Edit distance of run-length encoded strings

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Edit Distance

Definition:
• Given: strings $x=[x_1,x_2,\ldots,x_n]$ and $y=[y_1,y_2,\ldots,y_m]$.  
• Edit distance is minimum number of operations that are required to transform $x$ into $y$.  
• The operations are:
  - **substitution**: replace ‘c’ with ‘e’
    
    \[
    \begin{array}{cccc}
    a & b & c & d \\
    \downarrow & & & \\
    a & b & e & d \\
    \end{array}
    \]
  - **deletion**: delete ‘c’
    
    \[
    \begin{array}{cccc}
    a & b & c & d \\
    \downarrow & & & \\
    a & b & \_ & d \\
    \end{array}
    \]
  - **insertion**: insert ‘c’ between ‘b’ and ‘d’
    
    \[
    \begin{array}{cccc}
    a & b & \_ & d \\
    \downarrow & & & \\
    a & b & c & d \\
    \end{array}
    \]
Edit Distance

So the edit distance between “bread” and “beer” is 3
Trivial Solution

Build a matrix using:

- $A[i,j] = \min \{ A[i,j-1]+1, A[i-1,j]+1, A[i-1,j-1]+t_{i,j} \}$
- $t_{i,j} = \begin{cases} 
0, & \text{if } x_i = y_j \\
1, & \text{otherwise}
\end{cases}$

- For all $i$: $A[i,0] = j$
- For all $j$: $A[0,j] = j$

$A[i,j]$ contains the edit distance between sub-strings $[x_1,x_2,\ldots,x_i]$ and $[y_1,y_2,\ldots,y_j]$

Running time $O(n*m)$
Run-length encoded string

• A run-length encoded string is described as an ordered sequence of pairs $(\sigma,i)$, written as $\sigma^i$.

• The meaning of pair: character $\sigma$ appears $i$ consecutive times.

• Example: the string “aaaabbbbbccccabbbbc” will be encoded as: “$a^4b^4c^3a^1b^4c^2$”.

• Usage: Image compression.
Run-length encoded string

Given sequences:
X=[x₁,x₂,…,xₙ] with encoded length k (k pairs (σ,i) )
Y=[y₁,y₂,…,yₘ] with encoded length l (l pairs (σ,i) )

We’ll present the algorithm for finding edit distance between X and Y in running time $O(n*l+m*k)$
Matrix Partition

• We divide the matrix $A$ into “blocks”.
• **Block** is a sub-matrix $A[i_{\text{top}} \ldots i_{\text{bottom}}, j_{\text{left}} \ldots j_{\text{right}}]$ consisting of two runs one of $X$ and one of $Y$:
  - $x_{i_{\text{top}}-1} \neq x_{i_{\text{top}}} = x_{i_{\text{top}}+1} = \cdots = x_{i_{\text{bottom}}} \neq x_{i_{\text{bottom}}+1}$
  - $y_{j_{\text{left}}-1} \neq y_{j_{\text{left}}} = y_{j_{\text{left}}+1} = \cdots = y_{j_{\text{right}}} \neq y_{j_{\text{right}}+1}$
• **Black block**: for all $i$ and $j$ in the block $x_i = y_j$.
• **White block**: for all $i$ and $j$ in the block $x_i \neq y_j$. 
Matrix Partition

The block represents two substrings:
x1, x2, x3 = aaa
y3, y4, y5 = bbb
Matrix Filling

• Given the **top-left frame** we will fill the bottom row and the right column of the block.
• The top-left frame consists of bottom row and right column of the neighbouring blocks.
Matrix Filling

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• In order to fill the matrix in the $O(n*l+m*k)$ running time we shall use few theorems and definitions.
Theorem 1 (Ukkonen)

(1) \( A[i, j] - A[i - 1, j - 1] \in \{0, 1\} \)

\( 1 \leq i \leq n; 1 \leq j \leq m \)

\[
\begin{array}{|c|c|}
\hline
2 & 2 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
2 & 3 \\
\hline
\end{array}
\]

(2) \( A[i, j] - A[i - 1, j] , A[i, j] - A[i, j - 1] \in \{-1, 0, 1\} \)

