Methods in Computer Vision: Introduction to Projective Geometry and Camera Geometry

Oren Freifeld
Computer Science, Ben-Gurion University

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Further Reading

Multiple View Geometry in computer vision

Richard Hartley and Andrew Zisserman
• A point in the plane may be represented by the pair of coordinates:

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} = (x, y) \in \mathbb{R}^2
\]

• Thus, we usually identify the plane with \( \mathbb{R}^2 \).

• Considering \( \mathbb{R}^2 \) as a vector space, the coordinate pair \((x, y)\) is a vector; i.e., we identify points with vectors.
Homogeneous Representation of Lines

- A line in the plane may be represented by an equation such as
  \[ ax + by + c = 0 \]

- Thus, a line may naturally be represented by the coefficient vector
  \[ \begin{bmatrix} a & b & c \end{bmatrix}^T \]

- The correspondence between such vectors and lines is not one-to-one: for any nonzero \( k \in \mathbb{R} \),
  \[ \begin{align*}
  ax + by + c &= 0 \\
  (ka)x + (kb)y + (kc) &= 0
  \end{align*} \]
  represent the same line. Thus,
  \[ \begin{bmatrix} a & b & c \end{bmatrix}^T \text{ and } \begin{bmatrix} ka & kb & kc \end{bmatrix}^T \]
  define the same line.
Two such vectors related by an overall scaling are considered equivalent.

An equivalence class of vectors under this equivalence relationship is known as a homogeneous vector. Any particular nonzero vector \( \begin{bmatrix} a & b & c \end{bmatrix}^T \) is a representative of the equivalence class.

The set of equivalence classes of vectors in \( \mathbb{R}^3 \setminus \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \) forms the projective space \( \mathbb{P}^2 \).
More generally, the projective space $\mathbb{P}^n$ is defined similarly via the set of equivalence classes of homogeneous vectors in $\mathbb{R}^{n+1} \setminus \mathbf{0}_{(n+1) \times 1}$.

Remark: this is our old friend, $\text{Gr}(1, n + 1)$, the Grassmann manifold of order $(n + 1, 1)$; namely, the set of all 1D linear subspaces of $\mathbb{R}^{n+1}$. 
Homogeneous Representation of Points

- Fix the line $l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.
- A point $x = \begin{bmatrix} x \\ y \end{bmatrix}^T$ lies on the line $l \iff ax + by + c = 0$; i.e.

$$
\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} l = 0
$$

- Note that for any non-zero constant $k$ and line $l$, the equation $\begin{bmatrix} x & y & 1 \end{bmatrix} l = 0$ holds if and only if $\begin{bmatrix} kx & ky & k \end{bmatrix} l = 0$. 

Homogeneous Representation of Points

It is natural, therefore, to consider the set of vectors \[ \begin{bmatrix} kx & ky & k \end{bmatrix}^T \]
for varying values of non-zero \( k \) to be a representation of the point 
\( \mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \).

Thus, just as with lines, points are represented by homogeneous vectors.
Homogeneous Representation of Points

- An arbitrary homogeneous vector representative of a point is of the form
  \[ \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T, \]
  representing the point \[ \frac{x_1}{x_3} \frac{x_2}{x_3} \begin{bmatrix} x_1 \ x_2 \ x_3 \end{bmatrix}^T \in \mathbb{R}^2. \]
- Points, then, as homogeneous vectors, are also elements of \( \mathbb{P}^2 \).
- The point \( \mathbf{x} \) lies on the line \( \mathbf{l} \) if and only if
  \[ \mathbf{x}^T \mathbf{l} = 0 \]
  where \( \mathbf{x} \) and \( \mathbf{l} \) are understood to be written in their homogeneous representation.
- We distinguish between the homogeneous coordinates
  \[ \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \]
  of a point, which is a 3-vector, and the inhomogeneous coordinates
  \[ \begin{bmatrix} x & y \end{bmatrix}^T, \]
  which is a 2-vector.
- Since \( \mathbf{x}^T \mathbf{l} \) is a scalar:
  \[ \mathbf{x}^T \mathbf{l} = \mathbf{l}^T \mathbf{x} \]
Intersection of Lines

- Given two lines, $l = [a \ b \ c]^T$ and $l' = [a' \ b' \ c']^T$ we wish to find their intersection.
- Let $x \triangleq l \times l'$ (where $\times$ is the cross product).
- From the identity
  \[ l^T (l \times l') = l'^T (l \times l') = 0 \]
  we see that $l^T x = l'^T x = 0$.
- Thus, if $x$ is thought of as representing a point, then $x$ lies on both lines, and hence is the intersection of the two lines. This shows:

**Fact**

The intersection of two lines $l$ and $l'$ is the point $x = l \times l'$.

- Note that the simplicity of this expression for the intersection of the two lines is a direct consequence of the use of homogeneous vector representations of lines and points.
Intersection of Lines

Example

The intersection of the lines $x = 1$ and $y = 1$; i.e., the lines

$$-x + 1 = \begin{bmatrix} l \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \quad \text{and} \quad -y + 1 = \begin{bmatrix} l' \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0,$$

is given by

$$x = l \times l' = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = i \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} - j \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} + k \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = i + j + k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T,$$

which is a homogeneous representation of the point $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$. 
A Line Joining two Points

Analogously:

Fact

The line joining two points $x$ and $x'$ is

$$ l = x \times x' $$

Again: the simplicity of this result too is related to the homogeneous representation.
Parallel Lines

- In Euclidean geometry, parallel lines do not intersect (unless they are identical).
- In projective geometry, however, parallel lines do intersect.
Intersection of Parallel Lines

The lines $ax + by + c = 0$ and $ax + by + c' = 0$, are represented by 
\[ l = \begin{bmatrix} a & b & c \end{bmatrix}^T \] and \[ l' = \begin{bmatrix} a & b & c' \end{bmatrix}^T \] (i.e., disagreement only in the third coefficient).

