NA181: Numerical Examples for Errors in Multiplication

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Abstract
This document contains numerical examples for errors in multiplication, including what is called “interval arithmetics”. I touched upon the latter in my lectures, as this gives another way to think about the uncertainty that is caused by the errors when we do multiplication. If you haven’t seen interval arithmetics in the lectures (in effect, if you are in one of the other groups) don’t worry about it and feel free to ignore this document.

Most of what follows is based on Section 0.5 from Chapter 0 in Gerald and Wheatley’s Applied Numerical Analysis.

In terms of intervals, where the situation for multiplication is more complicated than what we saw for addition and subtraction, it is useful to define the following set of numbers,

\[
S = \{ (\tilde{x} - \varepsilon_x)(\tilde{y} - \varepsilon_y), (\tilde{x} - \varepsilon_x)(\tilde{y} + \varepsilon_y), (\tilde{x} + \varepsilon_x)(\tilde{y} - \varepsilon_y), (\tilde{x} + \varepsilon_x)(\tilde{y} + \varepsilon_y) \}.
\]

(1)

and let \( S_L = \min(S) \) and \( S_R = \max(S) \). We then get the following result:

\[
x \in [\tilde{x} - \varepsilon_x, \tilde{x} + \varepsilon_x]
\]

(2)

\[
y \in [\tilde{y} - \varepsilon_y, \tilde{y} + \varepsilon_y]
\]

(3)

\[
\Rightarrow xy \in [S_L, S_R],
\]

(4)
or, in a more compact notation,
\[
[\tilde{x} - \varepsilon_x, \tilde{x} + \varepsilon_x] \ast [\tilde{y} - \varepsilon_y, \tilde{y} + \varepsilon_y] = [S_L, S_R].
\] (5)

Example 1

Let
\[
\tilde{x} = 0.65 \quad \varepsilon_x = 0.15 \quad \tilde{y} = -0.55 \quad \varepsilon_y = 0.65.
\] (6)

\Rightarrow
\[
\tilde{x} - \varepsilon_x = 0.5 \quad \tilde{x} + \varepsilon_x = 0.8 \quad \tilde{y} - \varepsilon_y = -1.2 \quad \tilde{y} + \varepsilon_y = 0.1.
\] (7)

For \([0.5, 0.8] \times [-1.2, 0.1]\) we have
\[
S = \{(\tilde{x} - \varepsilon_x)(\tilde{y} - \varepsilon_y), (\tilde{x} - \varepsilon_x)(\tilde{y} + \varepsilon_y), (\tilde{x} + \varepsilon_x)(\tilde{y} - \varepsilon_y), (\tilde{x} + \varepsilon_x)(\tilde{y} + \varepsilon_y)\}
\]
\[
= \{(0.5)(-1.2), (0.5)(0.1), (0.8)(-1.2), (0.8)(0.1)\}
\]
\[
= \{-0.6, 0.05, -0.96, 0.08\} \tag{8}
\]
\Rightarrow \[S_L, S_R] = [-0.96, 0.08]. \tag{9}

Thus,
\[
[0.5, 0.8] \ast [-1.2, 0.1] = [-0.96, 0.08]. \tag{10}
\]

The width of the resulting interval (which equals to \(2\Delta(\tilde{x}\tilde{y})\)) is 1.04,

while the original intervals had lengths 0.3 (which equals to \(2\Delta\tilde{x} = 2\varepsilon_x\)) and 1.3 (which equals to \(2\Delta\tilde{y} = 2\varepsilon_y\)). There is no obvious relation between the various widths. Let’s see what happens with the relative errors. Suppose \(x\) and \(y\) are the left endpoints of the original intervals: \(x = 0.5\) and \(y = -1.2\). The sum of the relative errors is:

\[
\frac{\varepsilon_x}{|x|} + \frac{\varepsilon_y}{|y|} = \frac{0.3/2}{0.5} + \frac{1.3/2}{1.2} = 0.3 + 0.5416 \ldots = 0.8416 \ldots. \tag{11}
\]

Note that
\[
d(\tilde{x}\tilde{y}) = \frac{(S_R - S_L)/2}{|xy|} = \frac{(0.08 + 0.96)/2}{0.5 \times 1.2} = 0.866 \ldots \tag{12}
\]
In the following example the absolute errors are smaller than in the previous example (while the values of $|x|$ and $|y|$ are larger than before). Thus, the approximation of the relative error of the result (via the sum of the relative errors of inputs) will be even better.

**Example 2** Let

$$\tilde{x} = 10.01 \quad \epsilon_x = 0.01 \quad \tilde{y} = 20.01 \quad \epsilon_y = 0.01. \quad (13)$$

$$\Rightarrow$$

$$\tilde{x} - \epsilon_x = 10.0 \quad \tilde{x} + \epsilon_x = 10.02 \quad \tilde{y} - \epsilon_y = 20.0 \quad \tilde{y} + \epsilon_y = 20.02. \quad (14)$$

For $[10.0, 10.02] \ast [20, 20.02]$ we have

$$S = (200, 200.2, 200.4, 200.6004) \quad (15)$$

$$S_L = 200 \quad S_R = 200.6004. \quad (16)$$

Thus, $[10.0, 10.02] \ast [20, 20.02] = [200, 200.6004]$. Suppose $x$ and $y$ are the left endpoints of the original intervals: $x = 10.0$ and $y = 20.0$. The sum of the relative errors is:

$$\frac{0.02/2}{10} + \frac{0.02/2}{20} = 0.001 + 0.0005 = 0.0015 \quad (17)$$

Note that

$$\frac{(S_R - S_L)/2}{|xy|} = \frac{0.6004/2}{200} = 0.001501\ldots \quad (18)$$