Theorem 1.1 (Nonsingularity of SDD Matrices)

Strictly diagonally dominant matrices are always nonsingular.

Proof

Suppose that matrix \( A_{nn} \) is SDD and singular, then there exists a \( u \in u_n \) such that \( Au = b \) where \( b \) is the 0 vector while \( u \neq 0 \) (Definition NM[67]).

\[
A = \begin{bmatrix}
A_{11} & A_{12} & \ldots & A_{1n} \\
A_{21} & A_{22} & \ldots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \ldots & A_{nn}
\end{bmatrix}
\quad u = \begin{bmatrix}
u_1 \\
v_2 \\
\vdots \\
u_n
\end{bmatrix}
\quad b = 0
\]

In the vector \( u \) there is a "dominant element" in position \( u_i \) where its absolute value is either equal to or larger than the absolute value any other element in \( u \). Let's call this maximum value \( \alpha \).

Every element in \( u \) cannot be \( \alpha \). If this were the case then row \( i \) multiplied by \( u \) would not result in a 0 element for \( b \) which is needed in order for \( b \) to be the 0 vector.

(1) \( |u_1| = |u_2| = \ldots = |u_n| = \alpha \) \hspace{1cm} (premise)

(2) \( A_{i1}u_1 + A_{i2}u_2 + \ldots + A_{in}u_n = 0 \) \hspace{1cm} (row, column multiplication)

(3) \( \pm A_{i1} \alpha + \pm A_{i2} \alpha + \ldots + \pm A_{in} \alpha = 0 \) \hspace{1cm} (substitution)

(4) \( (\pm A_{i1} \pm A_{i2} \pm \ldots \pm A_{in}) = 0 \) \hspace{1cm} (distributivity)

(5) \( \pm A_{ij} \pm A_{ij} \pm \ldots \pm A_{ij} = 0 \) \hspace{1cm} (multiplicative inverse)

(6) \( |A_{ii}| = |A_{i2}| + \ldots + |A_{in}| \) \hspace{1cm} (check for SDD)

(6) contradicts the premise that \( A_{nn} \) is SDD thus every entry of \( u_n \) cannot be \( \alpha \).

In order for \( b_1 \) to equal 0 then \( \sum_{i=1}^{n} A_{ij}u_i = 0 \). Because \( |A_{11}| > \sum_{j \neq k} |A_{ij}| \), then position \( u_1 \) cannot be \( \alpha \). If \( u_1 \) cannot be \( \alpha \) then what about position \( u_2 \)? For the same reason \( u_1 \) cannot be \( \alpha \) due to the magnitude of \( A_{11} \) in row 1 of \( A \), \( u_2 \) cannot be \( \alpha \) due to the magnitude of \( A_{21} \) in row 2 of \( A \). This logic then continues from \( u_2 \) until \( u_n \). As a result no element in \( u \) can be the maximum element and all elements in \( u \) cannot be the maximum element. Therefore there is no vector \( u \) that we can create such that \( Au = 0 \).

If there is no \( u \) other than the 0 vector that can be created such that \( Au = 0 \), then \( A \) is nonsingular (Definition NM[67]), a contradiction to our premise.