Apply the same computation from before to obtain their intersection:

\[
l \times l' = \begin{vmatrix} i & j & k \\ a & b & c \\ a & b & c' \end{vmatrix} = i \begin{vmatrix} b & c \\ b & c' \end{vmatrix} - j \begin{vmatrix} a & c \\ a & c' \end{vmatrix} + k \begin{vmatrix} a & b \\ a & b \end{vmatrix} = (c' - c) \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}.
\]

Ignoring the nonzero scale, $(c' - c)$, this is the point \[ \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}^T. \]

Its inhomogeneous representation, \[ \begin{bmatrix} b/0 & -a/0 \end{bmatrix} \] is not defined.

Informally, this is consistent with “parallel lines meet only at $\infty$”
Intersection of Parallel Lines

Example

Consider the two parallel lines, \( x = 1 \) and \( x = 2 \). In homogeneous notation, these lines are \( l = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T \) and \( l' = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}^T \). The intersection is:

\[
\begin{vmatrix}
  i & j & k \\
  -1 & 0 & 1 \\
  -1 & 0 & 2
\end{vmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}
\]

which is the point at \( \infty \) in the direction of the \( y \) axis.
Ideal Points and $l_\infty$, the Line at $\infty$

- Homogeneous vectors $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ such that $x_3 \neq 0$ correspond to (finite) points in $\mathbb{R}^2$.
- One may augment $\mathbb{R}^2$ by adding points with last coordinate $x_3 = 0$. The resulting space is the set of all homogeneous 3-vectors, namely the projective space, $\mathbb{P}^2$.
- The points with last coordinate $x_3 = 0$ are known as ideal points, or points at infinity.
- The set of all ideal points may be written as $\begin{bmatrix} x_1 & x_2 & 0 \end{bmatrix}^T$.
- Note that this set lies on a single line, the line at infinity, denoted by the vector $l_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Indeed,

$$l_\infty^T \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = 0$$
Intersection of a Line with $l_{\infty}$

- The line $l = \begin{bmatrix} a & b & c \end{bmatrix}^T$ intersects $l_{\infty}$ at $\begin{bmatrix} b & -a & 0 \end{bmatrix}^T$ since
  \[
  \begin{bmatrix} b & -a & 0 \end{bmatrix} l = 0.
  \]

- The line $l' = \begin{bmatrix} a & b & c' \end{bmatrix}^T$, parallel to $l'$, intersects $l_{\infty}$ at the same point, irrespective of the value of $c'$.

- In inhomogeneous notation, $\begin{bmatrix} b & -a \end{bmatrix}^T$ is a vector tangent to the line, and orthogonal to the line’s normal, $\begin{bmatrix} a & b \end{bmatrix}$, and so represents the line’s direction.

- As the line’s direction varies, the ideal point $\begin{bmatrix} b & -a & 0 \end{bmatrix}^T$ varies over $l_{\infty}$

- For these reasons the line at infinity can be thought of as the set of directions of lines in the plane.
Note how the introduction of the concept of points at infinity simplifies the intersection properties of points and lines. In the projective plane $\mathbb{P}^2$, one may state without qualification that two distinct lines meet in a single point and two distinct points lie on a single line. This is not true in the standard Euclidean geometry of $\mathbb{R}^2$ in which parallel lines require special attention.
A Model for the Projective Plane

A useful way of thinking of $\mathbb{P}^2$ is as a set of rays in $\mathbb{R}^3$. 

Fig. 2.1. A model of the projective plane. Points and lines of $\mathbb{P}^2$ are represented by rays and planes, respectively, through the origin in $\mathbb{R}^3$. Lines lying in the $x_1x_2$-plane represent ideal points, and the $x_1x_2$-plane represents $l_\infty$. 

Figure from Hartley and Zisserman
Duality of Points and Planes

The basic incidence equation \( l^T x = 0 \) for line and point is symmetric, since \( l^T x = 0 \) implies \( x^T l = 0 \), in which the positions of line and point are swapped. Similarly, the results we saw for the intersection of two lines and the line through two points are essentially the same, with the roles of points and lines swapped.

More generally, the **duality principle** is as follows. To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem.
Projective Transformations

- Projective transformations are more general than (invertible) affine transformations.
- Affine transformations: parallel lines remain parallel lines.
- Projective transformations: lines remain lines.
A projectivity in $\mathbb{P}^2$ is an invertible mapping from points in $\mathbb{P}^2$ (i.e., homogeneous 3-vectors) to points in $\mathbb{P}^2$ that maps lines to lines. More precisely:

**Definition (A projectivity in $\mathbb{P}^2$)**

A projectivity is an invertible mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ such that three points $x_1, x_2$ and $x_3$ lie on the same line if and only if $h(x_1), h(x_2)$ and $h(x_3)$ do.