\( 1 \leq i \leq n; 1 \leq j \leq m \)

\[
\begin{array}{|c|c|}
\hline
2 & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
2 & 2 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
2 & 3 \\
\hline
\end{array}
\]
Theorem 1 examples

\[ A[i,j] - A[i,j-1] = -1 \]

\[
\begin{array}{ccc}
  & a & b \\
 a &  &  \\
 c &  &  \\
\end{array}
\]

\[ A[i,j-1] \]

\[ A[i,j] \]

\[ A[i,j-1] \]

\[ A[i,j] \]

\[ A[i,j] - A[i,j-1] = 0 \]

\[
\begin{array}{ccc}
  & a & b \\
 a &  &  \\
 d &  &  \\
\end{array}
\]

\[ A[i,j-1] \]

\[ A[i,j] \]

\[ A[i,j] \]

\[ A[i,j] \]

\[ A[i,j] - A[i-1,j] = 1 \]

\[
\begin{array}{cccc}
  & a & b & c \\
 a &  &  &  \\
 a &  &  &  \\
\end{array}
\]

\[ A[i-1,j] \]

\[ A[i,j] \]

\[ A[i-1,j] \]

\[ A[i,j] \]
Given $A[i,j]$ define **diagonal** $d = j - i$
Black Box

Lemma 2:
For every element $A[i, j]$ of black block:

Proof:
• $A[i, j] := \min A[i, j - 1] + 1, A[i - 1, j] + 1, A[i - 1, j - 1] + t_{i,j}$

• $t_{i,j} = 0$ (because $x_i$ and $y_j$ on the black block) $\Rightarrow A[i-1,j-1]+t_{i,j}=A[i-1,j-1]$ [*1]
• [*1] [*5] $\Rightarrow A[i - 1, j - 1] + t_{i,j} < A[i, j - 1] + 1$

We would choose $A[i-1,j-1]+t_{i,j}$ in contradiction to the assumption
Corollary 1:
Given the upper-left boundary of a black block.
A[i, j] on its bottom-right is equals to the element on the intersection between the upper-left boundary and the diagonal.
Black Box

Time Complexity:

• Given the indices of the black block and coordinates \((i, j)\), one can find in \(O(1)\) time the location of the element on the upper-left boundary that has to be copied into \(A[i, j]\).

• So given a black block the values of the right column and the bottom row are computed in \(O((j_{\text{right}}-j_{\text{left}}+1)+(i_{\text{bottom}}-i_{\text{top}}+1))\) time.
White Blocks

• We will show how to find the values of the right column of the block (bottom row is done is done similarly).
• We will assume that the upper, left and upper-left neighbour blocks are black blocks.
White Blocks

Definition:
• Given: A[a, b] and A[p, q] with a ≤ p, b ≤ q
• \( \text{dis}(A[a, b], A[p, q]) \) edit distance between \([x_a, \ldots, x_p], [y_b, \ldots, y_q]\). 

Example: \( \text{dis}(A[2,2], A[3,4]) \)

\[
\text{dis}(A[2,2], A[3,4]) = \text{EditDistance}(cd, bcd) = 1
\]
White Blocks

The cells $A[i, j_{\text{right}}]$ in the right column can be calculated using the formula:

$$A[i, j_{\text{right}}] = \min_{p,q} (A[p, q] + \text{dis}(A[p, q], A[i, j_{\text{right}}]))$$

where $(p, q)$ ranges over all the indices with $p \leq i$ corresponding to the upper-left frame of the block.
White Blocks

* Green cell represents the edit distance between two sub-strings $T_1=\ldots[a$ and $S_1=\ldots[a$ is 7.
* Yellow cell represents the edit distance between two sub-strings $T_2=\ldots[abbb$ and $S_2=\ldots[aaaaaa$.

The dis between green and yellow cells is 5 and it represents:
- 3 substitutions of ‘a’ with ‘b’
- 2 deletions of ‘a’
White Blocks

The diagonal $d = j_{\text{right}} - i$ goes from $A[i, j_{\text{right}}]$ and divides the top-left frame into 3 zones:

- **Zone I:**
  - If $\text{Diagonal}(i - j_{\text{right}})$ crosses the top frame, then Zone I consists of elements $A[i_{\text{top}}, j_{\text{diagonal}} \ldots j_{\text{right}}]$;
  - otherwise, Zone I consists of elements $A[i_{\text{top}}, j_{\text{left}} \ldots j_{\text{right}}]$.

