Numerical Analysis: Interpolation – Part 2

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(slides based mostly on Prof. Ben-Shahar’s notes)

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Optimal Placement of the Roots
We saw

\[ E_n(x) \triangleq f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} \prod_{i=0}^{n} (x - x_i) \]

In the product, \( \prod_{i=0}^{n} (x - x_i) \), both \( n \) and the locations of the nodes play a role. Thus, for a fixed \( n \), a smart choice of the locations can greatly influence the maximal error.

If we define

\[ Q(x) = \prod_{i=0}^{n} (x - x_i) \]

then, for a fixed \( n \), the optimal choice of \( (x_i)_{i=0}^{n} \) is the one that will minimize the maximum of \( |Q(x)| \):

\[ \max_{x \in [a,b]} |Q(x)| \Rightarrow \text{minimum} \]
So here is the plan.

We will build a certain polynomial, of degree $n$, and will get an analytic expression for its roots.

As it will turn out, these roots will minimize $\max_{x \in [a,b]} |Q(x)|$.

This method is due to Chebysehv.

The derivation in the next slides is based on Burden and Faires’ textbook.
We will start by assuming our interval is $[-1, 1]$. This will be generalized later.

Define, for any nonnegative integer $n$,

$$T_n : [-1, 1] \rightarrow \mathbb{R} \quad T_n : x \mapsto \cos(n \arccos(x)) \quad n \geq 0$$

It is not obvious, but $T_n(x)$ is in fact a polynomial in $x$, as we now show.
Optimal Placement of the Roots

\[ T_n(x) = \cos(n \arccos(x)) \]

First, \[ T_0(x) = \cos(0 \arccos(x)) = \cos(0) = 1 \].
Thus, \( T_0(x) \) is a zeroth-order polynomial.

Second, \[ T_1(x) = \cos(1 \arccos(x)) = \cos(\arccos(x)) = x \].
Thus, \( T_1(x) \) is a first-order polynomial.

So only need to show \( T_n \) is a polynomial for the case \( n > 1 \).
Optimal Placement of the Roots

\[ T_n(x) = \cos(n \arccos(x)) \]

- Let
  \[ \theta \triangleq \arccos(x) \]
- Thus,
  \[ \cos(\theta) = x \]
- The expression becomes
  \[ \hat{T}_n(\theta) \triangleq T_n(\cos(\theta)) = \cos(n \arccos(\cos(\theta))) = \cos(n\theta) \quad \theta \in [0, \pi] \]
- Observe:
  \[
  \hat{T}_{n+1}(\theta) = \cos((n + 1)\theta) = \cos(\theta) \cos(n\theta) - \sin(\theta) \sin(n\theta) \\
  \hat{T}_{n-1}(\theta) = \cos((n - 1)\theta) = \cos(\theta) \cos(n\theta) + \sin(\theta) \sin(n\theta)
  \]
  Adding
  \[ \Rightarrow \hat{T}_{n+1}(\theta) = 2 \cos(\theta) \cos(n\theta) - \hat{T}_{n-1}(\theta) \]
  or
  \[ T_{n+1}(x) = 2x \cos(n \arccos(x)) - T_{n-1}(x) \]
  \[ \Rightarrow T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) \]
Just saw:

\[
\begin{align*}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \quad n > 1
\end{align*}
\]

It follows that \( T_n(x) \) is a polynomial of order \( n \).

These polynomials are called **Chebysehv polynomials**.
The recurrence also implies that $T_n(x)$ is a polynomial of degree $n$ whose leading coefficient is $2^{n-1}$. 
Roots of $T_n(x)$

**Theorem**

The Chebyshev polynomial $T_n(x)$ of degree $n \geq 1$ has $n$ simple roots in $[1, 1]$ at

$$\bar{x}_k = \cos \left( \frac{2k - 1}{2n} \pi \right) \quad \text{for each } k = 1, 2, \ldots, n.$$
Proof.

Let
\[ \bar{x}_k = \cos \left( \frac{2k - 1}{2n} \pi \right) \]
for each \( k = 1, 2, \ldots, n \).