A projectivity is also called a collineation (a helpful name), a projective transformation or a homography: the terms are synonymous.
Some examples for geometric concepts that are not preserved under a projective transformation:

- **Shape** (e.g., circle may become an ellipse).
- **Lengths** (e.g., two perpendicular radii of a circle are stretched by different amounts)
- **Angles**
- **Distance**
- **Ratios of distances**
Under the binary operation of composition, projectivities form a group since the inverse of a projectivity is also a projectivity, and so is the composition of two projectivities.
Projective Transformation

Theorem (proof is omitted)

A mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a projectivity if and only if there exists a non-singular $3 \times 3$ matrix $H$ such that for any point in $\mathbb{P}^2$ represented by a vector $\mathbf{x}$ it is true that

$$h(\mathbf{x}) = H\mathbf{x}$$

- Interpretation: Any point in $\mathbb{P}^2$ is represented as a homogeneous 3-vector, $\mathbf{x}$, and $H\mathbf{x}$ is a linear mapping of homogeneous coordinates. The theorem asserts that any projectivity arises as such a linear transformation in homogeneous coordinates, and that conversely any such mapping is a projectivity.

- Note that if a matrix $H$ defines a projectivity $h$, then for every nonzero $k$, the matrix $kH$ defines the same $H$. 
An Alternative Definition of a Projective Transformation

As a result of this theorem, one may give an alternative definition of a projective transformation as follows.

**Definition**

A planar projective transformation is a linear transformation on homogeneous 3-vectors represented by a non-singular 3 × 3 matrix:

\[
\begin{bmatrix}
    x'_1 \\
    x'_2 \\
    x'_3 
\end{bmatrix} =
\begin{bmatrix}
    h_{11} & h_{12} & h_{13} \\
    h_{21} & h_{22} & h_{23} \\
    h_{31} & h_{32} & h_{33} 
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 
\end{bmatrix}
\]

or, more briefly,

\[x' = Hx\]

- Again, note that \(H\) may be multiplied by any nonzero scalar.
- We thus call \(H\) a homogeneous matrix, and say it has only 8 degrees of freedom (not 9).
If a projective transformation between two planes can be interpreted as a projection along rays through a common point (called the center of projection), the transformation is called a perspective transformation (AKA a perspectivity). So, a perspectivity is still an invertible map between projective spaces of the same dimensions. For example, as a map between $\mathbb{P}^2$ and $\mathbb{P}^2$, it can be written using an invertible $3 \times 3$ $H$.

In contrast, in a Perspective Camera Model, we map $\mathbb{P}^3$ (i.e., homogeneous 4-vectors) to $\mathbb{P}^2$ (i.e., homogeneous 3-vectors) via

$$x = PX \quad x \in \mathbb{P}^2 \quad X \in \mathbb{P}^3$$

for some $3 \times 4$ rank-3 matrix $P = K \begin{bmatrix} R & t \end{bmatrix}$, where the $3 \times 3$ $K$ is a perspective transformation.
Fig. A7.3. A line perspectivity. The lines joining corresponding points (a, A etc.) are concurrent. Compare with figure A7.4.

Fig. A7.4. A line projectivity. Points \( \{a, b, c\} \) are related to points \( \{A, B, C\} \) by a line–to–line perspectivity. Points \( \{a', b', c'\} \) are also related to points \( \{A, B, C\} \) by a perspectivity. However, points \( \{a, b, c\} \) are related to points \( \{a', b', c'\} \) by a projectivity; they are not related by a perspectivity because lines joining corresponding points are not concurrent. In fact the pairwise intersections result in three distinct points \( \{p, q, r\} \).

Figures from Hartley and Zisserman
Central projection, a projection along rays through a common point (the center of projection) defines a mapping from one plane to another.

The projection also maps lines to lines as may be seen by considering a plane through the projection centre which intersects with the two planes $\pi$ and $\pi'$. Since lines are mapped to lines, central projection is a projectivity and may be represented by a linear mapping of homogeneous coordinates $x' = Hx$.

Figure 2.3. Central projection maps points on one plane to points on another plane. The projection also maps lines to lines as may be seen by considering a plane through the projection centre which intersects with the two planes $\pi$ and $\pi'$. Since lines are mapped to lines, central projection is a projectivity and may be represented by a linear mapping of homogeneous coordinates $x' = Hx$.  

Figure from Hartley and Zisserman
Fig. 2.4. **Removing perspective distortion.** (a) The original image with perspective distortion – the lines of the windows clearly converge at a finite point. (b) Synthesized frontal orthogonal view of the front wall. The image (a) of the wall is related via a projective transformation to the true geometry of the wall. The inverse transformation is computed by mapping the four imaged window corners to corners of an appropriately sized rectangle. The four point correspondences determine the transformation. The transformation is then applied to the whole image. Note that sections of the image of the ground are subject to a further projective distortion. This can also be removed by a projective transformation.

