Assignment 5

1. Show that the finite difference scheme for solving a second-order ODE is:

   \[ I_{\text{num}} = \frac{h^2}{12} \left( 2f(x_0) + 5f(x_1) + 2f(x_2) \right) \]

   You can use the error analysis of the Simpson's rule to justify your result.

2. Given: \( g(x) = e^{2x} \)

   a. Find the second derivative \( g''(x) \) and \( O(h^n) \) map for \( g''(x) \).
   b. Show that the coefficients of the Taylor expansion of \( g''(x) \) are:
   c. Use the coefficients from part (b) to define the integration method for \( g''(x) \).

3. Solve the Model Problem

   \[ \frac{dy}{dx} = x + y; \quad y(0) = 1 \]

   using Euler's Method (EM) with \( h=0.1 \) and the 2nd order Runge-Kutta (RK2) with \( \lambda=2/3, h=0.1 \).

   a. Compare the solution with the exact solution \( y = 2e^x - x - 1 \) at \( x \) values between 0 and 1, with a step-size of 0.1 (i.e., \( x=0; x=0.1; x=0.2; \ldots; x=0.8 \), \( x=0.9; x=1.0 \)).
   b. Compare between RK2 and EM.
   c. Count the number of times \( f(x,y) \) was evaluated in both methods.
   d. How many time \( f(x,y) \) is called with Euler's method at \( h=0.05 \).
   e. Compare both methods for accuracy and efficiency: to be fair to both methods, adjust the value of \( h \) in EM so that both EM and RK2 use the same number of function evaluations. Now, compare the accuracy between the two methods.
Solve the Model Problem \( \frac{d^2 y}{dx^2} - (1 - \frac{x}{S}) y = x \); \( y(1) = 2 \); \( y(3) = -1 \) using the Rayleigh-Ritz method. Approximate \( y(x) \) with \( u(x) = \sum_{i=4}^{5} c_i N_i(x) \) where \( N_i(x) \) are the triangular basis functions defined in class. Using the variational method find the coefficients \( C_1, C_2, C_3, C_4, C_5 \) (the first and last are given as the boundary conditions).