

Sparsity and ℓ_p -Restricted Isometry

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Abstract

A matrix A is said to have the ℓ_p -Restricted Isometry Property (ℓ_p -RIP) if for all vectors x of up to some sparsity k , $\|Ax\|_p$ is roughly proportional to $\|x\|_p$. We study this property for $m \times n$ matrices of rank proportional to n and $k = \Theta(n)$. In this parameter regime, ℓ_p -RIP matrices are closely connected to Euclidean sections, and are “real analogs” of testing matrices for locally testable codes.

It is known that with high probability, random dense $m \times n$ matrices (e.g., with i.i.d. ± 1 entries) are ℓ_2 -RIP with $k \approx m/\log n$, and sparse random matrices are ℓ_p -RIP for $p \in [1, 2)$ when $k, m = \Theta(n)$. However, when $m = \Theta(n)$, sparse random matrices are known to *not* be ℓ_2 -RIP with high probability.

Against this backdrop, we show that sparse matrices *cannot* be ℓ_2 -RIP in our parameter regime. On the other hand, for $p \neq 2$, we show that every ℓ_p -RIP matrix *must* be sparse. Thus, sparsity is incompatible with ℓ_2 -RIP, but necessary for ℓ_p -RIP for $p \neq 2$.

Under a suitable interpretation, our negative result about ℓ_2 -RIP gives an impossibility result for a certain continuous analog of “ c^3 -LTCs”—locally testable codes of constant rate, constant distance and constant locality that were constructed in recent breakthroughs.

Keywords: Restricted Isometry Property, Sparse Matrices, Spread Subspaces, Euclidean Sections, Compressed sensing, Locally Testable Codes

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1 Introduction

A random (dense) $0.01n \times n$ matrix A with independent ± 1 entries acts on all $o(n)$ -sparse¹ vectors approximately isometrically. That is, $\|Ax\|_2 \approx \sqrt{0.01n} \|x\|_2$ for all $o(n)$ -sparse vectors x . Such a matrix A is said to satisfy the Restricted Isometry Property (RIP) for the ℓ_2 -norm. More generally, one can extend the definition of RIP to any ℓ_p -norm.

Definition 1.1. Let $p \geq 1$. A matrix $A \in \mathbb{R}^{m \times n}$ is (k, D) - ℓ_p -RIP if there exists $K > 0$ such that for every k -sparse $x \in \mathbb{R}^n$, it holds that

$$K \|x\|_p \leq \|Ax\|_p \leq D \cdot K \|x\|_p .$$

The original rise to prominence of the RIP is due to connections to compressed sensing [Don06] unearthed in several works [CT05; CT06; CRT06], where it was referred to as the Uniform Uncertainty Principle (UUP). The ℓ_p -RIP can be used to achieve the so-called “ ℓ_p - ℓ_1 ” guarantee for the sparse recovery problem: if A is $(k, 1 + \varepsilon)$ - ℓ_p -RIP and x is δ -close in ℓ_1 -distance to a sparse vector y , then one can approximately recover x given the noisy measurement $Ax + b$, where A is the measurement matrix and b is the noise vector. Formally, one has the guarantee that the recovered \hat{x} satisfies $\|x - \hat{x}\|_p \leq O(k^{-(1-\frac{1}{p})} \delta + \|b\|_p)$ [AGR15].

The most well-studied case of ℓ_p -RIP is for $p = 2$. When $p = 2$, the RIP is equivalent to saying that the eigenvalues of $A_S^\top A_S$ are roughly equal for any small set $S \subseteq [n]$, where A_S denotes the restriction of A to the columns in S . It is well-known that for $m \geq O(k \log(n/k))$, a (dense) random $m \times n$ matrix with independent ± 1 entries, or more generally any distribution that has the JL property (e.g., a subgaussian distribution) [BDDW08], is $(k, O(1))$ - ℓ_2 -RIP.

The RIP also has close connections to *spread subspaces*, an analog of error-correcting codes over the reals, as well as to Euclidean sections. A simple proof shows that the kernel of an $(\Omega(n), O(1))$ - ℓ_p -RIP matrix is a subspace with the “ ℓ_p -spread property”: any $x \in \ker(A)$ with $\|x\|_p = 1$ is $\Omega(1)$ -far in ℓ_p -distance from all $o(n)$ -sparse vectors [GMM22, Prop. 3.8]. For $p = 2$, this corresponds to $\ker(A)$ being a good Euclidean section of ℓ_1 , a notion that has been studied in classical works [FLM77; Kas77; GG84], and more recently in [KT07; GLW08; GLR10; Kar11; GMM22].

RIP matrices can also be interpreted as continuous analogs for testing matrices of linear locally testable codes. The testing matrix of a code is simply a matrix A where each row corresponds to one of the possible linear local tests performed by the tester. The code equals $\ker(A)$ and thus has dimension $n - \text{rank}(A)$. The number of queries made by the tester is the row sparsity of A . The probability that x fails the test is thus proportional to the Hamming weight $\text{wt}(Ax)$ of the “syndrome” Ax . In particular, the testing matrix A of a (strong) locally testable code of constant rate ρ , constant locality q and constant distance δ (“ c^3 -LTC”) will have rank $(1 - \rho)n$, row sparsity q , and $\text{wt}(Ax) \geq \beta \cdot \text{wt}(x)$ for all x with $\text{wt}(x) \leq \delta n/2$, where $\beta \in (0, 1)$ is a constant. Such codes were recently constructed in two breakthrough works [DELLM22; PK22].

In the continuous analog of codes, the testing matrix A for a c^3 -LTC is analogous to an $(\Omega(n), O(1))$ - ℓ_p -RIP matrix A of rank αn (for a constant $\alpha \in (0, 1)$) whose rows have constant

¹A vector x is k -sparse if the number of nonzero entries in x is at most k .

sparsity. The “code” is $\ker(A)$ and its “distance” corresponds to the fact that $\ker(A)$ is well-spread and in particular has no δn -sparse vectors. A testing matrix should then, in particular, satisfy the following: for all nonzero x which are $\delta n/2$ -sparse, the “syndrome” Ax has sizeable norm, specifically $\|Ax\|_p \geq \beta \|x\|_p$. Thus the analogy between RIP matrices and LTC testing matrices is achieved by replacing the finite field \mathbb{F}_2 with \mathbb{R} , and by replacing the Hamming “metric” $\text{wt}(\cdot)$ with the ℓ_p -norm $\|\cdot\|_p$.²

In this work we study the sparsity of (k, D) - ℓ_p -RIP matrices $A \in \mathbb{R}^{m \times n}$ in the regime where $k \geq \Omega(n)$, $D = O(1)$, and $\text{rank}(A) \geq \Omega(n)$. This parameter setting of k , D , and $\text{rank}(A)$ is naturally induced by the aforementioned relation of the RIP to ℓ_p -spread subspaces and to testing of codes, but is less common in the context of compressed sensing and the JL property, where one typically has $k = o(n)$ and $\text{rank}(A) \leq m = o(n)$. Sparsity naturally arises from the connection to testing of codes, where the testing matrix must be row sparse, as well as the connection to Euclidean sections/ ℓ_p -spread subspaces, where the best known explicit constructions, for all $p \in [1, 2]$, come from the kernels of sparse matrices [GLW08; GLR10; Kar11]. For $p \neq 2$, it is known that random sparse $m \times n$ matrices are ℓ_p -RIP [AGR15; GMM22], and explicit constructions, for $p \in [1, 2]$, are also sparse [BGKS08; GMM22]. However, [GMM22] also show that such matrices are *not* ℓ_2 -RIP in the aforementioned parameter regime, when $k \geq \Omega(n)$ and $D = O(1)$.