- **Zone II:**
  - If $\text{Diagonal}(i - j_{\text{right}})$ crosses the top frame, then Zone II consists of elements $A[i_{\text{top}} \ldots i, j_{\text{left}}]$;
  - otherwise, Zone II consists of elements $A[i_{\text{diagonal}} \ldots i, j_{\text{left}}]$.

- **Zone III:** consists of all of the elements on the upper-left frame between Zone I and Zone II.
White Blocks

zone I

zone III

zone II

A[i,j]

A[i,j_{right}]
White Blocks

Lemma 3
• The distance between each element in Zone I and $A[i, j_{right}]$ is $i - i_{top}$
• The distance between each element in Zone II and $A[i, j_{right}]$ is $j_{right} - j_{left}$

Proof:
• The distance between two distinct words is the length of the longer one.
• We have distance between two words which first letters are the same.
→ So the distance equals to the length of the longest word minus 1 which is $i - i_{top}$

Similarly for Zone II, except that the longer string is $j_{right} - j_{left}$

$X = a\ a$
$Y = a\ b\ b$

The edit distance is $\text{length}(X) - 1 = 2$
White Blocks

**Lemma 4:** Zone III is unnecessary.

**Proof:** We will show that \(A_{itop, diag}\) is preferred on any cell in Zone III

1. By theorem 1 the difference between two adjacent cells values is not more than 1.

\[A_{itop, diag} - A_{itop,c} \leq \text{diag} - c\]

2. \(\text{dis}(A_{itop,c}, A[i, j_{right}]) = \text{dis}(A_{itop, diag}, A[i, j_{right}]) + (\text{diag} - c)\)

\((1),(2) \Rightarrow A_{itop, diag} + \text{dis}(A_{itop, diag}, A[i, j_{right}]) \leq A_{itop,c} + \text{dis}(A_{itop,c}, A[i, j_{right}])\)
White Blocks

Reminder:

For all cells $A[i,j_{\text{diagonal}}]$ of the right column:
- $A[i, j_{\text{right}}] = \min_{p,q}(A[p, q]+ \text{dis}(A[p, q], A[i, j_{\text{right}}]),$
- The distance to all cells of Zone I are identical.
- The distance to all cells of Zone II are identical.
- The Zone III is unnecessary.
So in order to find the $A[i,j_{\text{right}}]$ we have to:

- Find the cell $A[i_{\text{top}},j_{\text{min}}]$ with minimal value among the cells in Zone I
- Find the cell $A[i_{\text{min}},j_{\text{left}}]$ with minimal value among the cells in Zone II
- Set $A[i,j_{\text{right}}] = \min\{ A[i_{\text{top}},j_{\text{min}}] + (i - i_{\text{top}}), A[i_{\text{min}},j_{\text{left}}] + (j_{\text{right}} - j_{\text{left}}) \}$

$$j_{\text{right}} - j_{\text{left}} = 9 \quad \text{and} \quad i - i_{\text{top}} = 2$$

$$A[i,j_{\text{right}}] = \min\{1 + 2, 2 + 9\} = 3$$
White Blocks

• We will see two algorithms:
  - **Algorithm 1**: matches for all right column element the minimum value element in Zone I.
  - **Algorithm 2**: matches for all right column element the minimum value element in Zone II.

