Numerical Analysis: Basics of Error Calculus

Computer Science, Ben-Gurion University

(slides based mostly on Prof. Ben-Shahar’s notes)

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1 Basic Definitions

2 Error Propagation
Absolute Error

Definition (absolute error)

The **absolute error** (in approximating $x$ by $\tilde{x}$) is

$$\Delta \tilde{x} \triangleq |x - \tilde{x}|.$$  \hfill (1)

Remark

Some other commonly-used symbols for the absolute error: $\varepsilon; e; E$. 

Relative Error

Definition (relative error)

Assuming $x \neq 0$, the relative error is

$$
\hat{\delta}x \triangleq \frac{\Delta \tilde{x}}{|x|} = \frac{|x - \tilde{x}|}{|x|}.
$$

(2)

It may also be represented in terms of percentage, i.e., $(\hat{\delta}x \cdot 100)\%$.

Remark

Some other commonly-used symbols for the relative error: $\rho$, $R$. 
Basic Definitions

Absolute and Relative Errors

Example (similar to Example 0.7 from Gerald & Wheatly)

Let $x = \frac{1}{3}$. Calculate the absolute and relative errors for the following approximation: $\tilde{x} = 0.333$. Solution:

$$\Delta x = \left| \frac{1}{3} - 0.333 \right| = \frac{1}{3} - \frac{333}{1000} = \frac{1000 - 999}{3000} = \frac{1}{3000}. \quad (3)$$

$$\delta \tilde{x} = \frac{1}{3000} \cdot \frac{1}{1/3} = 0.001 = 0.1\%. \quad (4)$$
Relative Sensitivity, AKA Condition Number

Definition (relative sensitivity; condition number)
If \( x \) and \( \tilde{x} \) are associated with the input, and \( y \) and \( \tilde{y} \) are associated with the output, then the \textbf{relative sensitivity} of the method to input errors is measured by

\[
C_\rho \triangleq \frac{\delta\tilde{y}}{\delta\tilde{x}} = \frac{\Delta\tilde{y}/|y|}{\Delta\tilde{x}/|x|}.
\]  

(5)
This number is also called the \textbf{condition number}.

We want condition numbers to be small.

Definition (ill-conditioned problems)
\textbf{Ill-conditioned problems} are problems with \( C_\rho \gg 1 \).
Definition ($d$ significant figures/digits)

The real number $\tilde{x}$ approximates $x$ within $d$ significant figures (AKA significant digits) if $d$ is the largest non-negative integer such that

$$\delta \tilde{x} \overset{\text{by def.}}{=} \frac{|x - \tilde{x}|}{|x|} \leq \frac{10^{1-d}}{2}.$$  

(6)
Example (significant figures in base 10)

\[ x = \pi / 1000 = 0.003141592 \ldots \]

\[ \tilde{x} = 0.0031414 \]

\[ \Delta \tilde{x} = |x - \tilde{x}| = 0.00000019265 \ldots \]

\[ \delta \tilde{x} = \frac{\Delta \tilde{x}}{|x|} = \frac{0.00000019265 \ldots}{0.003141592 \ldots} = 0.0000613224 \ldots . \]

<table>
<thead>
<tr>
<th>d</th>
<th>$10^{1-d}/2$</th>
<th>$\delta \tilde{x} \leq \frac{10^{1-d}}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>yes</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>yes</td>
</tr>
<tr>
<td>3</td>
<td>0.005</td>
<td>yes</td>
</tr>
<tr>
<td>4</td>
<td>0.0005</td>
<td>yes</td>
</tr>
<tr>
<td>5</td>
<td>0.00005</td>
<td>no</td>
</tr>
</tbody>
</table>

Thus the approximation is within 4 significant digits: \( \tilde{x} = 0.0031414 \).
Note leading zeros are ignored.
Example (significant figures in base 10)

\[ x = \pi / 1000 = 0.003141592 \ldots \]

\[ \tilde{x} = 0.0031416 \quad \text{(last digit is now 6)} \]

\[ \Delta \tilde{x} = |x - \tilde{x}| = 0.000000007346 \ldots \]

\[ \delta \tilde{x} = \frac{\Delta \tilde{x}}{|x|} = \frac{0.000000007346 \ldots}{0.003141592 \ldots} = 0.0000023384349 \ldots . \]