In this work, we thus ask the following questions: are ℓ_2 -RIP matrices necessarily dense? And is sparsity inherent to ℓ_p -RIP matrices?

1.1 Our results

We prove two theorems about the sparsity of ℓ_p -RIP matrices. For $p = 2$ we show that any ℓ_2 -RIP matrix must contain a large number of rows with superconstant density, and for $p < 2$ we show that the rows of A must be “analytically sparse on average”. Taken together, our results show that sparsity is incompatible for ℓ_2 -RIP, but is necessary for ℓ_p -RIP.

Our result for ℓ_2 -RIP is stated informally below.

Theorem 1 (Simplified Theorem 3). *Let $m, n \in \mathbb{N}$. Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank αn where $0 < \alpha < 1$ is a constant. Suppose that A is (k, D) - ℓ_2 -RIP for some $\Omega(n) \leq k \leq n$ and $1 \leq D \leq O(1)$. Then for any constant $\eta \in (0, 1)$, if we let T denote the matrix consisting of all rows in A that are not s -sparse for some $s \geq \Omega(\eta \log n)$, then: (1) T contains at least $\Omega(n^{1-\eta})$ rows, and (2) $\|T\|_F^2 \geq \Omega(n^{-\eta} \cdot \|A\|_F^2)$, where $\|\cdot\|_F$ denotes the Frobenius norm.*

Theorem 1 is not the only sparsity lower bound for ℓ_2 -RIP matrices. A simple argument given in [AGR15] (based on an argument in [Cha10]) shows that a (k, D) - ℓ_2 -RIP matrix A with column sparsity t must either have $m > n/k$ or $t \geq k/D^2$. This implies that if k is small, e.g. $O(\log n)$, then the matrix A must either have superconstant column sparsity ($\Omega(\log n)$) or have many rows ($\Omega(n/\log n)$, far larger than the polylog(n) rows required for dense matrices).

Theorem 1 is incomparable to the argument of [Cha10; AGR15], as it applies in the different parameter regime of $k = \Theta(n)$, which is the regime of importance for Euclidean sections and testing

²One may observe that the analogy only requires the lower bound in the RIP. However, the upper bound in the RIP also holds, as it is a simple consequence of the constant row sparsity, as the entries of A are bounded by a constant.

of codes. Indeed, the argument of [Cha10; AGR15] does not give *any* bound on the sparsity in the regime of $k = \Theta(n)$, as it only implies that the number of rows m is at least some constant. This limitation is inherent to the proof technique, and so the argument of [Cha10; AGR15] does not extend to the $k = \Theta(n)$ regime. Moreover, Theorem 1 still holds even when $m \geq n$ (assuming that $\text{rank}(A) = \alpha n$ for some constant $\alpha \in (0, 1)$), whereas the argument of [Cha10; AGR15] does not give any bound on the sparsity when $m \geq n$ (even if $k = 1$).

Theorem 1 rules out the existence of an ℓ_2 -RIP analog of c^3 -LTCs in the sense discussed earlier. Indeed, the theorem implies that a parity check matrix A with “constant rate” ($\dim \ker A \geq \Omega(n)$) and “constant distance” (A is $(\Omega(n), O(1))$ - ℓ_2 -RIP) must have rows of Hamming weight $\Omega(\log n)$.

We now turn to our theorem about ℓ_p -RIP matrices for $p < 2$.

Theorem 2 (Simplified Theorem 5). *Fix a constant $p \in [1, 2]$ and let $A \in \mathbb{R}^{m \times n}$ be a (k, D) - ℓ_p -RIP matrix for some $\Omega(n) \leq k \leq n$ and $1 \leq D \leq O(1)$. Let A_{1*}, \dots, A_{m*} denote the rows of A . Then,*

$$\sum_{i=1}^m \|A_{i*}\|_2^p = \Theta \left(\sum_{i=1}^m \|A_{i*}\|_p^p \right).$$

We note that the “ O ” part of Theorem 2 is trivial, as $\sum_{i=1}^m \|A_{i*}\|_2^p \leq \sum_{i=1}^m \|A_{i*}\|_p^p$ always holds since $\|x\|_2 \leq \|x\|_p$ for $p \in [1, 2]$. Thus, the “ Ω ” part is the nontrivial statement.

Theorem 2 is an analytic statement about the norms of the rows of A . To explain its implications for the sparsity of A , let us first consider the following simple case where every row of A has exactly s nonzero entries, each of magnitude 1. Then, any row A_{i*} satisfies $\|A_{i*}\|_p = s^{1/p-1/2} \|A_{i*}\|_2$, which implies that $O(1) \sum_{i=1}^m \|A_{i*}\|_2^p \geq \sum_{i=1}^m \|A_{i*}\|_p^p = s^{1-p/2} \sum_{i=1}^m \|A_{i*}\|_2^p$, where the first inequality is due to Theorem 2. So, Theorem 2 implies that $s^{1-p/2} \leq O(1)$, i.e., s is constant for any constant $p < 2$.

More generally, for $p \in [1, 2)$ and any vector $x \in \mathbb{R}^n$, Hölder’s inequality implies that $\|x\|_2 \leq \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{2}} \|x\|_2$, with the lower inequality achieved when $x = e_i$ is a standard basis vector, i.e., very sparse, and the upper inequality achieved when $x = 1^n$, i.e., very dense. Thus, the ratio $\frac{\|x\|_p}{\|x\|_2}$ can be viewed as a notion of analytic sparsity for the vector x . In particular, if $\|x\|_p = \Theta(\|x\|_2)$, then a constant fraction of the ℓ_p -mass of x must lie on a constant number of coordinates.³ Theorem 2 informally says that $\|A_{i*}\|_p = \Theta(\|A_{i*}\|_2)$ “on average”. One can thus interpret Theorem 2 to say that the rows of A must have “constant analytic sparsity on average”, where “analytic sparsity” is in the ℓ_p vs. ℓ_2 sense stated above.

We note that because Theorem 1 is a statement about the *average* of the rows, rather than a worst case statement about all rows, we cannot rule out the existence of an $\Omega(\log n)$ -sparse matrix A that is simultaneously $(\Omega(n), O(1))$ - ℓ_2 -RIP and $(\Omega(n), O(1))$ - ℓ_p -RIP for some $1 \leq p < 2$. Indeed, if Theorem 2 instead implied that $\|A_{i*}\|_p = \Theta(\|A_{i*}\|_2)$ for *all* rows $i \in [m]$, then the rows of the submatrix T from Theorem 1 would violate this conclusion, implying that A cannot be both ℓ_2 and ℓ_p -RIP. However, because Theorem 2 is only a statement that holds on average, our results do not prove this.

³See Section 2.2 and Proposition 2.4 for a more detailed discussion and formal statement.

1.2 Proof overview

The proof of Theorem 1 has two key ideas. First, we define a new property, the *analytic restricted isometry property* (ARIP, Definition 3.1), that strengthens Definition 1.1 by requiring not only that $\|Ax\|_2 \approx \|x\|_2$ for every k -sparse x , but also for vectors x that are *close* to k -sparse. We then show that one can convert between ARIP and RIP with some small loss in parameters when $k = \Theta(n)$ (Proposition 3.2). Hence, we can execute the proof of Theorem 1 assuming that A is ARIP.

