20225701 Advanced Seminar in Machine Learning **Lecturer:** Aryeh Kontorovich
Pajor's lemma November 14, 2016

This proof of Pajor's lemma is taken from here: `http://tinyurl.com/pssvc-pajor`; this version has fewer words and more symbols and of course any mistakes are mine alone.

We treat a subset $S \subseteq \mathcal{X}$ and a binary function $f : \mathcal{X} \to \{0, 1\}$ as the same object. (I mention binary functions only to be consistent with the notation used in class; for this note, the set notation is more convenient.) Let $\mathcal{X}$ be a set and $\mathcal{F} \subseteq \{0, 1\}^{\mathcal{X}}$ be a family of subsets. We say that $\mathcal{F}$ **shatters** a subset $S \subseteq \mathcal{X}$ if $|\mathcal{F}(S)| = 2^{|S|}$, where $\mathcal{F}(S)$ is the **range** of $\mathcal{F}$ on S:

$$\mathcal{F}(S) = \{S \cap T : T \in \mathcal{F}\}.$$

Define the **shards** $\mathcal{F}^{\$}$ — my own non-standard term! — of $\mathcal{F}$ to be the collection of all subsets of $\mathcal{X}$ that it shatters:

$$\mathcal{F}^{\$} = \left\{ S \subseteq \mathcal{X} : |\mathcal{F}(S)| = 2^{|S|} \right\}.$$

Theorem 0.1 *For all finite $\mathcal{F}$,*

$$|\mathcal{F}^{\$}| \geq |\mathcal{F}|.$$

Proof: We proceed by induction on $|\mathcal{F}|$. The base case is $\mathcal{F} = \{T\}$, and this singleton shatters the empty set, so the claim holds.

Now assuming $|\mathcal{F}| \geq 2$, there must be some $x \in \mathcal{X}$ that belongs to some but not all $T \in \mathcal{F}$. Split $\mathcal{F} = \mathcal{F}_x \cup \bar{\mathcal{F}}_x$ into two nonempty disjoint collections of sets: those that contain x and those that do not:

$$\mathcal{F}_x = \{T \in \mathcal{F} : x \in T\}.$$

The main insight of the proof is the claim that

$$|\mathcal{F}^{\$}| \geq |\mathcal{F}^{\$}_x| + |\bar{\mathcal{F}}^{\$}_x|. \qquad (*)$$

As a sanity check, note that for all $\mathcal{A}, \mathcal{B} \subseteq \{0, 1\}^{\mathcal{X}}$, we have $(\mathcal{A} \cup \mathcal{B})^{\$} \supseteq \mathcal{A}^{\$} \cup \mathcal{B}^{\$}$ (why?) and hence $|\mathcal{F}^{\$}| \geq |\mathcal{F}^{\$}_x \cup \bar{\mathcal{F}}^{\$}_x|$. However, we might have $|\mathcal{F}^{\$}_x \cup \bar{\mathcal{F}}^{\$}_x| < |\mathcal{F}^{\$}_x| + |\bar{\mathcal{F}}^{\$}_x|$, since there might be sets $T \subseteq \mathcal{X}$ that are shattered by *both* $\mathcal{F}_x$ and $\bar{\mathcal{F}}_x$. To prove (*), we must account for sets $T \in \mathcal{F}^{\$} \setminus \left(\mathcal{F}^{\$}_x \cup \bar{\mathcal{F}}^{\$}_x\right)$. Such sets necessarily exist, since for all $T \in \mathcal{F}^{\$}_x \cup \bar{\mathcal{F}}^{\$}_x$, we have $x \notin T$ (why?). Next, convince yourself that

$$T \in \mathcal{F}^{\$}_x \cap \bar{\mathcal{F}}^{\$}_x \implies T \cup \{x\} \in \mathcal{F}^{\$}.$$

Hence, every $T \in \mathcal{F}^{\$}_x \cap \bar{\mathcal{F}}^{\$}_x$ contributes two distinct sets to $\mathcal{F}^{\$}$: T itself, and $T \cup \{x\}$. Of course, every $T \in \mathcal{F}^{\$}_x$ contributes a set to $\mathcal{F}^{\$}$, as does every $T \in \bar{\mathcal{F}}^{\$}_x$. It follows that

$$|\mathcal{F}^{\$}| \geq |\mathcal{F}^{\$}_x \cup \bar{\mathcal{F}}^{\$}_x| + |\mathcal{F}^{\$}_x \cap \bar{\mathcal{F}}^{\$}_x| = |\mathcal{F}^{\$}_x| + |\bar{\mathcal{F}}^{\$}_x|,$$

which proves (*). We are now done, since by induction, $|\mathcal{F}^{\$}_x| \geq |\mathcal{F}_x|$ and $|\bar{\mathcal{F}}^{\$}_x| \geq |\bar{\mathcal{F}}_x|$. ∎

Let's end the note by showing how Pajor's lemma implies the bound

$$|\mathcal{F}(\mathcal{X})| \le \sum_{i=0}^{d} \binom{|\mathcal{X}|}{i}, \qquad (**)$$

where d is the VC-dimension of $\mathcal{F}$. By Pajor's lemma, we have $|\mathcal{F}(\mathcal{X})^{\$}| \ge |\mathcal{F}(\mathcal{X})|$. But $\mathcal{F}$ can only shatter sets S with $|S| \le d$, and the number of such sets is upper-bounded by the right-hand side of (**).