Now, unlike before, \( \delta \tilde{x} \leq \frac{10^{1-5}}{2} = 0.00005 \) \ldots Should we stop here? (e.g., \( \delta \tilde{x} \leq \frac{10^{1-6}}{2} = 0.000005 \) is also true). There is ambiguity here: Since 

\[ 0.0031416 = 0.0031416000000 \ldots \]

the answer depends on whether the trailing zeros are just placeholders to indicate the scale of the number or actual measurements that happen to be zeros. If it is the former, then the approximation is within **5 significant digits** since \( \tilde{x} \) has only 5 figures (leading zeros, as usual, are ignored):

\[ 0.0031416. \]
Significant Figures

Remark

The definition above assumes we work in base 10. More generally, to find significant figures in base $B$, we will work with

$$
\delta \tilde{x} \quad \text{by def.} \quad \frac{|x - \tilde{x}|}{|x|} \leq \frac{B^{1-d}}{2}
$$

(7)

instead. If $B$ is not explicitly specified, we will usually assume $B = 10$. 

Error Propagation

So far we saw absolute error, relative error, and a measure for sensitivity of a computation to errors in the input. We will now consider the way in which error is propagated through simple computations: addition, subtraction, and multiplication.

Let \( x, y \in \mathbb{R} \) denote the true values. Let \( \tilde{x} \) and \( \tilde{y} \) denote their approximations, with absolute errors \( \Delta \tilde{x} = \varepsilon_x \) and \( \Delta \tilde{y} = \varepsilon_y \). In other words:

\[
x = \tilde{x} \pm \varepsilon_x, \quad y = \tilde{y} \pm \varepsilon_y.
\]
Remark

Earlier we mentioned that $\Delta \tilde{x}$ is a function of $\tilde{x}$ (and $x$ of course). To emphasize this, we could also write $\Delta(\tilde{x})$ instead of (the shorter) $\Delta \tilde{x}$. We will usually avoid the $\Delta(\tilde{x})$ notation, but this kind of notation is useful, e.g., when using the symbol $\Delta(\tilde{x} + \tilde{y})$ to denote the absolute error of $\tilde{x} + \tilde{y}$ in approximating $x + y$. 
Addition

\[ x + y = (\tilde{x} \pm \varepsilon_x) + (\tilde{y} \pm \varepsilon_y) = (\tilde{x} + \tilde{y}) \pm (\varepsilon_x \pm \varepsilon_y). \]

Assuming the worst case (i.e., if \( x - \tilde{x} \) and \( y - \tilde{y} \) have the same sign), we get

\[ \Delta(\tilde{x} + \tilde{y}) = \Delta\tilde{x} + \Delta\tilde{y}. \]
Addition

Another way to view this is through the use of intervals:

$$x \in [\tilde{x} - \varepsilon_x, \tilde{x} + \varepsilon_x]$$
$$y \in [\tilde{y} - \varepsilon_y, \tilde{y} + \varepsilon_y]$$

$$\Rightarrow x + y \in [(\tilde{x} + \tilde{y}) - (\varepsilon_x + \varepsilon_y), (\tilde{x} + \tilde{y}) + (\varepsilon_x + \varepsilon_y)].$$

More compactly, this is written in terms of interval arithmetics\(^1\):

$$[\tilde{x} - \varepsilon_x, \tilde{x} + \varepsilon_x] + [\tilde{y} - \varepsilon_y, \tilde{y} + \varepsilon_y] =$$
$$[(\tilde{x} + \tilde{y}) - (\varepsilon_x + \varepsilon_y), (\tilde{x} + \tilde{y}) + (\varepsilon_x + \varepsilon_y)]$$

\(^1\)See, e.g., [Gerald & Wheatly], Chapter 0.
Example

\[ \tilde{x} = 0.65 \quad \varepsilon_x = 0.15 \quad \tilde{y} = -0.55 \quad \varepsilon_y = 0.65 \]

\[ \Rightarrow x + y \in [0.5, 0.8] + [-1.2, 0.1] = [-0.7, 0.9]. \]

Notice that the width of the sum is the sum of the widths of the intervals being added.
Subtraction

As we have no control on the sign of the error, the exact same analysis hold for subtraction:

\[ \Delta(\tilde{x} - \tilde{y}) = \Delta\tilde{x} + \Delta\tilde{y}. \]