Now, given that we assume that A is ARIP, it suffices to show that if A is too sparse, then there is a vector x that is close to a k -sparse vector and $\|Ax\|_2 \ll \|x\|_2$. Our second key idea (Lemma 3.4) is to choose $x = e^{-tA^\top A} e_i$, where e_i is a standard basis vector chosen so that $\|\Pi e_i\|_2$ is large, where Π is the orthogonal projection onto $\ker(A)$ and $t > 0$ is a parameter that we will pick carefully. In the eigenbasis of $A^\top A$, the transformation $e^{-tA^\top A}$ significantly attenuates the magnitude of eigenvectors with large eigenvalues but preserves the eigenvectors with eigenvalue 0, i.e., those in $\ker(A)$. Thus, as $t \rightarrow \infty$, we have $x \rightarrow \Pi e_i$, i.e., x becomes the projection of e_i onto $\ker(A)$. So, as $t \rightarrow \infty$, we have $\|Ax\|_2 \rightarrow 0$, but $\|x\|_2$ will still be large because e_i was chosen to have large projection onto $\ker(A)$.

To violate the ARIP of A , we additionally need to have that x is close to k -sparse, and if we take $t \rightarrow \infty$ it becomes hard to control this quantity. But, on the other extreme when $t = 0$, we have $x = e_i$, which is 1-sparse. By carefully choosing t to be an intermediate quantity, we can simultaneously ensure that $\|Ax\|_2$ is small, $\|x\|_2$ is large, and x is close to sparse. We control the “approximate sparsity” of x by bounding $\|x\|_1$, and here we crucially use the sparsity of A .

The full proof of Theorem 1 requires some additional steps, but Lemma 3.4 captures the core of the argument.

The proof of Theorem 2 is simpler. We probe the matrix A with a random standard coordinate vector e_j and a random k -sparse vector g with Gaussian entries. We show that $\|Ae_j\|_p^p / \|e_j\|_p^p = t$ behaves very differently in expectation compared to $\|Ag\|_p^p / \|g\|_p^p$, and this allows us to bound $\sum_{i=1}^m \|A_{i*}\|_2^p$ in terms of $\sum_{i=1}^m \|A_{i*}\|_p^p$.

2 Preliminaries

2.1 Notation and conventions

The implicit factor in asymptotic notations is an absolute constant, unless stated otherwise.

For a matrix $A \in \mathbb{R}^{m \times n}$, we denote the i -th row and the j -th column of A by A_{i*} and A_{*j} , respectively.

Let $p \geq 1$. For a vector $x \in \mathbb{R}^n$, we let $\|x\|_p$ denote the ℓ_p -norm of x , i.e., $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. For a matrix $A \in \mathbb{R}^{m \times n}$, we let $\|A\|_{\ell_p \rightarrow \ell_p}$ denote the $\ell_p \rightarrow \ell_p$ operator norm of a matrix A , which is defined by

$$\|A\|_{\ell_p \rightarrow \ell_p} = \sup \left\{ \|Ax\|_p \mid x \in \mathbb{R}^n \wedge \|x\|_p = 1 \right\}.$$

The Frobenius norm $\|A\|_F$ is defined as $\sqrt{\sum_{i,j}|A_{i,j}|^2}$. The Frobenius norm satisfies

$$\|A\|_F^2 = \text{tr}(AA^\top) = \text{tr}(A^\top A) . \quad (1)$$

2.2 Sparse, Compressible and Distorted vectors

Definition 2.1. Let $x \in \mathbb{R}^n$ and let $1 \leq q \leq p$. We define the (ℓ_q, ℓ_p) -distortion of x as

$$\Delta_{q,p}(x) = \frac{\|x\|_p \cdot n^{1/q-1/p}}{\|x\|_q} .$$

Hölder's inequality yields

$$1 \leq \left(\frac{n}{|\text{supp}(x)|} \right)^{1/q-1/p} \leq \Delta_{q,p}(x) \leq n^{1/q-1/p} . \quad (2)$$

In particular, sparse vectors have large distortion. As we next discuss, a certain converse of this fact also holds.

Definition 2.2. Fix $p \geq 1$. Let $k \leq n \in \mathbb{N}$ and $\varepsilon > 0$. A vectors $x \in \mathbb{R}^n \setminus \{0\}$ is (k, ε) - ℓ_p -compressible if there exists a k -sparse $y \in \mathbb{R}^n$ such that $\|x - y\|_p \leq \varepsilon \|x\|_p$.

Remark 2.3. Without loss of generality y can be taken to be equal to x in the k entries of x that have the largest absolute value, and 0 everywhere else.

Proposition 2.4 ([GMM22, Prop. 3.11]). Fix $1 \leq q < p$, $k \leq n \in \mathbb{N}$ and $x \in \mathbb{R}^n$. The following holds.

1. Let $\varepsilon > 0$. If x is (k, ε) - ℓ_p -compressible then $\Delta_{q,p}(x) \geq \frac{1}{\left(\frac{k}{n}\right)^{1/q-1/p} + \varepsilon}$.
2. The vector x is $\left(k, \frac{\left(\frac{n}{k}\right)^{1/q}}{\Delta_{q,p}(x)}\right)$ - ℓ_p -compressible.

We can now state the full version of the main theorems and prove that they imply their specialized versions from Section 1.

3 ℓ_2 -RIP Matrices Cannot Be Sparse

In this section, we prove Theorem 3, stated below, which is the formal version of Theorem 1.

Theorem 3. Fix $m, n \in \mathbb{N}$. Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank $r = \alpha n$ for some $0 < \alpha < 1$. Suppose that A is (k, D) - ℓ_2 -RIP for some $k \geq 1$ and $D \geq 1$. Fix $\eta > 0$ such that $\frac{C_1 D^8}{(1-\alpha)^4} \leq n^\eta \leq \frac{C_2 k^3}{D^4 n^2}$, where C_1, C_2 are universal positive constants. Then, there is a submatrix $T \in \mathbb{R}^{t \times n}$ of rows of A , for some $t \leq m$, with the following properties.

1. $\frac{\|T\|_F^2}{\|A\|_F^2} \geq n^{-\eta}$.

2. $t \geq \Omega\left(\frac{k^3}{D^4 n^2 + \eta}\right)$.

3. Every row x of T satisfies

$$\Delta_{1,2}(x) \leq O\left(\left(\frac{D^3 n}{\eta \sqrt{1-\alpha} \log n}\right)^{1/2}\right).$$

In particular, every row is $\sqrt{\frac{s}{n}}$ -far from all s -sparse vectors, where $s = \Omega\left(\frac{\eta \sqrt{1-\alpha} \log n}{D^3}\right)$.

3.1 The Analytic Restricted Isometry Property and its equivalence to RIP

To prove Theorem 3, we introduce the Analytic Restricted Isometry Property (ARIP) — a natural strengthening of the RIP where we require that $\|Ax\|_2 \approx \|x\|_2$ for not just all k -sparse x , but all x that is “analytically k -sparse”, i.e., $\Delta_{1,2}(x) \geq (n/k)^{1/2}$.⁴ Using Proposition 2.4, it is straightforward to show that the two notions are related, and are essentially equivalent in a suitable parameter regime. We then prove Theorem 3 for matrices that are ARIP, and use the equivalence to complete the proof.

We now define the ARIP below. For simplicity, we will focus here on the ℓ_2 -norm, although the definition makes sense for any ℓ_p -norm as well.

Definition 3.1. A matrix $A \in \mathbb{R}^{m \times n}$ is (k, D) - ℓ_2 -ARIP if there exists $K > 0$ such that for every $x \in \mathbb{R}^n$ with $\Delta_{1,2}(x) \geq \left(\frac{n}{k}\right)^{1/2}$ it holds that

$$K \|x\|_2 \leq \|Ax\|_2 \leq D \cdot K \|x\|_2 .$$

By Eq. (2), ARIP immediately implies RIP. As we show next, a reverse implication also holds at a certain cost to the parameters.

