**2017 - אנליזה נומרית**

**Piezoelectric Interpolation, Spline, Curve fitting - 7.**

*Approximation via curve fitting*:

1. **Shekel's Question**: Approximation via curve fitting

In the laboratory, we attempted to find the relationship between the number of electric pulses measured during the sliding of two materials against each other, knowing that the surfaces of the materials are alternately conductive and insulating, connected to a voltage source.

In the laboratory, we measured the following samples (number of pulses per second as a function of time):

- (2, 4), (3, 4.5), (6, 6), (8, 7).

The main scientist in the laboratory believes that the model that best describes this relationship is the function:

\[ f(x) = a \times x + b + c \times \sin(x) \]

Find the function that best approximates the set of measured samples.

**Solution:**

We want to find the parameters \(a, b, c\). Such that the error will be minimal.

At the square root of the error is:

\[ E_2(a_1, a_2, \ldots) = \sqrt{\sum_{i=1}^{n} (y_i - f(a_1, a_2, \ldots))^2} \]

We choose the function of the form best fit to the parameters of the model:

\[ E_2(a, b, c) = \sum_{i=1}^{n} (y_i - (ax_i + b + c \sin(x_i)))^2 \]

Consider the derivatives of the error with respect to \(a, b, c\):

\[ \frac{\partial E_2}{\partial a} = 2 \sum_{i=1}^{n} (y_i - (ax_i + b + c \sin(x_i)))(-x_i) = 0 \]

\[ \sum_{i=1}^{n} y_i x_i - a \sum_{i=1}^{n} x_i^2 - b \sum_{i=1}^{n} x_i - c \sum_{i=1}^{n} x_i \sin(x_i) = 0 \]

\[ a \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i + c \sum_{i=1}^{n} x_i \sin(x_i) = \sum_{i=1}^{n} y_i x_i \]

**Note:**

\[ \frac{\partial E_2}{\partial b} = 2 \sum_{i=1}^{n} (y_i - (ax_i + b + c \sin(x_i)))(-1) = 0 \]

\[ \sum_{i=1}^{n} y_i - a \sum_{i=1}^{n} x_i + b4 - c \sum_{i=1}^{n} \sin(x_i) = 0 \]

\[ a \sum_{i=1}^{n} x_i + b4 + c \sum_{i=1}^{n} \sin(x_i) = \sum_{i=1}^{n} y_i \]

**Note:**

\[ \frac{\partial E_2}{\partial c} = 2 \sum_{i=1}^{n} (y_i - (ax_i + b + c \sin(x_i)))(\sin(x_i)) = 0 \]

\[ \sum_{i=1}^{n} y_i \sin(x_i) - a \sum_{i=1}^{n} x_i \sin(x_i) - b \sum_{i=1}^{n} \sin(x_i) - c \sum_{i=1}^{n} \sin(x_i)^2 = 0 \]
\[
a \sum_{i=1}^{4} x_i \sin(x_i) + b \sum_{i=1}^{4} \sin(x_i) + c \sum_{i=1}^{4} \sin(x_i)^2 = \sum_{i=1}^{4} y_i \sin(x_i)
\]

The Normal Equations are:

\[
\begin{align*}
\sum_{i=1}^{4} x_i^2 &= 2^2 + 3^2 + 6^2 + 8^2 = 113 \\
\sum_{i=1}^{4} x_i &= 2 + 3 + 6 + 8 = 19 \\
\sum_{i=1}^{4} x_i \sin(x_i) &= 8.480327862 \\
\sum_{i=1}^{4} y_i x_i &= 113.5 \\
\sum_{i=1}^{4} \sin(x_i) &= 1.760360183 \\
\sum_{i=1}^{4} y_i &= 21.5 \\
\sum_{i=1}^{4} \sin(x_i)^2 &= 1.903639428 \\
\sum_{i=1}^{4} y_i \sin(x_i) &= 9.5212
\end{align*}
\]

The equations can be written as a matrix equation:

\[
\begin{bmatrix}
113 19 8.48 & 113.5 \\
19 4 1.76 & 21.5 \\
8.48 1.76 1.9 & 9.52
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= 
\begin{bmatrix}
100 \\
010 \\
001
\end{bmatrix}
\begin{bmatrix}
0.5 \\
0 \\
1
\end{bmatrix}
\]

This gives us the coefficients:

\[
f(x) = a \cdot x + b + c \cdot \sin(x) = \frac{1}{2} x + 3
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(Lease Square Fitting) 2

A. 

In a certain experiment, the measurements taken at the equipment, which operation can be measured in a linear manner, were received.

\[ p_1 = (1, 1) \]
\[ p_2 = (2, 3) \]
\[ p_3 = (3, 3) \]

A. Find the best parameters (in the sense of Least Square Error) of the given equipment model.

B. If adding two more measurements would cause a change to the measurement of the accepted model from the first set of measurements? If yes, find the updated model parameters. If no, explain why.

\[ p_4 = (4, 4 \frac{1}{2}) \]
\[ p_5 = (-1, -\frac{2}{3}) \]

Solution:

A. We want to find the line \( y = Ax + B \) that best fits the data:

\[
A = \frac{N \sum_{i=1}^{N} x_i y_i^{N} - \left( \sum_{i=1}^{N} x_i \right) \sum_{i=1}^{N} y_i^{N}}{N \sum_{i=1}^{N} x_i^{N} - (\sum_{i=1}^{N} x_i)^2} = \frac{3 \times 16 - 6 \times 7}{3 \times 14 - 36} = 1
\]

\[
B = \frac{\sum_{i=1}^{N} x_i^{N} y_i^{N} - \left( \sum_{i=1}^{N} x_i \right) \sum_{i=1}^{N} x_i y_i^{N}}{N \sum_{i=1}^{N} x_i^{N} - (\sum_{i=1}^{N} x_i)^2} = \frac{14 \times 7 - 16 \times 6}{3 \times 14 - 36} = \frac{1}{3}
\]

Obtained line is:
\[
y = x + \frac{1}{3}
\]

B. In order to connect to the line of the equation to the next equation, it is required: for the equation to be established, the line is not valid. It is not valid in all cases.