Likewise:

\[ x \in [\tilde{x} - \varepsilon_x, \tilde{x} + \varepsilon_x] \]
\[ y \in [\tilde{y} - \varepsilon_y, \tilde{y} + \varepsilon_y] \]
\[ \Rightarrow x - y \in [(\tilde{x} - \tilde{y}) - (\varepsilon_x + \varepsilon_y), (\tilde{x} - \tilde{y}) + (\varepsilon_x + \varepsilon_y)] \]

and

\[ [\tilde{x} - \varepsilon_x, \tilde{x} + \varepsilon_x] - [\tilde{y} - \varepsilon_y, \tilde{y} + \varepsilon_y] = \\
[ (\tilde{x} - \tilde{y}) - (\varepsilon_x + \varepsilon_y), (\tilde{x} - \tilde{y}) + (\varepsilon_x + \varepsilon_y) ]. \]
Example

\[ \tilde{x} = 0.65 \quad \varepsilon_x = 0.15 \quad \tilde{y} = -0.55 \quad \varepsilon_y = 0.65 \]

\[ \Rightarrow x - y \in [0.5, 0.8] - [-1.2, 0.1] = [0.5, 0.8] + [-0.1, 1.2] = [0.4, 2.0] \]

Again, notice that the resulting width is the sum of the widths of the intervals (even though the operation is subtraction).
Multiplication

With multiplication, analyzing the **absolute error** is slightly more involved.

\[
x y = (\tilde{x} \pm \varepsilon_x)(\tilde{y} \pm \varepsilon_y) = \tilde{x}\tilde{y} + (\tilde{x}\varepsilon_y \pm \tilde{y}\varepsilon_x \pm \varepsilon_x\varepsilon_y)
\]

So in the worst-case scenario,

\[
\Delta(\tilde{x}\tilde{y}) = |\tilde{x}\varepsilon_y| + |\tilde{y}\varepsilon_x| + \varepsilon_x\varepsilon_y = |\tilde{x}|\varepsilon_y + |\tilde{y}|\varepsilon_x + \varepsilon_x\varepsilon_y.
\]

Thus, if \( |\tilde{x}| > 1 \) or \( |\tilde{y}| > 1 \), it is possible the error will be magnified.
We can learn more by inspecting the relative error that follows from the expression above. First, let $\rho_x = \delta \tilde{x} = \frac{\varepsilon_x}{|x|}$ and $\rho_y = \delta \tilde{y} = \frac{\varepsilon_y}{|y|}$. Now:

$$
\left| \frac{xy - \tilde{x}\tilde{y}}{xy} \right| = \left| \frac{\tilde{x}\varepsilon_y \pm \tilde{y}\varepsilon_x \pm \varepsilon_x \varepsilon_y}{xy} \right| = \left| \frac{\tilde{x}\varepsilon_y}{xy} \pm \frac{\tilde{y}\varepsilon_x}{yx} \pm \frac{\varepsilon_x \varepsilon_y}{xy} \right|
$$

$$
= \left| \pm \frac{\tilde{x}}{x} \rho_y \pm \frac{\tilde{y}}{y} \rho_x \pm \rho_y \rho_x \right|.
$$

So, adopting a pessimistic view (i.e., all errors have the same sign so nothing cancels out):

$$
\delta(\tilde{x}\tilde{y}) = \left| \frac{\tilde{x}}{x} \right| \rho_y + \left| \frac{\tilde{y}}{y} \right| \rho_x + \rho_y \rho_x.
$$
$$\delta(\tilde{x}\tilde{y}) = \left|\frac{\tilde{x}}{x}\right| \rho_y + \left|\frac{\tilde{y}}{y}\right| \rho_x + \rho_y\rho_x,$$

If we now assume that the approximations, $\tilde{x}$ and $\tilde{y}$, are sufficiently good (so the errors are small), it means that $\tilde{x}/x \approx 1$ and $\tilde{y}/y \approx 1$. Moreover:

$$\rho_y\rho_x \approx 0,$$

and so

$$\delta(\tilde{x}\tilde{y}) \approx \delta\tilde{x} + \delta\tilde{y}.$$

In words: in multiplication, assuming small errors, the relative errors add up (while in addition or subtraction, regardless if the errors were small or not, it were the absolute errors who summed up).
The above result for relative errors can be derived more easily when we express the relative error as follows. Recall $\rho_x = \left| \frac{\tilde{x} - x}{x} \right|.$