Proposition 3.2 (Equivalence between RIP and ARIP). *Fix $k > 0$ and $D \geq 1$, and let $A \in \mathbb{R}^{m \times n}$. The following holds.*

1. If A is (k, D) - ℓ_2 -ARIP then it is also (k, D) - ℓ_2 -RIP.
2. Let $D' > D$ and suppose that A is (k, D) - ℓ_2 -RIP. Then, A is (k', D') - ℓ_2 -ARIP for $k' = \frac{(D'-D)^2 k^3}{(D'D+D'+D)^2 n^2}$. In particular, if $k \geq \Omega(n)$ and $D \leq O(1)$ then A is $(\Omega(n), 2D)$ - ℓ_2 -ARIP.

Proof. The first claim follows immediately from Eq. (2). We turn to proving the second claim.

We assume without loss of generality that A satisfies Definition 1.1 with $K = 1$. Fix $k' > 0$. Let $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$ and suppose that $\Delta_{1,2}(x) \geq \left(\frac{n}{k'}\right)^{1/2}$. Our goal is to show that

$$1 - (D + 1) \cdot \frac{n \cdot k'^{1/2}}{k^{3/2}} \leq \|Ax\|_2 \leq D \left(1 + \frac{n \cdot k'^{1/2}}{k^{3/2}}\right) . \quad (3)$$

Eq. (3) yields the claim, since, taking k' as in the proposition statement, the ratio between the right-hand side and the left-hand side of Eq. (3) becomes at most D' . This implies that A is (k', D') - ℓ_2 -ARIP. We turn to proving Eq. (3).

⁴Note that if x is k -sparse, then $\|x\|_1 \leq \sqrt{k} \|x\|_2$, which implies that $\Delta_{1,2}(x) \geq (n/k)^{1/2}$.

Suppose, without loss of generality, that the entries of x are sorted in order of non-increasing absolute value. Write $x = \sum_{j=1}^{\lceil n/k \rceil} y^j$, where y^j is the k -sparse vector defined by

$$y_i^j = \begin{cases} x_i & \text{if } (j-1)k < i \leq jk \\ 0 & \text{otherwise.} \end{cases}$$

Denote $y' = x - y^1 = \sum_{j=2}^{\lceil n/k \rceil} y^j$. By Proposition 2.4, x is $(k, \frac{\sqrt{nk'}}{k})$ - ℓ_2 -compressible, so Remark 2.3 yields

$$\|y'\|_2 = \|x - y^1\|_2 \leq \frac{\sqrt{nk'}}{k}. \quad (4)$$

Now, by the triangle inequality,

$$\|Ay^1\|_2 - \|Ay'\|_2 \leq \|Ax\|_2 \leq \|Ay^1\|_2 + \|Ay'\|_2. \quad (5)$$

By the RIP assumption,

$$1 - \|y'\|_2 = \|x\|_2 - \|y'\|_2 \leq \|y^1\|_2 \leq \|Ay^1\|_2 \leq D \|y^1\|_2 \leq D \|x\|_2 = D. \quad (6)$$

Also, by the RIP assumption and Hölder's inequality,

$$\|Ay'\|_2 \leq \sum_{j=2}^{\lceil n/k \rceil} \|Ay^j\|_2 \leq D \sum_{j=2}^{\lceil n/k \rceil} \|y^j\|_2 \leq D \left(\frac{n}{k}\right)^{1/2} \|y'\|_2. \quad (7)$$

Together, Eqs. (5) to (7) yield

$$1 - (D+1) \left(\frac{n}{k}\right)^{1/2} \|y'\|_2 \leq 1 - \left(1 + D \left(\frac{n}{k}\right)^{1/2}\right) \|y'\|_2 \leq \|Ax\|_2 \leq D + D \left(\frac{n}{k}\right)^{1/2} \|y'\|_2.$$

Eq. (3) follows from the above and Eq. (4). \square

We shall now state Theorem 4 – a version of Theorem 3 for ARIP matrices – and immediately prove that the former implies the latter. The rest of this section will be devoted to proving Theorem 4.

Theorem 4 (Theorem 3 for ARIP matrices). *Fix $m, n \in \mathbb{N}$. Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank $r = \alpha n$ for some $0 < \alpha < 1$. Suppose that A is (k, D) - ℓ_2 -ARIP for some $k \geq 1$ and $D \geq 1$. Fix $\eta > 0$ such that $\frac{64D^8}{(1-\alpha)^4} \leq n^\eta \leq \frac{k}{D^2}$. Then, there is a submatrix $T \in \mathbb{R}^{t \times n}$ of rows of A , for some $t \leq m$, with the following properties.*

1. $\frac{\|T\|_F^2}{\|A\|_F^2} \geq n^{-\eta}$.
2. $t \geq \frac{k}{D^2 n^\eta}$.

3. Every row x of T satisfies

$$\Delta_{1,2}(x) \leq O\left(\left(\frac{D^3 n}{\eta \sqrt{1-\alpha} \log n}\right)^{1/2}\right). \quad (8)$$

In particular, every row is $\sqrt{\frac{s}{n}}$ -far from all s -sparse vectors, where $s = \Omega\left(\frac{\eta \sqrt{1-\alpha} \log n}{D^3}\right)$.

Proof of Theorem 3 given Theorem 4. Let $A \in \mathbb{R}^{n \times m}$ be (k, D) - ℓ_2 -RIP. By Proposition 3.2, A is (k', D') - ℓ_2 -ARIP for $D' = 2D$ and $k' = \frac{D^2 k^3}{(2D^2 + 3D)^2 n^2} \geq \Omega\left(\frac{k^3}{D^2 n^2}\right)$. The conclusion of Theorem 3 follows by applying Theorem 4 to A . \square

3.2 Distortion bounds for ℓ_2 -ARIP matrices

We next develop the necessary tools to prove Theorem 4. The following lemma states several simple but useful facts about ARIP matrices.

Lemma 3.3. *Let $A \in \mathbb{R}^{m \times n}$ be (k, D) - ℓ_2 -ARIP. Assume that $\|A\|_F = \sqrt{n}$. The following then holds.*

1. There exist some $i_{\text{less}}, i_{\text{more}} \in \{1, \dots, n\}$ such that $\|Ae_{i_{\text{less}}}\|_2 \leq 1 \leq \|Ae_{i_{\text{more}}}\|_2$.
2. Let $\Pi \in \mathbb{R}^{n \times n}$ be the projection matrix for the orthogonal projection onto $\ker(A)$. Then, there exists some $i_{\ker} \in \{1, \dots, n\}$ such that $\|\Pi e_{i_{\ker}}\|_2 \geq \sqrt{1 - \frac{\text{rank}(A)}{n}}$.
3. For every $x \in \mathbb{R}^n$ with $\Delta_{1,2}(x) \geq \left(\frac{n}{k}\right)^{1/2}$, it holds that $\frac{\|x\|_2}{D} \leq \|Ax\|_2 \leq D \|x\|_2$.
4. Each row of A is of ℓ_2 -norm at most $D \left(\frac{n}{k}\right)^{1/2}$.
5. Each column of A is of ℓ_2 -norm at most D .

Proof. We begin with Items 1 and 2. Let i' be uniformly sampled from $\{1, \dots, n\}$. Then,

$$\mathbb{E}\|Ae_{i'}\|_2^2 = \frac{1}{n} \sum_{i=1}^n \|Ae_i\|_2^2 = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^m A_{i,j}^2 = \frac{\|A\|_F^2}{n} = 1.$$

which yields Item 1 since there must exist choices of i' for which $\|Ae_{i'}\|_2^2$ is at most (resp. at least) its expectation. Item 2 follows similarly from

$$\mathbb{E}\|\Pi e_{i'}\|_2^2 = \frac{\|\Pi\|_F^2}{n} = \frac{\dim(\ker(A))}{n} = 1 - \frac{\text{rank}(A)}{n}.$$

For Item 3 recall that according to the ARIP assumption, there is some $K > 0$ such that

$$K \|x\|_2 \leq \|Ax\|_2 \leq D \cdot K \|x\|_2$$

for all $x \in \mathbb{R}^n$ with $\Delta_{1,2}(x) \geq \left(\frac{n}{k}\right)^{1/2}$. It thus suffices to show that $K \leq 1 \leq DK$. We use Item 1, substituting e_{less} for x . This yields

$$K = K \|e_{\text{less}}\|_2 \leq \|Ae_{\text{less}}\|_2 \leq 1.$$

Similarly,

$$KD = KD \|e_{\text{more}}\|_2 \geq \|Ae_{\text{more}}\|_2 \geq 1 ,$$

proving the claim.

