Numerical Analysis:
Finding a Square Root using Newton’s Method

Computer Science, Ben-Gurion University

(slides based mostly on Prof. Ben-Shahar’s notes)

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Goal: Given $M > 0$, compute $\sqrt{M}$ up to any given accuracy, using only arithmetic operations.

Method: Newton’s iterations.
\( \sqrt{M} \) is a root of

\[
f(x) = x^2 - M
\]

In other words, we want to solve the equation

\[
x^2 - M = 0.
\]

According to Newton’s method, the iteration is

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

\[
\Rightarrow x_{i+1} = x_i - \frac{x_i^2 - M}{2x_i}
\]

\[
\Rightarrow x_{i+1} = \frac{1}{2} \left( x_i + \frac{M}{x_i} \right)
\]

This computation requires:

- 1 division
- 1 addition
- 1 division by 2 (shifts bits to the right)
How to Pick an Initial Guess?

- Given $M$, we want an initial guess that will satisfy the conditions that guarantee convergence.

\[ g(x) = \frac{1}{2} \left( x + \frac{M}{x} \right) \quad \Rightarrow \quad |g'(x)| = \frac{1}{2} \left| 1 - \frac{M}{x^2} \right| \]

- Want:

\[ \frac{1}{2} \left| 1 - \frac{M}{x^2} \right| < 1 \]
Equivalently, want:

\[-2 < 1 - \frac{M}{x^2} < 2\]

Inspecting the right inequality:

\[1 - \frac{M}{x^2} < 2 \iff -\frac{M}{x^2} < 1 \iff \frac{M}{x^2} > -1 \iff -M < x^2\]

so this inequality always holds.
As for the left inequality:

\[-2 < 1 - \frac{M}{x^2} \iff -3 < -\frac{M}{x^2} \iff 3 > \frac{M}{x^2} \iff x > \sqrt{\frac{M}{3}}\]
So, it seems we have solved our problem: we “just” need to start in some $x_0 > \sqrt{\frac{M}{3}}$

The problem, of course, is that requires us to compute a square root – which is exactly the problem we are trying to solve.
Idea: transform the problem to a range where it is easy to pick a safe initial guess. Apply Newton’s method to the new problem, and then transform the result back to the original problem.
For every $M > 0$, there exist a real number $m$ such that $1/4 \leq m \leq 1$ and an integer number $e$ such that

$$M = m \cdot 4^e$$

Thus,

$$\sqrt{M} = \sqrt{m} \cdot 2^e$$
Since $1/4 \leq m \leq 1$ we know that

$$1/2 \leq \sqrt{m} \leq 1$$

We can get a good initial guess for finding $\sqrt{m}$ using a linear approximation w/o having to use $\sqrt{}$.

The line between the two points

$$(m = 1/4, \sqrt{m} = 1/2)$$

and

$$(m = 1, \sqrt{m} = 1)$$

is

$$\frac{1}{3}(2m + 1)$$
Taking \( x_0 = \frac{1}{3}(2m + 1) \) ensures \( x_0 > \sqrt{\frac{m}{3}} \).

Indeed:

\[
\frac{1}{3}(2m + 1) > \sqrt{\frac{m}{3}}
\]

\[
\frac{1}{9}(2m + 1)^2 > \frac{m}{3}
\]

\[
(2m + 1)^2 > 3m
\]

\[
4m^2 + 4m + 1 > 3m
\]

\[
4m^2 + m + 1 > 0
\]

but this always holds since \( m > 0 \).

Thus, convergence of Newton’s iterations is guaranteed.
How Many Iterations Are Required?

\[ e_0 = \sqrt{m} - x_0 = m^{1/2} - \frac{1}{3} 2m + 1 \]

\[ e'_0 = \frac{1}{2} m^{-1/2} - \frac{2}{3} \]

Set \( e'_0 \) to zero find \( m \) that will give the maximal error:

\[ m_{\text{max}}^{-1/2} = \frac{4}{3} \Rightarrow m_{\text{max}} = \frac{9}{16} \Rightarrow \sqrt{m_{\text{max}}} = \frac{3}{4} \]

In which case, \( e_0^{\text{max}} = \frac{3}{4} - \frac{1}{3} \left( 2 \cdot \frac{9}{16} + 1 \right) \approx 0.042 \)
We know the convergence rate is quadratic.

\[ |e_{n+1}| = |x_{i+1} - \sqrt{m}| = \left| \frac{1}{2}(x_i + m/x_i) - \sqrt{m} \right| \]

\[ = \frac{1}{2x_i}(x_i - \sqrt{m})^2 = \frac{1}{2x_i}e_i^2 \]

Since \( 1/2 \leq x_i \leq 1 \) we get that \( |e_{n+1}| \leq e_i^2 \).

Thus,

\[ e_3 \leq e_2^2 \leq e_1^2 \leq e_0^4 \leq 0.042^8 \approx 10^{-11} \]

Point: even just three iterations suffice for accuracy of at least 10 digits!
Version Log

- 12/11/2018, ver 1.00.