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Figure from Hartley and Zisserman
Removing Perspective Distortion

- Each point correspondence, \(((x, y), (x', y'))\), where
  
  \[(x, y) \leftrightarrow (kx, ky, k) \quad \text{and} \quad (x', y') \leftrightarrow (k'x, k'y', k')\],

provides two equations:

\[
\begin{bmatrix}
  k'x' \\
k'y' \\
k'
\end{bmatrix}
= 
H
\begin{bmatrix}
  kx \\
k'y \\
k
\end{bmatrix}
= 
\begin{bmatrix}
  h_{11} & h_{12} & h_{13} \\
h_{21} & h_{22} & h_{23} \\
h_{31} & h_{32} & h_{33}
\end{bmatrix}
\begin{bmatrix}
  kx \\
k'y \\
k
\end{bmatrix}
\Rightarrow
\]

\[
x' = \frac{k'x'}{k'} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \quad y' = \frac{k'y'}{k'} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}
\Rightarrow
\]

\[
x'(h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13}
y'(h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23}
\]
Removing Perspective Distortion

\[ x'(h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13} \]
\[ y'(h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23} \]
\[ \Rightarrow \]
\[ x'(h_{31}x + h_{32}y + h_{33}) - h_{11}x - h_{12}y - h_{13} = 0 \]
\[ y'(h_{31}x + h_{32}y + h_{33}) - h_{21}x - h_{22}y - h_{23} = 0 \]
\[ \Rightarrow \]

\[
\begin{bmatrix}
-x & -y & -1 & 0 & 0 & 0 & x'x & x'y & x' \\
0 & 0 & 0 & -x & -y & -1 & y'x & y'y & y'
\end{bmatrix}
\begin{bmatrix}
h_{11} \\
h_{12} \\
h_{13} \\
h_{21} \\
h_{22} \\
h_{23} \\
h_{31} \\
h_{32} \\
h_{33}
\end{bmatrix}
\]

This is linear in the entries of \( H \).
Removing Perspective Distortion

- If have 4 pairs of correspondence:

\[
\begin{bmatrix}
-x_1 & -y_1 & -1 & 0 & 0 & 0 & x'_1 & x'_1 & x'_1 \\
0 & 0 & 0 & -x_1 & -y_1 & -1 & y'_1 & y'_1 & y'_1 \\
-x_2 & -y_2 & -1 & 0 & 0 & 0 & x'_2 & x'_2 & x'_2 \\
0 & 0 & 0 & -x_2 & -y_2 & -1 & y'_2 & y'_2 & y'_2 \\
-x_3 & -y_3 & -1 & 0 & 0 & 0 & x'_3 & x'_3 & x'_3 \\
0 & 0 & 0 & -x_3 & -y_3 & -1 & y'_3 & y'_3 & y'_3 \\
-x_4 & -y_4 & -1 & 0 & 0 & 0 & x'_4 & x'_4 & x'_4 \\
0 & 0 & 0 & -x_4 & -y_4 & -1 & y'_4 & y'_4 & y'_4 \\
\end{bmatrix}
\begin{bmatrix}
h_{11} \\
h_{12} \\
h_{13} \\
h_{21} \\
h_{22} \\
h_{23} \\
h_{31} \\
h_{32} \\
h_{33} \\
\end{bmatrix}
\]

- If no three of the points are collinear, then since $H$ is defined up to scale, can set $h_{33} = 1$ and obtain an invertible system of the form $Ax = b$ where $A$ is $8 \times 8$ and $x, b \in \mathbb{R}^8$.

- If have more than 4 pairs: can use least-squares estimation in a linear model.
The computation of the rectifying transformation $H$ in this way does not require knowledge of any of the cameras parameters or the pose of the plane.

It is not always necessary to know coordinates for four points in order to remove projective distortion: there are alternative approaches which require less, and different types of, information.

There exist superior (and preferred) methods for computing projective transformations.
Transformation of Lines

Fact

If \( x \) lies on line \( l \), then under a projective transformation, the transformed point \( x' = Hx \) lies on the line \( l' = H^{-T}l \).

Indeed:

\[
    l'^T x' = (H^{-T}l)^T H x = l^T H^{-1} H x = l^T x = 0.
\]

Note that points transform according to \( H \), while lines transform according to \( H^{-T} \).
Dropping from 3D world to a 2D image is a projection process in which we lose one dimension.

The usual way of modeling this process is by central projection in which a ray from a point in space is drawn from a 3D world point through a fixed point in space, the center of projection.

This ray will intersect a specific plane in space chosen as the image plane.
Perspective Camera Model

Left: Image formation – the image points $x_i$ are the intersection of a plane with rays from the space points $X_i$ through the camera center $C$.

Right: If the space points are coplanar then there is a **projective transformation** between the world and image planes, $x_i = H_{3 \times 3} X_i$.

Figure from Hartley and Zisserman
Left: Images of the same camera center are related by a projective transformation, \( x'_i = H_{3 \times 3} x_i \). Both here and the right figure in the previous slides, planes are mapped to one another by rays through a center. There, the mapping was between a scene and image plane, here it is between two image planes.

Right: If we have two different camera centers, then the images are in general not related by a projective transformation.