We turn to Item 4. Suppose towards contradiction that A has a row A_{i_*} with $\|A_{i_*}\|_2 > D \cdot \sqrt{\frac{n}{k}}$. Let J be a set of k coordinates such that the respective entries of A_{i_*} are the k largest in absolute value. Define $b \in \mathbb{R}^n$ by

$$b_j = \begin{cases} A_{i_*,j} & \text{if } j \in J \\ 0 & \text{otherwise.} \end{cases}$$

In particular, b is k -sparse and so $\Delta_{1,2}(b) \geq \left(\frac{n}{k}\right)^{1/2}$ due to Eq. (2). Hence,

$$\frac{\|Ab\|_2}{\|b\|_2} \geq \frac{(Ab)_i}{\|b\|_2} = \frac{\langle b, b \rangle}{\|b\|_2} = \|b\|_2 = \left(\sum_{j \in J} A_{i_*,j}^2 \right)^{1/2} \geq \left(\frac{k}{n} \sum_{j=1}^n A_{i_*,j}^2 \right)^{1/2} = \sqrt{\frac{k}{n}} \cdot \|A_{i_*}\|_2 > D ,$$

which contradicts Item 3.

Finally, for Item 5, observe that Item 3 yields $\|A_{*,i}\|_2 = \|Ae_i\|_2 \leq D$ for all $1 \leq i \leq n$. \square

With Lemma 3.3, we are now ready to prove our key technical lemma, Lemma 3.4. This lemma contains the technical core of our argument.

Lemma 3.4. *Fix $m, n \in \mathbb{N}$. Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank $r = \alpha n$ for some $0 < \alpha < 1$. Suppose that A is (k, D) - ℓ_2 -ARIP for some $k \geq 1$ and $D \geq 1$. Then,*

$$\frac{\|A^\top A\|_{\ell_1 \rightarrow \ell_1}}{\|A\|_F^2} \geq \frac{e\sqrt{1-\alpha} \ln(k(1-\alpha))}{Dn} . \quad (9)$$

In particular, for $D \leq O(1)$, $1 - \alpha \geq \Omega(1)$ and $k \geq \Omega(n)$ we have

$$\frac{\|A^\top A\|_{\ell_1 \rightarrow \ell_1}}{\|A\|_F^2} \geq \Omega\left(\frac{\log n}{n}\right) . \quad (10)$$

Remark 3.5. To understand the statement of the lemma, it is helpful to consider the special case where A is an $(\Omega(n), O(1))$ - ℓ_2 -ARIP matrix with entries of magnitude 1, $m = \Theta(n)$, and exactly s (resp. t) nonzero entries per row (resp. column). For such a matrix A , each entry of $A^\top A$ is at most st , and $\|A\|_F^2 = tn$. Eq. (10) then implies that $s \geq \Omega(\log n)$. In particular, A cannot have constant row sparsity.

Proof. Let B denote the positive semi-definite matrix $A^\top A \in \mathbb{R}^{n \times n}$. Write $\lambda_1, \dots, \lambda_n$ for the eigenvalues of B , and v_1, \dots, v_n for the respective eigenvectors. Note that B has exactly r nonzero eigenvalues, so we may assume that $\lambda_1, \dots, \lambda_r$ are positive, while $\lambda_{r+1} = \dots = \lambda_n = 0$. We further assume, without loss of generality, that $\|A\|_F = \sqrt{n}$, and consequently, $\text{tr}(B) = \sum_{i=1}^n \lambda_i = \|A\|_F^2 = n$.

Eq. (10) clearly follows from Eq. (9), so it suffices to prove the latter. Our proof proceeds as outlined in Section 1.2.

Let $\Pi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the orthogonal projection map onto $\ker(A)$. By Lemma 3.3(2), there exists some $1 \leq i_{\ker} \leq n$ such that $\|\Pi e_{i_{\ker}}\|_2 \geq \sqrt{1-\alpha}$. We take $x = e^{-tB} e_{i_{\ker}}$ for some $t > 0$ to be determined later. Recall that e^{-tB} is defined to be the matrix $\sum_{j=0}^{\infty} \frac{(-tB)^j}{j!}$. We claim that x satisfies the following properties.

1. $\|x\|_2 \geq \sqrt{1-\alpha}$
2. $\|x\|_1 \leq e^{t\|B\|_{\ell_1 \rightarrow \ell_1}}$
3. $\|Ax\|_2 \leq \frac{1}{2te}$

Before proving these properties, we show that they yield Eq. (9) and, consequently, the lemma. Take $t = \frac{\ln(\sqrt{k(1-\alpha)})}{\|B\|_{\ell_1 \rightarrow \ell_1}}$. Then,

$$\Delta_{1,2}(x) = \frac{\sqrt{n} \|x\|_2}{\|x\|_1} \geq \frac{\sqrt{n(1-\alpha)}}{e^{t\|B\|_{\ell_1 \rightarrow \ell_1}}} = \sqrt{\frac{n}{k}}.$$

Therefore, by Lemma 3.3(3),

$$D \geq \frac{\|x\|_2}{\|Ax\|_2} \geq 2te\sqrt{1-\alpha} = \frac{e\sqrt{1-\alpha} \ln(k(1-\alpha))}{\|B\|_{\ell_1 \rightarrow \ell_1}} = \frac{e\sqrt{1-\alpha} \ln(k(1-\alpha))}{\|A^\top A\|_{\ell_1 \rightarrow \ell_1}}.$$

Eq. (9) follows due to the assumption that $\|A\|_F = \sqrt{n}$.

We turn to proving Properties 1 to 3. For $1 \leq j \leq n$, denote $a_j = \langle e_{i_{\ker}}, v_j \rangle$. Observe that $x = \sum_{j=1}^n e^{-t\lambda_j} a_j v_j$. Hence,

$$\|x\|_2^2 = \sum_{j=1}^n e^{-2t\lambda_j} a_j^2 \geq \sum_{j=r+1}^n e^{-2t\lambda_j} a_j^2 = \sum_{j=r+1}^n a_j^2 = \|\Pi e_{i_{\ker}}\|_2^2 \geq 1-\alpha,$$

proving Property 1.

For Property 2, we have

$$\|x\|_1 = \|e^{tB} e_{i_{\ker}}\|_1 \leq \|e^{tB}\|_{\ell_1 \rightarrow \ell_1} \|e_{i_{\ker}}\|_1 = \|e^{tB}\|_{\ell_1 \rightarrow \ell_1} \leq \sum_{j=0}^{\infty} \frac{t^j \|B\|_{\ell_1 \rightarrow \ell_1}^j}{j!} = e^{t\|B\|_{\ell_1 \rightarrow \ell_1}}.$$

Finally, for Property 3 we use the inequality

$$\lambda \cdot e^{-2t\lambda} \leq \frac{1}{2te} \tag{11}$$

for all $\lambda \geq 0$, which is readily verified via derivation of the left-hand side by λ . Eq. (11) yields Property 3 since

$$\|Ax\|_2^2 = x^\top Bx = \sum_{j=1}^n \lambda_j a_j^2 e^{-2t\lambda_j} \leq \frac{1}{2te} \sum_{j=1}^n a_j^2 = \frac{1}{2te}.$$

This finishes the proof. \square

Our next lemma, Lemma 3.6 below, strengthens Lemma 3.4 by replacing $\|B\|_{\ell_1 \rightarrow \ell_1}$ with a sharper quantity. Concretely, it could be the case that $\|B\|_{\ell_1 \rightarrow \ell_1}$ is large because a small number of columns of B have large ℓ_1 -norm. By removing these columns from A , we can obtain a submatrix A' of A that is still ARIP but has $\|B'\|_{\ell_1 \rightarrow \ell_1}$ smaller than $\|B\|_{\ell_1 \rightarrow \ell_1}$, and this idea yields Lemma 3.6.