Case I: If $\rho_x = \frac{\tilde{x} - x}{x}$ then

\[
x \rho_x = \tilde{x} - x
\]

\[
x + x \rho_x = \tilde{x}
\]

\[
x(1 + \rho_x) = \tilde{x}.
\]

Case II: If $\rho_x = \frac{x - \tilde{x}}{x}$ then

\[
x \rho_x = x - \tilde{x}
\]

\[
\tilde{x} = x - x \rho_x
\]

\[
\tilde{x} = x(1 - \rho_x).
\]

Taken both cases together:

\[
\tilde{x} = x(1 \pm \rho_x).
\]

Likewise:

\[
\tilde{y} = y(1 \pm \rho_y).
\]
Thus:

\[ \tilde{x}\tilde{y} = xy(1 \pm \rho_x)(1 \pm \rho_y) = xy(1 \pm \rho_x \pm \rho_y \pm \rho_x\rho_y) . \]

Under a small-error assumption \( \rho_x\rho_y \) is negligible so

\[ \tilde{x}\tilde{y} \approx xy(1 \pm \rho_x \pm \rho_y) . \]

With the usual pessimism, we will assume the errors have the same sign:

\[ \tilde{x}\tilde{y} \approx xy(1 \pm (\rho_x + \rho_y)) . \]

In other words, assuming small errors, we again obtain that

\[ \delta(\tilde{x}\tilde{y}) \approx \delta\tilde{x} + \delta\tilde{y} . \]
In terms of intervals, where the situation for multiplication is more complicated than what we saw for addition and subtraction, it is useful to define the following set of numbers,

\[ S = \{(\tilde{x} - \varepsilon_x)(\tilde{y} - \varepsilon_y), \]
\[ \quad (\tilde{x} - \varepsilon_x)(\tilde{y} + \varepsilon_y), \]
\[ \quad (\tilde{x} + \varepsilon_x)(\tilde{y} - \varepsilon_y), \]
\[ \quad (\tilde{x} + \varepsilon_x)(\tilde{y} + \varepsilon_y)\} . \]

and let \( S_L = \min(S) \) and \( S_R = \max(S) \).
We then get the following result:

\[ x \in [\tilde{x} - \varepsilon_x, \tilde{x} + \varepsilon_x] \]
\[ y \in [\tilde{y} - \varepsilon_y, \tilde{y} + \varepsilon_y] \]
\[ \Rightarrow xy \in [S_L, S_R] . \]
or, in a more compact notation,

\[[\tilde{x} - \varepsilon_x, \tilde{x} + \varepsilon_x] \ast [\tilde{y} - \varepsilon_y, \tilde{y} + \varepsilon_y] = [S_L, S_R].\]
Example

\[ \tilde{x} = 0.65 \quad \varepsilon_x = 0.15 \quad \tilde{y} = -0.55 \quad \varepsilon_y = 0.65. \]
\[ \Rightarrow \tilde{x} - \varepsilon_x = 0.5 \quad \tilde{x} + \varepsilon_x = 0.8 \quad \tilde{y} - \varepsilon_y = -1.2 \quad \tilde{y} + \varepsilon_y = 0.1. \]

For \([0.5, 0.8] \times [-1.2, 0.1]\) we have

\[
S = \{ (\tilde{x} - \varepsilon_x)(\tilde{y} - \varepsilon_y), (\tilde{x} - \varepsilon_x)(\tilde{y} + \varepsilon_y), (\tilde{x} + \varepsilon_x)(\tilde{y} - \varepsilon_y), (\tilde{x} + \varepsilon_x)(\tilde{y} + \varepsilon_y) \} \\
= \{ (0.5)(-1.2), (0.5)(0.1), (0.8)(-1.2), (0.8)(0.1) \} \\
= \{ -0.6, 0.05, -0.96, 0.08 \} \\
\Rightarrow [S_L, S_R] = [-0.96, 0.08].
\]

Thus,

\[ [0.5, 0.8] \ast [-1.2, 0.1] = [-0.96, 0.08]. \]
In that example, the width of the resulting interval (which equals to $2\Delta(\tilde{x}\tilde{y})$) is 1.04, while the original intervals had lengths 0.3 (which equals to $2\Delta\tilde{x} = 2\varepsilon_x$) and 1.3 (which equals to $2\Delta\tilde{y} = 2\varepsilon_y$).