Then,
\[
T_n(\bar{x}_k) = \cos \left( n \arccos(\bar{x}_k) \right) = \cos \left( n \arccos \left( \cos \left( \frac{2k - 1}{2n} \pi \right) \right) \right)
\]
\[ = \cos \left( n \frac{2k - 1}{2n} \pi \right) = \cos \left( \frac{2k - 1}{2} \pi \right) = 0 \]
(since we get \( \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \))

So we got \( n \) such roots. Since these roots are distinct, and \( T_n \) is of degree \( n \), these are all its roots.
Optimal Placement of the Roots

Extreme Points of $T_n(x)$

Theorem

The Chebyshev polynomial $T_n(x)$ of degree $n \geq 1$ assumes its absolute extrema at

$$\bar{x}_k' = \cos \left( \frac{k\pi}{n} \right) \quad \text{with} \quad T_n(\bar{x}_k') = (-1)^k \quad \text{for each} \quad k = 0, 1, \ldots, n.$$
Proof.

\[
\frac{d}{dx} T_n(x) = \frac{d}{dx} \left( \cos(n \arccos(x)) \right) = -\sin(n \arccos(x)) \frac{d}{dx} \left( n \arccos(x) \right)
\]

\[
= -n \sin(n \arccos(x)) \frac{d}{dx} \left( \arccos(x) \right)
\]

\[
= -n \sin(n \arccos(x)) \left( -\frac{1}{\sqrt{1-x^2}} \right) = \frac{n \sin(n \arccos(x))}{\sqrt{1-x^2}}
\]

For \( k = 1, 2, \ldots, n - 1 \):

\[
\frac{d}{dx} T_n(\bar{x}'_k) = \frac{n \sin(n \arccos(\cos(k\pi/n)))}{\sqrt{1-\left( \cos\left(\frac{k\pi}{n}\right) \right)^2}} = \frac{n \sin(k\pi)}{\sin\left(\frac{k\pi}{n}\right)} = 0
\]

\( T_n \) is a polynomial of degree \( n \) ⇒ \( \frac{d}{dx} T_n \) is a polynomial of degree \( n - 1 \). And we just found \( n - 1 \) distinct roots of the latter, so it has no others. The only other extrema of \( T_n \) can occur at the endpoints, \{±1\}. And these coincide with \( \bar{x}'_0 = -1 \) and \( \bar{x}'_n = 1 \). Finally, for \( k = 0, 1, \ldots, n \),

\[
T_n(\bar{x}_k) = \cos \left( n \arccos \left( \cos \left( \frac{k\pi}{n} \right) \right) \right) = \cos(k\pi) = (-1)^k.
\]
The Monic Chebysehv Polynomials

The Monic (polynomials whose leading coefficient is 1) Polynomials, denoted by $\tilde{T}_n$, derived from the Chebysehv Polynomials, $T_n$, by dividing the latter by $2^{n-1}$:

$$\tilde{T}_0 \equiv 1 \quad \text{and} \quad \tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x) \quad \text{for each } n \geq 1.$$ 

It follows that

$$\tilde{T}_2(x) = x\tilde{T}_1(x) - \frac{1}{2} \tilde{T}_0(x) \quad \text{and}$$

$$\tilde{T}_{n+1}(x) = x\tilde{T}_n(x) - \frac{1}{4} \tilde{T}_{n-1}(x) \quad \text{for each } n \geq 2.$$
Since $\tilde{T}_n$ is just a multiple of $T_n$, the zeros of $\tilde{T}_n$ (for $n \geq 1$, occur at

$$\bar{x}_k = \cos \left( \frac{2k - 1}{2n} \pi \right) \quad \text{for each } k = 1, 2, \ldots, n,$$

and its extreme values, for $n \geq 1$, occur at

$$\bar{x}'_k = \cos \left( \frac{k}{n} \pi \right) \quad \text{for each } k = 1, 2, \ldots, n,$$

with

$$\tilde{T}_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}}.$$
Let $\tilde{\Pi}_n$ denote the set of all monic polynomials of degree $n$ on $[-1, 1]$.