Lemma 3.6. *Fix m, n, \mathbb{N} and $s > 0$. Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank αn for some $0 < \alpha < 1$. Suppose that A is (k, D) - ℓ_2 -ARIP for some $k \geq 1$ and $D \geq \frac{1}{\sqrt{1-\alpha}}$ such that $k(1-\alpha) \geq 2$. Then,*

$$\frac{\sum_{i=1}^m \|A_{i\star}\|_1^2}{\|A\|_F^2} \geq \Omega\left(\frac{\sqrt{1-\alpha} \log(k(1-\alpha))}{D^3}\right). \quad (12)$$

In particular, if $D \leq O(1)$, $1-\alpha \geq \Omega(1)$ and $k \geq \Omega(n)$ then it must hold that

$$\frac{\sum_{i=1}^m \|A_{i\star}\|_1^2}{\|A\|_F^2} \geq \Omega(\log n).$$

Proof. Assume without loss of generality that $\|A\|_F = \sqrt{n}$. Write $B = A^\top A \in \mathbb{R}^{n \times n}$. Write $W = \sum_{i=1}^m \|A_{i\star}\|_1^2$ and $H = \frac{2WD^2}{n}$. Let $J = \{j \in \{1, \dots, n\} \mid \|B_{\star j}\|_1 \geq H\}$ indicate the ℓ_1 -heavy columns of B . Let $A' \in \mathbb{R}^{m \times (n-|J|)}$ be the matrix A without the columns indicated by J . Note that the ARIP property is preserved by column-removal operations, so A' is (k, D) - ℓ_2 -ARIP as well. Write $\alpha' = \frac{\text{rank}(A')}{n-|J|}$. Lemma 3.4 yields

$$\frac{\|A'^\top A'\|_{\ell_1 \rightarrow \ell_1}}{\|A'\|_F^2} \geq \frac{e\sqrt{1-\alpha'} \ln(k(1-\alpha'))}{D(n-|J|)} \geq \frac{e\sqrt{1-\alpha'} \ln(k(1-\alpha'))}{Dn}. \quad (13)$$

To deduce the lemma, we shall prove that Eq. (13) implies Eq. (12). To do so, we bound some of the terms involved in Eq. (13).

Let $B' = A'^\top A'$ and note that B' is the result of removing from B the rows and columns indicated by J . Note that

$$\|A'^\top A'\|_{\ell_1 \rightarrow \ell_1} = \|B'\|_{\ell_1 \rightarrow \ell_1} = \max_{1 \leq j \leq n-|J|} \|B'_{\star j}\|_1 \leq \max_{j \in \{1, \dots, n\} \setminus J} \|B_{\star j}\|_1 \leq \frac{2WD^2}{n}.$$

We next bound $|J|$. Observe that

$$|J| \leq \frac{\sum_{j=1}^n \|B_{\star j}\|_1}{H} = \frac{\sum_{j=1}^n \sum_{i=1}^n |B_{i,j}|}{H} = \frac{\sum_{j=1}^n \sum_{j'=1}^n |\langle A_{\star j}, A_{\star j'} \rangle|}{H} \leq \frac{\sum_{j=1}^n \sum_{j'=1}^n \sum_{i=1}^m |A_{i,j}| |A_{i,j'}|}{H} = \frac{W}{H} = \frac{n}{2D^2}.$$

Also, by Lemma 3.3(5),

$$B_{i,i} = \|A_{\star i}\|_2^2 \leq D^2$$

for all $1 \leq i \leq n$. Hence,

$$\|A'\|_F^2 = \text{tr}(B') \geq \text{tr}(B) - |J|D^2 = n - |J|D^2 \geq \frac{n}{2}.$$

Next, observe that

$$1 - \alpha' = 1 - \frac{\text{rank } B'}{n - |J|} \geq 1 - \frac{\alpha n}{n - |J|} \geq 1 - \frac{\alpha n}{n - \frac{n}{2D^2}} = 1 - \frac{\alpha}{1 - \frac{1}{2D^2}} \geq 1 - \frac{\alpha}{1 - \frac{1-\alpha}{2}} \geq \frac{1-\alpha}{2} .$$

Eq. (13) therefore yields

$$\frac{2WD^2}{n^2} \geq \frac{\frac{e}{\sqrt{2}} \cdot \sqrt{1-\alpha} \left(\log(k(1-\alpha)) - \log \sqrt{2} \right)}{Dn} .$$

Eq. (12) follows since

$$\frac{\sum_{i=1}^m \|A_{i*}\|_1^2}{\|A\|_F^2} = \frac{W}{n} \geq \Omega \left(\frac{\sqrt{1-\alpha} \log(k(1-\alpha))}{D^3} \right) . \quad \square$$

The final tool needed for the proof of Theorem 4 is the following *row removal lemma*. Notice that ARIP is preserved by the removal of columns and addition of rows. In other words, removing rows makes it “harder” for a matrix to be ARIP, while removing columns makes it “easier”. In Lemma 3.7 we show it is possible to remove any “not too heavy” set of rows and preserve ARIP, if one also removes a suitable set of columns.

Lemma 3.7 (Row removal lemma). *Let $A \in \mathbb{R}^{m \times n}$ be (k, D) - ℓ_2 -ARIP. Fix $k' \leq k$ and $\delta > 0$. Fix a set $I \subseteq \{1, \dots, m\}$, and let A_I denote the restriction of A to the row set I . Then, there exists a column set $J \subseteq \{1, \dots, n\}$ with*

$$|J| \leq \frac{nD^2 \|A_I\|_F^2}{\delta^2 \|A\|_F^2}$$

such that the matrix $A_{I,J}$, defined as the matrix A without the row set I and the column set J , is $\left(k', \frac{D}{\sqrt{1-k'\delta^2}}\right)$ - ℓ_2 -ARIP.

Proof. Assume without loss of generality that $\|A\|_F = \sqrt{n}$. Fix $K > 0$ such that

$$K \|x\|_2 \leq \|Ax\|_2 \leq D \cdot K \|x\|_2$$

for all $x \in \mathbb{R}^n$ with $\Delta_{1,2}(x) \geq \left(\frac{n}{k}\right)^{1/2}$. By Lemma 3.3(3), we have $\frac{1}{D} \leq K \leq 1$.