**Note**: for two different elements in right column, Zone I and Zone II may consist of different elements.
Zone I Algorithm

- We start calculating the minimum from the top cell and proceeding down. At the beginning Zone I contains two elements so the cell’s value is a minimum among these two.
Zone I Algorithm

- In the first iterations whenever we move down one element on the rightmost boundary, one element is added to Zone 1. So the minimum value for the new element may be the previous minimum or the new element value.
Zone I Algorithm

- At the certain stage Zone I contains whole top frame and it stays static. So the minimum do not change after each iteration.
Zone I Algorithm

for $i = i_{\text{top}+1}$ to $i_{\text{bottom}}$ do
{
    if $(i - i_{\text{top}} \leq j_{\text{right}} - j_{\text{left}})$ then
        $\text{MinZoneI}[i] = \min(\text{MinZoneI}[i-1], A[i_{\text{top}},j_{\text{diagonal}}])$  \hspace{1cm} (a)
    else
        $\text{MinZoneI}[i] = \text{MinZoneI}[i-1]$  \hspace{1cm} (b)
}

(a) $A[i_{\text{top}},j_{\text{diagonal}}]$

(b) $A[i,j_{\text{right}}]$
Unlike Zone I, when Zone II reaches its maximal size, when we’re moving downward $A[i,j_{\text{right}}]$, one element is removed from Zone II and a new element is added.
Zone II Algorithm

• **Observation:** Let $s, w$ be two values in a given Zone II. Without loss of generality, we assume that $s < w$. Then, following Theorem 1, all of the values between $s$ and $w$ must also appear in the Zone II.

• So if the minimum value in the zone is $a$ and the maximum value is $b$, then the values $a, a + 1, \ldots, b - 1, b$ must appear in that Zone II.
Zone II Algorithm

• Variable **Min**: contains minimum value of Zone II.
• Variable **New**: the value of the new element that is added to the Zone II.
• Variable **Out**: the value of the new element that is deleted from the Zone II.
• Array **counters[ ]**: counts the appearances of each value in the Zone II (counters[x]=y means, the value x appears y times).
Zone II Algorithm

for i = i_{top} to i_{bottom} do
{
    counter[New] ++
    Min := min(Min, New)
    counter[Out] --
    if counter[Min] = 0 then Min ++
    MinZoneII[i] := Min
}

According to the observation Min+1 exists in Zone II
## Zone II Algorithm

<table>
<thead>
<tr>
<th>Counter</th>
<th>Min</th>
<th>Out</th>
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</thead>
<tbody>
<tr>
<td>0 1 0 0 0 0</td>
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<td>0 1 1 0 0 0</td>
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<td>0 1 1 2 1 0</td>
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</tbody>
</table>
White Block

Time Complexity:

- MinZone1 is computed using Algorithm 1 in $O(i_{\text{bottom}} - i_{\text{top}} + 1)$
- MinZone2 is computed using Algorithm 2 in $O(i_{\text{bottom}} - i_{\text{top}} + 1)$
- Using MinZone1 and MinZone2 we can calculate in $O(1)$ the value for each $(i, j_{\text{right}})$ for $i_{\text{top}} \leq i \leq i_{\text{bottom}}$ (Total $O(i_{\text{bottom}} - i_{\text{top}} + 1)$ time).

- Calculation of the bottom row is done similarly with running time $O(j_{\text{right}} - j_{\text{left}} + 1)$.

Hence:
- Given a white block the values of the right column and the bottom row are computed in $O((j_{\text{right}} - j_{\text{left}} + 1) + (i_{\text{bottom}} - i_{\text{top}} + 1))$ time.
Time Complexity

**Theorem 5:**
The edit distance between two run-length encoded strings, X and Y:
- X=[x_1,x_2,…,x_n] with encoded length k
- Y=[y_1,y_2,…,y_m] with encoded length l
can be computed in \( O(k \cdot m + l \cdot n) \) time.

**Proof:**
We have shown that the work for each block (black or white) is linear in the size of its bottom-right boundary. Hence, the total time complexity is linear in the total size of the boundaries of all blocks and that is exactly \( O(k \cdot m + l \cdot n) \).
Time Complexity

\[ n = 9 \quad k = 3 \]

\[ m = 5 \quad l = 2 \]

\[ O(n) \quad O(n) \quad O(n) \]

\[ O(m) \quad O(m) \quad O(m) \]
# Example

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- Example grid with shaded cells and labels.
Example

Zone I: 3+1
Zone II: 0+4

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Zone I: 2+2
Zone II: 0+4
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Zone I: 1+3
Zone II: 0+4
Example

Zone I: 0+4
Zone II: 1+4
Example

Zone I: 0+5
Zone II: 1+5
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#### Zone II: 2+5
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