Wireless Capacity with Oblivious Power in General Metrics

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Outline

1. Introduction
   - Definition of the Problem
2. Model and Preliminaries.
3. Algorithm
4. Range Assignment Problem

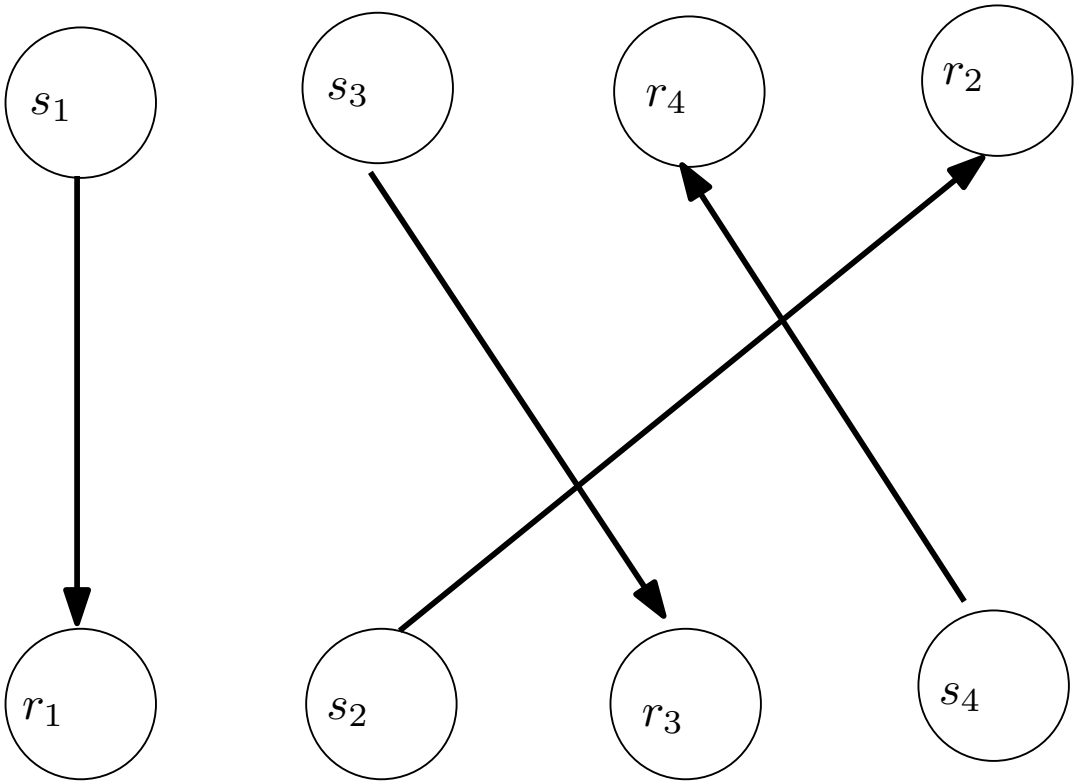
Definition of the Problem

How much communication can occur simultaneously in any given wireless network? This basic question addressed here of the *capacity* of a wireless network has attracted tremendous attention, led by the highly-cited work on non-constructive average-case analysis by Gupta and Kumar. Less is known about algorithmic results on arbitrary instances.

Definition of the Problem

s_v (sender) and r_v (receiver) are points in the metric space.
Link ℓ_v represents a directed communication request from a sender s_v to a receiver r_v.
A power P_v is associated with each link l_v.
We will use the *SINR* model to represented the *interference* between two links in the metric space.

The Problem



The problem

Given a set of n links we need to find the maximum number of links that can communicate simultaneosly.

Definitions

Given is a set $L = \{\ell_1, \ell_2, \ldots, \ell_n\}$ of links, where each link ℓ_v represents a communication request from a sender s_v to a receiver r_v.
$d(x, y)$ - distance between x and y.
$d_{vw} = d(s_v, r_w)$ - distance from link ℓ_v to link ℓ_w.
Let P_v denote the power assigned to link ℓ_v.

Assumptions

We will work with the *path loss radio propagation* model for the reception of signals.
Let α be the *path loss exponent*, then the power $\mathcal{P}$ is $\mathcal{P}/d(x,y)^{\alpha}$, ($\alpha > 0$).

Assumptions

We worked in the *SINR* model of interference, in which a node r_v successfully receives a message from a sender s_v if and only if the following condition holds:

$$\frac{P_v/\ell_v^\alpha}{\sum_{\ell_w \in S \setminus \{\ell_v\}} P_w/d_{wv}^\alpha + N} \geq \beta, \tag{1}$$

Where N is a constant noise, and $\beta \geq 1$ denotes the minimum SINR, and S is the set of concurrently scheduled links in the same slot.

Definitions

We say that S is *SINR-feasible* (or simply *feasible*) if (1) is satisfied for each link in S.
We want to find the maximum *SINR-feasible* set of links.