Figure from Hartley and Zisserman
However, if all the space points are coplanar, then even if we have two different camera centers then the images are related by a projective transformation.
Perspective Camera Model

- \( \mathbf{X}_w = (X_w, Y_w, Z_w) \): a 3D point in world coordinate system.
- \( \mathbf{x}_i = (x, y) \): a 2D point in image coordinates.
- \( \mathbf{P} \): a rank-3 \( 3 \times 4 \) matrix

Camera model:

\[
\begin{bmatrix}
\lambda x \\
\lambda y \\
\lambda
\end{bmatrix} = \lambda
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix} = \mathbf{P}
\begin{bmatrix}
\lambda' X_w \\
\lambda' Y_w \\
\lambda' Z_w \\
\lambda'
\end{bmatrix} = \mathbf{P} \lambda'
\]

- We might as well absorb \( \lambda' \) into \( \mathbf{P} \):

\[
\begin{bmatrix}
\lambda x \\
\lambda y \\
\lambda
\end{bmatrix} = \lambda
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix} = \mathbf{P}
\begin{bmatrix}
X_w \\
Y_w \\
Z_w \\
1
\end{bmatrix}
\]
Perspective Camera Model

- \( X_c = (X_c, Y_c, Z_c) = RX_w + t \): a 3D point in camera coordinate system

\[
\begin{bmatrix}
    t_1 \\
    t_2 \\
    t_3
\end{bmatrix}, \quad
\begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix}
\in \text{SO}(3) : RR^T = I_{3 \times 3}, \quad \det R = 1
\]

\[
\begin{bmatrix}
\lambda x \\
\lambda y \\
\lambda
\end{bmatrix}
= \underbrace{K}_{3 \times 3 \text{, invertible}}
\begin{bmatrix}
    r_{11} & r_{12} & r_{13} & t_1 \\
    r_{21} & r_{22} & r_{23} & t_2 \\
    r_{31} & r_{32} & r_{33} & t_3
\end{bmatrix}
\begin{bmatrix}
    X_w \\
    Y_w \\
    Z_w \\
    1
\end{bmatrix}
\]

- \((c_x, c_y)\): principal point. \((f_x, f_y)\): focal lengths.

- Extrinsic parameters: \( R \) and \( t \). Intrinsic parameters: \( K \)
Radial Distortion

- Often there is also radial distortion.

![No distortion](No distortion) ![Positive radial distortion (Barrel distortion)](Positive radial distortion) ![Negative radial distortion (Pincushion distortion)](Negative radial distortion)

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Figure: http://docs.opencv.org/2.4/modules/calib3d/doc/camera_calibration_and_3d_reconstruction.html
Camera calibration is the estimation of the intrinsic parameters, extrinsic parameters, and the radial distortion (which is also intrinsic in nature).

The intrinsics do not depend on the camera pose (i.e., $R$ and $t$).

There exist good toolboxes out there, e.g.: OpenCV; Jean-Yves Bouguet’s Matlab toolbox (http://www.vision.caltech.edu/bouguetj/calib_doc/)
Camera Calibration

Calibration is usually based on a planar calibration object and minimizing (sum of squared) "re-projection" errors; see next slide.

Figure: [http://www.vision.caltech.edu/bouguetj/calib_doc/](http://www.vision.caltech.edu/bouguetj/calib_doc/)
Camera Calibration

- Intrinsic, using “C cameras” (really a single camera and C images):

\[
\min_{R_c, t_c, K} \sum_{c=1}^{C} \sum_{n=1}^{N} \left\| \tilde{P}_c X_{w,n} - x_{i,c,n} \right\|^2
\]

- \(N\): the number of points in the calibration objects.
- \(n\): point index; \(c\): camera index
- \(R_c, t_c\): pose of camera \(c\)
- \(P_c = \text{func}(R_c, t_c, K)\): projection matrix for camera \(c\); note the same \(K\) is shared across cameras.
- \(x_{i,c,n}\): observed image point \(n\) in camera \(c\)
- \(X_{w,n}\): known world point \(n\)
- \(\tilde{P}_c X_{w,n}\): the inhomogeneous representation of \(P_c X_{w,n}\).

Here we are interested only in \(K\): the \((R_c, t_c)_{c=1}^C\) are just auxiliary variables to be discarded once \(K\) is found.
Camera Calibration

- Extrinsic (once $K$ is known):

$$
\min_{R,t} \sum_{n=1}^{N} \left\| \widehat{PX}_{w,n} - x_{i,n} \right\|^2
$$

- Now we have only one camera, so there is no $c$ index.

- In principle, as long we don’t change the camera’s focal length (“zoom in/out”), intrinsics can be done once. Remark: some cameras provide the intrinsics as meta data, but it is prudent to estimate these anyway.

- The extrinsic calibration should be done every time we move the camera.
Examples of Projectivities Arising in Perspective Images

Fig. 2.5. **Examples of a projective transformation,** $x' = Hx$, **arising in perspective images.** (a) The projective transformation between two images induced by a world plane (the concatenation of two projective transformations is a projective transformation); (b) The projective transformation between two images with the same camera centre (e.g. a camera rotating about its centre or a camera varying its focal length); (c) The projective transformation between the image of a plane (the end of the building) and the image of its shadow onto another plane (the ground plane). Figure (c) courtesy of Luc Van Gool.
Affine Transformations

- Projective transformations are more general than (invertible) affine transformations; e.g.,

\[
\begin{bmatrix}
\lambda x'

\lambda y'

\lambda
\end{bmatrix} =
\begin{bmatrix}
 h_{11} & h_{12} & h_{13} \\
 h_{21} & h_{22} & h_{23} \\
 h_{31} & h_{32} & h_{33}
\end{bmatrix}
\begin{bmatrix}
x
y
1
\end{bmatrix}
\quad \text{det } H \neq 0
\]