Let $J \subseteq \{1, \dots, n\}$ indicate the columns in A_I whose ℓ_2 -norm is larger than $K\delta$. Observe that

$$|J| \leq \frac{\|A_I\|_F^2}{\delta^2 K^2} = \frac{n \|A_I\|_F^2}{K^2 \delta^2 \|A\|_F^2} \leq \frac{nD^2 \|A_I\|_F^2}{\delta^2 \|A\|_F^2} .$$

Define B to be the matrix A without the row set I and the column set J . We need to show that B is $\left(k', \frac{D}{\sqrt{1-k'\delta^2}}\right)$ - ℓ_2 -ARIP. Let $A_{\bar{I}}$ denote the restriction of A to the rows indicated by $\{1, \dots, n\} \setminus I$. Let $x \in \mathbb{R}^n$ have $\|x\|_2 = 1$ and $\|x\|_1 \leq k'$, so that $\Delta_{1,2}(x) \geq \left(\frac{n}{k'}\right)^{1/2}$. Further assume that $x_j = 0$ for all $j \in J$. Note that it suffices to show that

$$K\sqrt{1-k'\delta^2} \leq \|A_{\bar{I}}x\|_2 \leq KD . \quad (14)$$

The right-hand inequality holds since $\Delta_{1,2}(x) \geq \left(\frac{\mu}{k'}\right)^{1/2} \geq \left(\frac{\mu}{k}\right)^{1/2}$, implying that $\|A_{\bar{I}}x\|_2 \leq \|Ax\|_2 \leq KD$. For the left-hand inequality of Eq. (14), we first note that

$$\|A_{\bar{I}}x\|_2^2 = \|Ax\|_2^2 - \|A_Ix\|_2^2 \geq K^2 - \|A_Ix\|_2^2. \quad (15)$$

Now, let c_1, \dots, c_n denote the columns of A_I , and recall that $\|c_j\|_2 \leq K\delta$ for all $j \in \{1, \dots, n\} \setminus J$. Consequently,

$$\begin{aligned} \|A_Ix\|_2 &= \left\| \sum_{j=1}^n x_j c_j \right\|_2 \leq \sum_{j=1}^n |x_j| \|c_j\|_2 = \sum_{j \in \{1, \dots, n\} \setminus J} |x_j| \|c_j\|_2 \leq \|x\|_1 \cdot \max_{j \in \{1, \dots, n\} \setminus J} \{\|c_j\|_2\} \leq \|x\|_1 \cdot K\delta \\ &\leq \sqrt{k'} \cdot K\delta. \end{aligned}$$

The left-hand inequality of Eq. (14) now follows from the above and Eq. (15). \square

3.3 Proof of Theorem 4

We finally turn to proving Theorem 4 using Lemmas 3.6 and 3.7.

Proof of Theorem 4. Suppose, without loss of generality, that $\|A\|_F = \sqrt{n}$ and that the rows A_{1*}, \dots, A_{m*} are sorted so that $\Delta_{1,2}(A_{i*})$ is non-decreasing in i . Let $1 \leq t \leq m$ be the minimal integer for which $\sum_{i=1}^t \|A_{i*}\|_2^2 \geq n^{1-\eta}$. We take T to be the matrix whose rows are A_{1*}, \dots, A_{t*} . By definition, $\frac{\|T\|_F^2}{\|A\|_F^2} \geq n^{-\eta}$. By Lemma 3.3(4),

$$n^{1-\eta} \leq \sum_{i=1}^t \|A_{i*}\|_2^2 \leq \frac{tD^2n}{k},$$

implying that $t \geq \frac{k}{D^2n^\eta}$. This proves that T satisfies Properties 1 and 2.

To prove Property 3, it suffices to show that

$$\Delta_{1,2}(A_{(t+1)*}) \leq O\left(\left(\frac{D^3n}{\eta\sqrt{1-\alpha}\log n}\right)^{1/2}\right). \quad (16)$$

Indeed, Eq. (16) implies Eq. (8) since $\Delta_{1,2}(A_{i*})$ is non-decreasing in i .

Let $A' \in \mathbb{R}^{(m-t) \times n}$ be the matrix whose rows are $A_{(t+1)*}, \dots, A_{m*}$. We apply Lemma 3.7 to the matrix A , with $I = \{1, \dots, t\}$, $k' = \frac{(1-\alpha)n^\eta}{8D^4}$ and $\delta = \sqrt{\frac{1}{2k'}}$. The lemma yields a $(k', \sqrt{2} \cdot D)$ - ℓ_2 -ARIP submatrix $S \in \mathbb{R}^{(m-t) \times (n-w)}$ of A' , where

$$w \leq \frac{nD^2\|T\|_F^2}{\delta^2\|A\|_F^2} = \frac{D^2\|T\|_F^2}{\delta^2} = 2D^2\|T\|_F^2k'.$$

By the minimality of t and Lemma 3.3(4),

$$\|T\|_F^2 \leq n^{1-\eta} + \|A_{t*}\|_2^2 \leq n^{1-\eta} + \frac{D^2n}{k} \leq 2n^{1-\eta},$$

where the last step uses the hypothesis $k \geq n^\eta D^2$. We therefore have

$$w \leq 4D^2 n^{1-\eta} k' = \frac{(1-\alpha)n}{2D^2} .$$

Let

$$\alpha' = \frac{\text{rank}(S)}{n-w} \leq \frac{\text{rank}(A)}{n-w} = \frac{\alpha n}{n-w}$$

and note that

$$1 - \alpha' \geq 1 - \frac{\alpha n}{n-w} \geq 1 - \frac{\alpha}{1 - \frac{1-\alpha}{2D^2}} \geq 1 - \frac{\alpha}{1 - \frac{1-\alpha}{2}} \geq \frac{1-\alpha}{2} .$$

Therefore, Lemma 3.6 yields

$$\frac{\sum_{i=1}^{m-t} \|S_{i*}\|_1^2}{\|S\|_F^2} \geq \Omega\left(\frac{\sqrt{1-\alpha'} \log(k'(1-\alpha'))}{D^3}\right) \geq \Omega\left(\frac{\sqrt{1-\alpha} \log\left(\frac{(1-\alpha)^2 n^\eta}{8D^4}\right)}{D^3}\right) \geq \Omega\left(\frac{\eta \sqrt{1-\alpha} \log n}{D^3}\right) . \quad (17)$$

Let $c_1, \dots, c_w \in \mathbb{R}^m$ denote the columns of A that are missing from S . By Lemma 3.3(5), $\|c_j\|_2 \leq D$ for all $1 \leq j \leq w$. Hence,

$$\|S\|_F^2 \geq \|A\|_F^2 - \|T\|_F^2 - \sum_{j=1}^w \|c_j\|_2^2 \geq n - 2n^{1-\eta} - wD^2 \geq n - 2n^{1-\eta} - \frac{n}{2} \geq \frac{n}{4} .$$

Consequently,

$$\begin{aligned} \frac{\sum_{i=1}^{m-t} \|S_{i*}\|_1^2}{\|S\|_F^2} &\leq \frac{4 \sum_{i=1}^{m-t} \|S_{i*}\|_1^2}{n} \leq \frac{4 \sum_{i=t+1}^m \|A_{i*}\|_1^2}{n} = \sum_{i=t+1}^m \frac{4 \|A_{i*}\|_2^2}{\Delta_{1,2}(A_{i*})^2} \leq \sum_{i=t+1}^m \frac{4 \|A_{i*}\|_2^2}{\Delta_{1,2}(A_{(t+1)*})^2} \\ &\leq \frac{4 \|A\|_F^2}{\Delta_{1,2}(A_{(t+1)*})^2} = \frac{4n}{\Delta_{1,2}(A_{(t+1)*})^2} . \end{aligned}$$

Eq. (16), which yields Property 3, follows from the above and Eq. (17). \square

4 For $p \neq 2$, ℓ_p -RIP Matrices Must Be Sparse

In this section, we prove Theorem 2, which we formally state below.

Theorem 5. *Let $A \in \mathbb{R}^{m \times n}$ be a (k, D) - ℓ_p -RIP matrix, and let A_{1*}, \dots, A_{m*} denote the rows of A . Then, if $1 \leq p < 2$, it holds that*

$$D^p \left(\frac{n}{k}\right)^{p(\frac{1}{p}-\frac{1}{2})} \sum_{i=1}^m \|A_{i*}\|_2^p \geq \sum_{i=1}^m \|A_{i*}\|_p^p ,$$

and if $p > 2$ it holds that

$$\sum_{i=1}^m \|A_{i*}\|_2^p \leq D^p \left(\frac{n}{k}\right)^{p(\frac{1}{2}-\frac{1}{p})} \sum_{i=1}^m \|A_{i*}\|_p^p .$$

We will need the following simple claim.

