum_{i=1}^{4} y_i \sin(x_i) \]

**Theorem:** For a given set of data, the normal equations are:

\[ X^T X \hat{\beta} = X^T y \] : Normal Equations

\[ X_{ij} = f_j(x_i) \text{ and } f_\beta(x) = \sum_{j=1}^{m} \beta_3 f_j(x) \]

Column by column solved for each column \( i \) in \( X \) and \( t \) gives \( X \) and \( t \) which satisfy the normal equations. The solution is:

\[
\begin{align*}
\sum_{i=1}^{4} x_i^2 &= 2^2 + 3^2 + 6^2 + 8^2 = 113 \\
\sum_{i=1}^{4} x_i &= 2 + 3 + 6 + 8 = 19 \\
\sum_{i=1}^{4} x_i \sin(x_i) &= 8.480327862 \\
\sum_{i=1}^{4} y_i x_i &= 113.5 \\
\sum_{i=1}^{4} \sin(x_i) &= 1.76036183 \\
\sum_{i=1}^{4} y_i &= 21.5 \\
\sum_{i=1}^{4} \sin(x_i)^2 &= 1.903639428 \\
\sum_{i=1}^{4} y_i \sin(x_i) &= 9.5212 \\
\end{align*}
\]

\[
\begin{pmatrix}
113 \beta_1 + 19 \beta_2 + 8.48 \beta_3 = 113.5 \\
19 \beta_1 + 4 \beta_2 + 1.76 \beta_3 = 21.5 \\
8.48 \beta_1 + 1.76 \beta_2 + 1.9 \beta_3 = 9.52
\end{pmatrix}
\]

After the matrix inversion and subsequent manipulation of the above:

\[
\begin{pmatrix}
113 & 19.84 & 113.5 \\
19 & 1.76 & 21.5 \\
8.48 & 1.76 & 9.52
\end{pmatrix}
\sim \begin{pmatrix}
1 & 0 & 0.5 \\
0 & 1 & 3 \\
0 & 0.01 & 0
\end{pmatrix}
\]

Reduced matrix solution:

\[
\begin{pmatrix}
113 & 19.84 & 113.5 \\
19 & 1.76 & 21.5 \\
8.48 & 1.76 & 9.52
\end{pmatrix}
\sim \begin{pmatrix}
1 & 0.01 & 3 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{pmatrix}
\]

Column of solutions: \( f_\beta(x) = \beta_1 x + \beta_2 + \beta_3 \sin(x) = \frac{1}{2} x + 3 \)

(Note: Linear columns of the matrix reached the minimum at the coordinate vector.)