There is no obvious relation between the various widths.
Let’s see what happens with the relative errors. Suppose \( x \) and \( y \) are the left endpoints of the original intervals: \( x = 0.5 \) and \( y = -1.2 \). The sum of the relative errors is:

\[
\frac{\varepsilon_x}{|x|} + \frac{\varepsilon_y}{|y|} = \frac{0.3}{2}{/0.5} + \frac{1.3}{2}{/1.2} = 0.3 + 0.5416\ldots = 0.8416\ldots.
\]

Note that

\[
\delta(\tilde{x}\tilde{y}) = \frac{(S_R - S_L)/2}{|xy|} = \frac{(0.08 + 0.96)/2}{0.5 \times 1.2} = 0.866\ldots
\]
In the following example the absolute errors are smaller than in the previous example (while the values of $|x|$ and $|y|$ are larger than before). Thus, the approximation of the relative error of the result (via the sum of the relative errors of inputs) will be even better.
Example

\[ \tilde{x} = 10.01 \quad \varepsilon_x = 0.01 \quad \tilde{y} = 20.01 \quad \varepsilon_y = 0.01. \]

\[ \Rightarrow \tilde{x} - \varepsilon_x = 10.0 \quad \tilde{x} + \varepsilon_x = 10.02 \quad \tilde{y} - \varepsilon_y = 20.0 \quad \tilde{y} + \varepsilon_y = 20.02. \]

For \([10.0, 10.02] \times [20, 20.02]\) we have

\[ S = (200, 200.2, 200.4, 200.6004) \]
\[ S_L = 200 \quad S_R = 200.6004. \]

Thus, \([10.0, 10.02] \times [20, 20.02] = [200, 200.6004]\). Suppose \(x\) and \(y\) are the left endpoints of the original intervals: \(x = 10.0\) and \(y = 20.0\). The sum of the relative errors is

\[ \frac{0.02/2}{10} + \frac{0.02/2}{20} = 0.001 + 0.0005 = 0.0015 \]

while

\[ \frac{(S_R - S_L)/2}{|xy|} = \frac{0.6004/2}{200} = 0.001501 \ldots \]
Division

Since

\[
x \div y = x \frac{1}{y}
\]

and we already know how to handle multiplication, we just need to understand what happens with

\[
\frac{1}{\tilde{y}},
\]

the reciprocal of \(\tilde{y}\).
Recall that $\Delta \tilde{y} = |\tilde{y} - y|$. Assume small errors and suppose that the sign of the error is such that $\tilde{y} = y + \Delta \tilde{y}$.

\[
\frac{1}{\tilde{y}} = \frac{1}{y + \Delta \tilde{y}} = \frac{1}{y(1 + \delta \tilde{y})} = \frac{1}{y(1 + \delta \tilde{y})} \frac{(1 - \delta \tilde{y})}{(1 - \delta \tilde{y})}
\]

\[
= \frac{1}{y(1 - (\delta \tilde{y})^2)} (1 - \delta \tilde{y}) \approx \frac{1}{y} (1 - \delta \tilde{y})
\]

\[
\tilde{y} = y + \Delta \tilde{y} \Rightarrow \frac{1}{\tilde{y}} \approx \frac{1}{y} (1 - \delta \tilde{y}) .
\]

Similarly,

\[
\tilde{y} = y - \Delta \tilde{y} \Rightarrow \frac{1}{\tilde{y}} \approx \frac{1}{y} (1 + \delta \tilde{y}) .
\]
We just saw:

\[
\frac{1}{\tilde{y}} \approx \frac{1}{y} (1 - \delta \tilde{y})
\]

\[
\frac{1}{\tilde{y}} \approx \frac{1}{y} (1 + \delta \tilde{y})
\]

Taken together:

\[
\frac{1}{\tilde{y}} \approx \frac{1}{y} (1 \pm \delta \tilde{y})
\]

Equivalently:

\[
\delta \left( \frac{1}{\tilde{y}} \right) = \delta \tilde{y}.
\]

In words: the relative error is preserved under the reciprocal operation.
Thus, using the result from multiplication, in division, under the small-error assumption, the relative error is the sum of the relative errors.
The General Case, in 1D

Start with the 1D case. Suppose we have a differentiable function, $y = f(x)$, $f : \mathbb{R} \to \mathbb{R}$. Can we say something about the propagation of the error, hence the sensitivity of the function?