Claim 4.1. Suppose X, Y are non-negative random variables, and $\Pr[Y = 0] = 0$. Then, $\Pr[X/Y \leq \mathbb{E}[X]/\mathbb{E}[Y]] > 0$ and $\Pr[X/Y \geq \mathbb{E}[X]/\mathbb{E}[Y]] > 0$.

Proof. Let $\alpha = \mathbb{E}[X]/\mathbb{E}[Y]$. Then, $\mathbb{E}[X - \alpha Y] = 0$. So, $\Pr[X - \alpha Y \leq 0] > 0$. Therefore, $\Pr[X - \alpha Y \leq 0 \wedge Y > 0] > 0$, and so $\Pr[X/Y \leq \alpha] > 0$. Similarly, $\Pr[X - \alpha Y \geq 0] > 0$, and so $\Pr[X - \alpha Y \geq 0 \wedge Y > 0] > 0$, which implies $\Pr[X/Y \geq \alpha] > 0$. This finishes the proof. \square

Proof of Theorem 5. For $j \in [n]$, let e_j denote the j -th standard basis vector. Without loss of generality, we shall assume that A satisfies Definition 1.1 with $K = 1$; otherwise, we can rescale A so that this holds. Let A_{*1}, \dots, A_{*n} be the columns of A . Observe that $\|Ae_j\|_p = \|A_{*j}\|_p$ for all $j \in [n]$. As $k \geq 1$, we thus have that $1 \leq \|A_{*j}\|_p \leq D$, for all $j \in [n]$. It thus follows that $n \leq \sum_{i=1}^m \|A_{i*}\|_p^p = \sum_{j=1}^n \|A_{*j}\|_p^p \leq nD^p$.

Now, let $S \subseteq [n]$, $|S| \leq k$. For $j \in S$, let $g_j \sim N(0, 1)$, and let $x = \sum_{j \in S} g_j e_j$. Note that $x \in \mathbb{R}^S$. We observe that $\|Ax\|_p^p$ and $\|x\|_p^p$ are nonnegative random variables, and $\Pr[\|x\|_p^p = 0] = 0$.

Next, we note that if $g \sim N(0, 1)$, then $\mathbb{E}[|g|^p] = \frac{2^{p/2}}{\sqrt{\pi}} \cdot \Gamma(\frac{1+p}{2}) =: f(p)$. By linearity of expectation, it then follows that $\mathbb{E}[\|x\|_p^p] = f(p)|S|$, and that $\mathbb{E}[\|Ax\|_p^p] = f(p) \sum_{i=1}^m \|A_{i,S}\|_2^p$, where $A_{i,S}$ denotes the i -th row restricted to the coordinates in S .

We now have two cases.

Case 1: $p < 2$. Applying Claim 4.1, we see that there exists $y \in \mathbb{R}^S \setminus \{0^S\}$ such that $\|Ay\|_p^p / \|y\|_p^p \leq \mathbb{E}[\|Ax\|_p^p] / \mathbb{E}[\|x\|_p^p]$.

It follows that there exists $y \in \mathbb{R}^S \setminus \{0^S\}$ with $\|y\|_p = 1$ such that $\|Ay\|_p^p \leq \sum_{i=1}^m \|A_{i,S}\|_2^p / |S|$. On the other hand, because y is k -sparse, we see that $\|Ay\|_p^p \geq 1$. Hence,

$$\sum_{i=1}^m \|A_{i,S}\|_2^p \geq |S|, \quad (18)$$

for every S of size at most k .

Now, fix i , and let X denote the random variable $\|A_{i,S}\|_2^p$, with randomness over the draw of $S \subseteq [n]$, $|S| = k$. By Hölder's inequality (and using that $p < 2$), we have $\mathbb{E}[X] \leq \mathbb{E}[X^{2/p}]^{p/2}$. Now, $\mathbb{E}[X^{2/p}] = \mathbb{E}_S[\|A_{i,S}\|_2^2] = \frac{k}{n} \|A_{i*}\|_2^2$. This is because each coordinate of n appears in a randomly chosen S with probability $\frac{k}{n}$. It thus follows that $\mathbb{E}[X] \leq (k/n)^{p/2} \|A_{i*}\|_2^p$.

Taking expectations of Eq. (18) over the choice of $|S| = k$, we now have that

$$(k/n)^{p/2} \sum_{i=1}^m \|A_{i*}\|_2^p \geq k.$$

Combining with the inequality $\sum_{i=1}^m \|A_{i*}\|_p^p \leq nD^p$, we thus have

$$(k/n)^{p/2} \sum_{i=1}^m \|A_{i*}\|_2^p \geq k \geq \frac{k}{n} D^{-p} \sum_{i=1}^m \|A_{i*}\|_p^p$$

$$\implies D^p \left(\frac{n}{k}\right)^{p(\frac{1}{p}-\frac{1}{2})} \sum_{i=1}^m \|A_{i\star}\|_2^p \geq \sum_{i=1}^m \|A_{i\star}\|_p^p ,$$

as required.

Case 2: $p > 2$. Applying Claim 4.1, we see that exists $y \in \mathbb{R}^S \setminus \{0^S\}$ such that $\|Ay\|_p^p / \|y\|_p^p \geq \mathbb{E}[\|Ax\|_p^p] / \mathbb{E}[\|x\|_p^p]$.

It follows that there exists $y \in \mathbb{R}^S \setminus \{0^S\}$ with $\|y\|_p = 1$ such that $\|Ay\|_p^p \geq \sum_{i=1}^m \|A_{i,S}\|_2^p / |S|$. On the other hand, because y is k -sparse, we see that $\|Ay\|_p^p \leq D^p$. Hence,

$$\sum_{i=1}^m \|A_{i,S}\|_2^p \leq |S| D^p , \quad (19)$$

for every S of size at most k .

Now, fix i , and let X denote the random variable $\|A_{i,S}\|_2^p$, with randomness over the draw of $S \subseteq [n]$, $|S| = k$. By Hölder's inequality (and using that $p > 2$), we have $\mathbb{E}[X] \geq \mathbb{E}[X^{2/p}]^{p/2}$. Now, $\mathbb{E}[X^{2/p}] = \mathbb{E}_S[\|A_{i,S}\|_2^2] = \frac{k}{n} \|A_{i\star}\|_2^2$. This is because each coordinate of n appears in a randomly chosen S with probability $\frac{k}{n}$. It thus follows that $\mathbb{E}[X] \geq (k/n)^{p/2} \|A_{i\star}\|_2^p$.

Taking expectations of Eq. (19) over the choice of $|S| = k$, we now have that

$$(k/n)^{p/2} \sum_{i=1}^m \|A_{i\star}\|_2^p \leq k D^p .$$

Combining with the inequality $\sum_{i=1}^m \|A_{i\star}\|_p^p \geq n$, we thus have

$$\begin{aligned} (k/n)^{p/2} \sum_{i=1}^m \|A_{i\star}\|_2^p &\leq k D^p \leq \frac{k}{n} D^p \sum_{i=1}^m \|A_{i\star}\|_p^p \\ \implies \sum_{i=1}^m \|A_{i\star}\|_2^p &\leq D^p \left(\frac{n}{k}\right)^{p(\frac{1}{2}-\frac{1}{p})} \sum_{i=1}^m \|A_{i\star}\|_p^p , \end{aligned}$$

as required. □

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