- In the affine case, \( h_{31} = h_{32} = 0 \) and \( h_{33} = 1 \). Thus,

\[
x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \quad y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}
\]

or, equivalently:

\[
\begin{bmatrix}
x'

y'
\end{bmatrix} =
\begin{bmatrix}
h_{11} & h_{12} \\
h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
x
y
\end{bmatrix} +
\begin{bmatrix}
h_{13} \\
h_{23}
\end{bmatrix}
\]
More generally, an affine transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be written, using homogeneous coordinates, as

$$
\begin{bmatrix}
  x' \\
  1
\end{bmatrix} = 
\begin{bmatrix}
  A & b \\
  0_{1 \times n} & 1
\end{bmatrix} 
\begin{bmatrix}
  x \\
  1
\end{bmatrix} \quad x, x' \in \mathbb{R}^n, \quad b \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}
$$

It is invertible if and only if $A$ is invertible.
An ideal point under an affine transformation:

\[
\begin{bmatrix}
A & t \\
0^T & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
0
\end{bmatrix}
= \begin{bmatrix}
A \\
0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
0
\end{bmatrix}
= \text{still an ideal point}
\]

An ideal point under a projective transformation:

\[
\begin{bmatrix}
A & t \\
v^T & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
0
\end{bmatrix}
= \begin{bmatrix}
A \\
v_1x_1 + v_2x_2
\end{bmatrix}
= \text{usually not an ideal point}
\]

where \(v^T = \begin{bmatrix} v_1 & v_2 \end{bmatrix}\)
A projective transformation may map ideal point to finite points.

\[ l_\infty \text{ may be mapped to a finite line.} \]

But if the projective transformation, \( H \), is affine, then \( Hl_\infty = l_\infty \)
Proof. If $H$ is affine then $Hl_\infty = l_\infty$.

To transform a line, we need to find $H^{-T}$.

\[
H = \begin{bmatrix}
    A & t \\
    0^T & 1
\end{bmatrix} \Rightarrow H^T = \begin{bmatrix}
    A^T & 0 \\
    t^T & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
    A^T & 0 \\
    t^T & 1
\end{bmatrix} \begin{bmatrix}
    A^{-T} & 0 \\
    -t^T A^{-T} & 1
\end{bmatrix} = \begin{bmatrix}
    A^T A^{-T} - 0(t^T A^{-T}) & A^T 0 + 0 \\
    t^T A^{-T} - t^T A^T & t^T 0 + 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    I_{n \times n} & 0 \\
    0^T & 1
\end{bmatrix} = I_{(n+1) \times (n+1)} \Rightarrow H^{-T} = \begin{bmatrix}
    A^{-T} & 0 \\
    -t^T A^{-T} & 1
\end{bmatrix}
\]

\[
l'_\infty = H^{-T} l_\infty = \begin{bmatrix}
    A^{-T} & 0 \\
    -t^T A^{-T} & 1
\end{bmatrix} \begin{bmatrix}
    0 \\
    0 \\
    1
\end{bmatrix}^T = \begin{bmatrix}
    0 \\
    0 \\
    1
\end{bmatrix} = l_\infty
\]
The converse is also true: an affine transformation is the most general linear transformation (of homogeneous vectors) that fixes \( l_\infty \): To map, say, \( \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \) to \( l_\infty \) requires \( h_{31} = 0 \). To map, say, \( \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \) to \( l_\infty \) requires \( h_{32} = 0 \). Thus, since \( H \) is invertible, \( h_{33} \) must be nonzero. We conclude the last row of \( H \) is \( \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \), so \( H \) is affine.
Together, we obtain the following fact:

**Fact**

$l_\infty$ is a fixed line under the projective transformation $H$ if and $H$ is affine.

However, $l_\infty$ is not fixed pointwise under an affine transformation; i.e., an affine transformation may map one ideal point to another ideal point.

As we will see, identifying $l_\infty$ allows the recovery of affine properties (e.g.: parallelism, ratio of areas)
2.7 Recovery of affine and metric properties from images

Fig. 2.12. **Affine rectification.** A projective transformation maps $l_\infty$ from $(0, 0, 1)^T$ on a Euclidean plane $\pi_1$ to a finite line $l$ on the plane $\pi_2$. If a projective transformation is constructed such that $l$ is mapped back to $(0, 0, 1)^T$ then from result 2.17 the transformation between the first and third planes must be an affine transformation since the canonical position of $l_\infty$ is preserved. This means that affine properties of the first plane can be measured from the third, i.e. the third plane is within an affinity of the first.

Figure from Hartley and Zisserman
The Line at Infinity

If \( l = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T \), then, provided \( l_3 \neq 0 \), the following projectivity maps \( l \) to \( l_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \):

\[
H = H_A \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
l_1 & l_2 & l_3 \end{bmatrix}
\]

\((H_A \text{ can be any affine transformation})\)
Proof.

\[
H^{-T} = \left( H_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} \right)^{-T} = \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}^T \cdot H_A^T \right)^{-1}
\]

\[
= H_A^{-T} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}^{-T} = H_A^{-T} \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & l_2 \\ 0 & 0 & l_3 \end{bmatrix}^{-1}
\]

\[
= H_A^{-T} \begin{bmatrix} 1 & 0 & -l_1/l_3 \\ 0 & 1 & -l_2/l_3 \\ 0 & 0 & 1/l_3 \end{bmatrix}
\]

\[
\Rightarrow H^{-T} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = H_A^{-T} \begin{bmatrix} 1 & 0 & -l_1/l_3 \\ 0 & 1 & -l_2/l_3 \\ 0 & 0 & 1/l_3 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = H_A^{-T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
= H_A^{-T} l_\infty = l_\infty
\]
Example (Affine Rectification)