By inspection, we observe that the higher the derivative of the function at point $x$ is, the more significant the error propagation is.
Let \( \frac{df}{dx} \bigg|_x \) denote \( \frac{df}{dx} \) evaluated at \( x \).

Let \( y = f(x) \) and \( \tilde{y} = f(\tilde{x}) \).

A first-order Taylor expansion:

\[
\tilde{y} \approx y + \frac{df}{dx} \bigg|_x (\tilde{x} - x)
\]

Equivalently:

\[
\tilde{y} - y \approx \frac{df}{dx} \bigg|_x (\tilde{x} - x)
\]

or

\[
\frac{\tilde{y} - y}{\tilde{x} - x} \approx \frac{df}{dx} \bigg|_x
\]

Thus, taking absolute values:

\[
\frac{\Delta \tilde{y}}{\Delta \tilde{x}} \approx \left| \frac{df}{dx} \bigg|_x \right| \quad \text{or, equivalently,} \quad \Delta \tilde{y} \approx \left| \frac{df}{dx} \bigg|_x \right| \Delta \tilde{x}
\]
The General Case

- In the multivariate case, \( x = (x_1, \ldots, x_n) \), \( y = f(x) = f(x_1, \ldots, x_n) \), \( f : \mathbb{R}^n \rightarrow \mathbb{R} \).
- \( x \in \mathbb{R}^n \), \( y \in \mathbb{R} \).
- \( \nabla f \), the gradient of \( f \), is the \( n \)-dimensional vector
  \[ \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \in \mathbb{R}^n. \]
- \( \nabla f \big|_x \) stands for evaluating all these partial derivatives at \( x \in \mathbb{R}^n \):
  \[ \nabla f \big|_x = \begin{bmatrix} \frac{\partial f}{\partial x_1} \big|_x \\ \vdots \\ \frac{\partial f}{\partial x_n} \big|_x \end{bmatrix} \in \mathbb{R}^n. \]
A first-order multivariate Taylor expansion:

\[
\tilde{y} - y \approx \nabla f \big|_x \cdot (\tilde{x} - x) = \begin{bmatrix}
\frac{\partial f}{\partial x_1} \big|_x \\
\vdots \\
\frac{\partial f}{\partial x_n} \big|_x
\end{bmatrix} \cdot \begin{bmatrix}
\tilde{x}_1 - x_1 \\
\vdots \\
\tilde{x}_n - x_n
\end{bmatrix} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \big|_x (\tilde{x}_i - x_i)
\]

Taking absolute values and a pessimistic view:

\[
\Rightarrow \Delta \tilde{y} \approx \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \big|_x \right| \Delta \tilde{x}_i.
\]
Example (revisiting addition)

- Addition of two real scalars,

\[ y = f(x_1, x_2) = x_1 + x_2 , \]

is a function from \( \mathbb{R}^2 \) to \( \mathbb{R} \).

- Since here \( \frac{\partial f}{\partial x_i} = 1 \) for \( i = 1, 2 \), we have

\[ \Delta \tilde{y} \approx \Delta \tilde{x}_1 + \Delta \tilde{x}_2 . \]

- This is consistent with the result we had earlier (where there \( x \) and \( y \) played the role of \( x_1 \) and \( x_2 \) here).

- In fact, we saw earlier there is no approximation – this is exact.
Example (revisiting multiplication)

- Multiplication of two real scalars,

\[ y = f(x_1, x_2) = x_1 x_2, \]

is a function from \( \mathbb{R}^2 \) to \( \mathbb{R} \).

- Given the above, and since here

\[ \frac{\partial f}{\partial x_1} = x_2 \quad \text{and} \quad \frac{\partial f}{\partial x_2} = x_1 \]

we have

\[ \Delta \tilde{y} \approx |x_2 \Delta \tilde{x}_1| + |x_1 \Delta \tilde{x}_2| \]

Again, this is consistent with the result we had earlier (where \( x \) and \( y \) played the role of \( x_1 \) and \( x_2 \) from here).
Version Log

- 18/10/2018. ver 1.02. Slides 31–32 (instances of $\delta y$ replaced with $\delta \tilde{y}$). Slides 37–39 (instances of $\Delta x$ replaced with $\Delta \tilde{x}$).
- 18/10/2018. ver 1.01. Slides 32 (added a clarification) and 37 ($\tilde{x} - x$ is a column vector).
- 17/10/2018, ver 1.00.