As we will see, the fact that the extreme values of Chebysehv’s monic polynomials, $\tilde{T}_n$ (for $n \geq 1$), are

$$\pm \frac{1}{2^{n-1}},$$

will lead to an important minimization property that sets Chebysehv’s monic polynomials apart from other members of $\tilde{\Pi}_n$. 
Theorem

The polynomials of the form $\tilde{T}_n$, when $n \geq 1$, satisfy the following property:

$$\frac{1}{2^{n-1}} = \max_{x \in [-1,1]} |\tilde{T}_n(x)| \leq \max_{x \in [-1,1]} |P_n(x)|, \quad \forall P_n \in \tilde{\Pi}_n.$$

Moreover, equality holds only if $P_n \equiv \tilde{T}_n$. 
**Proof.**

Let $P_n \in \tilde{\Pi}_n$. Suppose that

$$\max_{x \in [-1,1]} |P_n(x)| \leq \frac{1}{2^{n-1}} = \max_{x \in [-1,1]} |\tilde{T}_n(x)|.$$

Let $Q = \tilde{T}_n - P_n$. Since $\tilde{T}_n$ and $P_n$ are monic polynomials, $Q$ is a polynomial of degree $\leq n - 1$. At, $\bar{x}'_k$, the extreme points of $\tilde{T}_n$, we have

$$Q(\bar{x}'_k) = \tilde{T}_n(\bar{x}'_k) - P_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}} - P_n(\bar{x}'_k).$$

However, by assumption,

$$|P_n(\bar{x}'_k)| \leq \frac{1}{2^{n-1}} \quad \text{for each } k = 0, 1, \ldots, n.$$

So $Q(\bar{x}'_k) \leq 0$ when $k$ is odd $Q(\bar{x}'_k) \geq 0$ when $k$ is even. By continuity of $Q$, for each $j = 0, 1, \ldots, n - 1$, $Q$ has at least one root between $\bar{x}'_j$ and $\bar{x}'_{j+1} \Rightarrow Q$ has at least $n$ roots in $[-1, 1]$. It follows that $Q \equiv 0$. \qed
Recall the quantity of interest is \( |Q(x)| = \prod_{i=0}^{n} |x - x_i| \)

\( Q(x) = \prod_{i=0}^{n} (x - x_i) \) is a monic polynomial of order \( n + 1 \).

Based on what we just saw, the maximal value of \( |Q(x)| \) is smallest when the \((x_k)_{k=0}^{n}\) are chosen to be roots of the Chebysehv polynomial, \( \tilde{T}_{n+1}(x) \); i.e., when

\[
\prod_{i=0}^{n} |x - x_i| = \tilde{T}_{n+1}(x)
\]

Hence we choose

\[
\bar{x}_{k+1} \cos \left( \frac{2k + 1}{2n} \pi \right) \quad \text{for each } k = 0, 1, \ldots, n.
\]
We also know that, since \( \max_{x \in [-1,1]} \tilde{T}_{n+1}(x) = 2^{-n} \),

\[
2^{-n} = \max_{x \in [-1,1]} \prod_{i=1}^{n+1} |x - \bar{x}_i| \leq \max_{x \in [-1,1]} \prod_{i=0}^{n} |x - x_i|
\]

for any choice of \( (x_i)_{i=0}^{n} \subset [-1, 1] \).
Corollary

Let $P_n(x)$ be the interpolation polynomial of degree $\leq n$ with nodes at the roots of $T_{n+1}$. Then

$$\max_{x \in [-1,1]} |f(x) - P_n(x)| \leq \frac{1}{2^n(n + 1)!} \max_{x \in [-1,1]} |f^{(n+1)}(x)|$$

for every $f \in C^{(n+1)}([-1, 1])$. 
Generalizing from $[-1,1]$ to $[a, b]$

Set

$$x \mapsto \frac{1}{2}[(b-a)x + a + b]$$

to transform the numbers $\bar{x}_k$ from $[-1,1]$ to numbers in $[a,b]$. 
Version Log

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