$l_\infty$ on the world plane is imaged as the vanishing line, $l$. The latter may be computed by intersecting imaged parallel lines. The image is then rectified by applying a projective warping such that $l$ is mapped to $l_\infty$. 
Fig. 2.13. **Affine rectification via the vanishing line.** The vanishing line of the plane imaged in (a) is computed (c) from the intersection of two sets of imaged parallel lines. The image is then projectively warped to produce the affinely rectified image (b). In the affinely rectified image parallel lines are now parallel. However, angles do not have their veridical world value since they are affinely distorted. See also figure 2.17.
The plane is $z = f$. The point $[X \ Y \ Z]^T$ is mapped to the point $[fx/z \ fy/z \ f]^T$ on the image plane. Ignoring the final image coordinate, we get:

$$[X \ Y \ Z]^T \mapsto [fx/z \ fy/z]^T$$

In homogeneous coordinates:

$$[fx \ fy \ z]^T = \begin{bmatrix} f & 0 \\ f & 0 \\ 1 & 0 \end{bmatrix} [X \ Y \ Z \ 1]^T$$

Figures from Hartley and Zisserman
Camera Model

To this we saw we can add, e.g.:

- a “principal point offset”; i.e.,

\[
P = \begin{bmatrix}
f & 0 & cx & 0 \\
0 & f & cy & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
f & 0 & cx \\
0 & f & cy \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

- a world-to-camera rigid-body transformation

\[
P = \begin{bmatrix}
f & 0 & cx \\
0 & f & cy \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
R & t
\end{bmatrix}
\]

- More generally:

\[
P = K \begin{bmatrix}
R & t
\end{bmatrix}
\]
Camera Model

- Let $x_c$ and $x_w$ denote a point in the camera and world coordinate systems.
  \[
  x_c = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} x_w
  \]

- If $\tilde{c} \in \mathbb{R}^3$ is the camera origin in the world coordinate system, then $t = -R\tilde{c}$ and
  \[
  x_c = \begin{bmatrix} R & -R\tilde{c} \\ 0^T & 1 \end{bmatrix} x_w
  \]

Indeed:

\[
\begin{bmatrix} R & -R\tilde{c} \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} \tilde{c} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

- The image point, $x_i$, is then
  \[
  x_i = KR \begin{bmatrix} I & \tilde{c} \end{bmatrix} x_w
  \]
Affine Camera Model

- If all objects are at the same distance from the camera, we can approximate the full perspective camera model:

\[
\begin{bmatrix}
P_{11} & P_{12} & P_{13} & P_{14} \\
P_{21} & P_{22} & P_{23} & P_{24} \\
P_{31} & P_{32} & P_{33} & P_{34}
\end{bmatrix}
\]

using an affine camera model:

\[
\begin{bmatrix}
P_{11} & P_{12} & P_{13} & P_{14} \\
P_{21} & P_{22} & P_{23} & P_{24} \\
0 & 0 & 0 & P_{34}
\end{bmatrix}
\]
Orthographic and Scaled Orthographic

Orthographic:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
R \\
t
\end{bmatrix}
\]

Scaled Orthographic:

\[
\begin{bmatrix}
s & 0 & 0 & 0 \\
0 & s & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
R \\
t
\end{bmatrix}
\]
Fig. 6.8. **Perspective vs weak perspective projection.** The action of the weak perspective camera is equivalent to orthographic projection onto a plane (at \( Z = d_0 \)), followed by perspective projection from the plane. The difference between the perspective and weak perspective image point depends both on the distance \( \Delta \) of the point \( X \) from the plane, and the distance of the point from the principal ray.
Weak Perspective versus Perspective

Fig. 6.7. As the focal length increases and the distance between the camera and object also increases, the image remains the same size but perspective effects diminish.

Figures from Hartley and Zisserman
In $\mathbb{P}^3$ is the space of homogeneous 4-vectors.

An idea point: $\begin{bmatrix} x_1 & x_2 & x_3 & 0 \end{bmatrix}^T$

In $\mathbb{P}^3$, the Euclidean space $\mathbb{R}^3$ is augmented with a set of ideal points, which are on a plane at infinity, $\pi_\infty$.

Parallel planes intersect on $\pi_\infty$.

So there are similarities to what we had in $\mathbb{P}^2$. The extra dimension, however, adds additional properties.
A projective transformation acting on $\mathbb{P}^3$ is an invertible linear transformation on homogeneous 4-vectors represented by a non-singular $4 \times 4$ matrix:

$$x' = Hx$$

The matrix $H$ representing the transformation is homogeneous and has 15 degrees of freedom (ignoring the scale).

Again, the map is a collineation (namely, lines are mapped to lines) which preserves incidence relations such as the intersection point of a line with a plane and order of contact.

In $\mathbb{P}^3$ points and planes are dual, and their representation and development is analogous to the pointline duality in $\mathbb{P}^2$. Lines are self-dual in $\mathbb{P}^3$. 
A plane in $\mathbb{R}^3$ may be written as

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

Note scaling does not matter.

The homogeneous representation of the plane is

$$\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix}^T$$

$$x = \begin{bmatrix} \lambda x_1 & \lambda x_2 & \lambda x_3 & \lambda \end{bmatrix}^T$$, the homogeneous representation of $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \in \mathbb{R}^3$, is in $\pi$ if and only if

$$\pi^T x = x^T \pi = 0$$
The first 3 components of $\pi$ correspond to the plane normal of Euclidean geometry. Using inhomogeneous notation, $\pi^T x = 0$ becomes the familiar plane equation written as

$$n^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + d = 0$$

where $n = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}^T$ and $d = \pi_4$.

In this form, $d/\|n\|$ is the distance of the plane from the origin.
In $\mathbb{P}^3$:

- A plane is defined uniquely by the join of three (non-collinear) points, or the join of a line and point (as long as the points are not incident with the line).
- Two distinct planes intersect in a unique line.
- Three distinct planes intersect in a unique point.
Suppose $x_1$, $x_2$, and $x_3$ are all on the same plane, $\pi$:

$$
\begin{bmatrix}
    x_1^T \\
    x_2^T \\
    x_3^T
\end{bmatrix}
\pi = 0_{3 \times 1}
$$

The matrix has rank 3 (since the points are non-collinear).

Up to a scale, $\pi$ is the 1D (right) null space of this matrix.

We could have done a similar thing in $\mathbb{P}^2$ to find a line defined by two points, but there it was easier to use the cross product.

How do we find the null space?
Suppose $\pi$ is determined by three non-collinear points, $(x_i)_{i=1}^3$

Let $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ be another point and set

\[ M \triangleq \begin{bmatrix} x & x_1 & x_2 & x_3 \end{bmatrix} \]

$\det M = 0$ when $x$ lies on $\pi$ since the point $x$ is then expressible as a linear combination of the other three points. Thus,

$\det M = x_1 D_{234} - x_2 D_{134} + x_3 D_{124} - x_4 D_{123}$

where $D_{jkl}$ is the determinant formed from the $jkl$ rows of the $4 \times 3$ matrix $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$. $\Rightarrow$ If $x \in \pi$,

\[ \pi = \begin{bmatrix} D_{234}, -D_{134}, D_{124}, -D_{123}^T \end{bmatrix} \]

This is the sought-after null space.
Example (3 non-collinear points define a plane)

Suppose, for $i = 1, 2, 3$, that $\mathbf{x}_i = \left[ \begin{array}{c} \tilde{x}_i \ 1 \end{array} \right]$, $\tilde{x}_i = [ x_i \ y_i \ z_i ]^T \in \mathbb{R}^3$. Recall $\pi = [ D_{234}, -D_{134}, D_{124}, -D_{123}^T ]$. Then,

$$D_{234} = \begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{vmatrix} = y_1 z_2 - y_2 z_1 - (y_1 z_3 - z_1 y_3) + y_2 z_3 - z_2 y_3$$

$$= \begin{vmatrix} y_1 - y_3 & y_2 - y_3 & y_3 \\ z_1 - z_3 & z_2 - z_3 & z_3 \\ 0 & 0 & 1 \end{vmatrix} = ((\mathbf{x}_1 - \mathbf{x}_3) \times (\mathbf{x}_2 - \mathbf{x}_3))_1$$

and similarly for the other components, giving:

$$\pi = \left[ \begin{array}{c} (\mathbf{x}_1 - \mathbf{x}_3) \times (\mathbf{x}_2 - \mathbf{x}_3) \\ -\mathbf{x}_3^T (\mathbf{x}_1 \times \mathbf{x}_2) \end{array} \right]$$

This is a familiar result from Euclidean geometry where, e.g.,

$$(\mathbf{x}_1 - \mathbf{x}_3) \times (\mathbf{x}_2 - \mathbf{x}_3)$$

is the plane normal.
By duality of points and planes in $\mathbb{P}^3$, the intersection point $x$ of three planes $(\pi_i)_{i=1}^3$ can be found as the (right) null space of the $3 \times 4$ matrix composed of the planes as rows:

$$
\begin{bmatrix}
\pi_1^T \\
\pi_2^t \\
\pi_3^R
\end{bmatrix}
$$

We can apply a similar approach using sub-determinants as before, but numerically it is usually better to use a least-square approach.
In analogy to result from $\mathbb{P}^2$:

- Under the point transformation $x' = Hx$, a plane transforms as

$$\pi' = H^{-T}\pi$$
\( \pi_\infty \)

- \( \pi_\infty \) is useful to recover affine properties distorted by projective transformation.
- Two planes are parallel if, and only if, their line of intersection is on \( \pi_\infty \).
- A line is parallel to another line, or to a plane, if the point of intersection is on \( \pi_\infty \).
- We then have, in \( \mathbb{P}^3 \), that any pair of planes intersect in a line, with parallel planes intersecting in a line on the plane at infinity.
- \( \pi_\infty \) is a fixed plane under a projective transformation \( H \) if, and only if, \( H \) is affine.
- While \( \pi_\infty \) is fixed as a set under an affine transformation, it is not fixed pointwise.
- Under a particular affine transformation there may be planes, in addition to \( \pi_\infty \), which are fixed. However, only \( \pi_\infty \) is fixed under any affine transformation.