Affectance

The affectance $a_w^P(v)$ of link ℓ_v caused by another link ℓ_w, with a given power assignment P, is the interference of ℓ_w on ℓ_v relative to the power received, or

$$a_w^P(v) = c_v \frac{P_w/d_{wv}^\alpha}{P_v/\ell_v^\alpha} = c_v \frac{P_w}{P_v} \cdot \left(\frac{\ell_v}{d_{wv}}\right)^\alpha,$$

where $c_v = \beta/(1 - \beta N \ell_v^\alpha / P_v)$ is a constant depending only on the length and power of the link ℓ_v.
let $a_v^P(S) = \sum_{w \in S} a_v^P(w)$ and $a_S^P(v) = \sum_{w \in S} a_w^P(v)$.
Using this notation, Eqn. 1 can be rewritten as $a_S^P(v) \leq 1$.

Definitions for power assignments

Definition

A power assignment P is *length-monotone* if $P_v \geq P_w$ whenever $\ell_v \geq \ell_w$.

Definition

A power assignment P is *sub-linear* if $\frac{P_v}{\ell_v^\alpha} \leq \frac{P_w}{\ell_w^\alpha}$ whenever $\ell_v \geq \ell_w$.

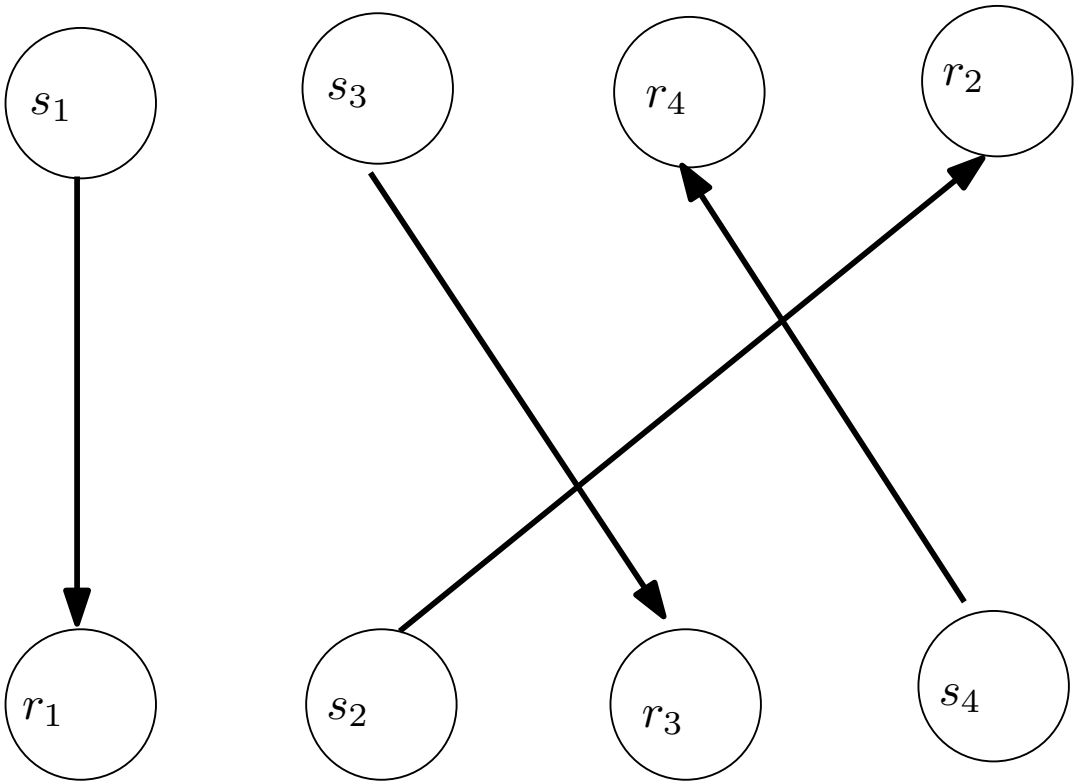
Algorithm Overview

We will see now the algorithm to the $\mathcal{P}$-Capacity problem for *fixed* power assignments.

Algorithm **C**(L,P):

Suppose the links $L = \{\ell_1, \ell_2, \ldots, \ell_n\}$ are in non-decreasing order of length.
Let $\gamma = 1/2$
$S \leftarrow \emptyset$
for $v \leftarrow 1$ to n do
    if $a_S^P(v) + a_v^P(S) < \gamma$
        then add ℓ_v to S
Output $X = \{\ell_v \in S : a_S^P(v) \leq 1\}$

Example



For the following input:
$s_1(0, 10), s_2(15, 0), s_3(5, 16), s_4(52, 0)$.
$r_1(0, 0), r_2(39, 32), r_3(17, 0), r_4(40, 16)$.

Distances:

R\S	1	2	3	4
1	10	45	20	40
2	15	40	2	30
3	17	38	20	35
4	52	35	35	20

For the following input:
$s_1(0,10), s_2(15,0), s_3(5,16), s_4(52,0)$.
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We assign for every link v, $P_v = l_v^3$

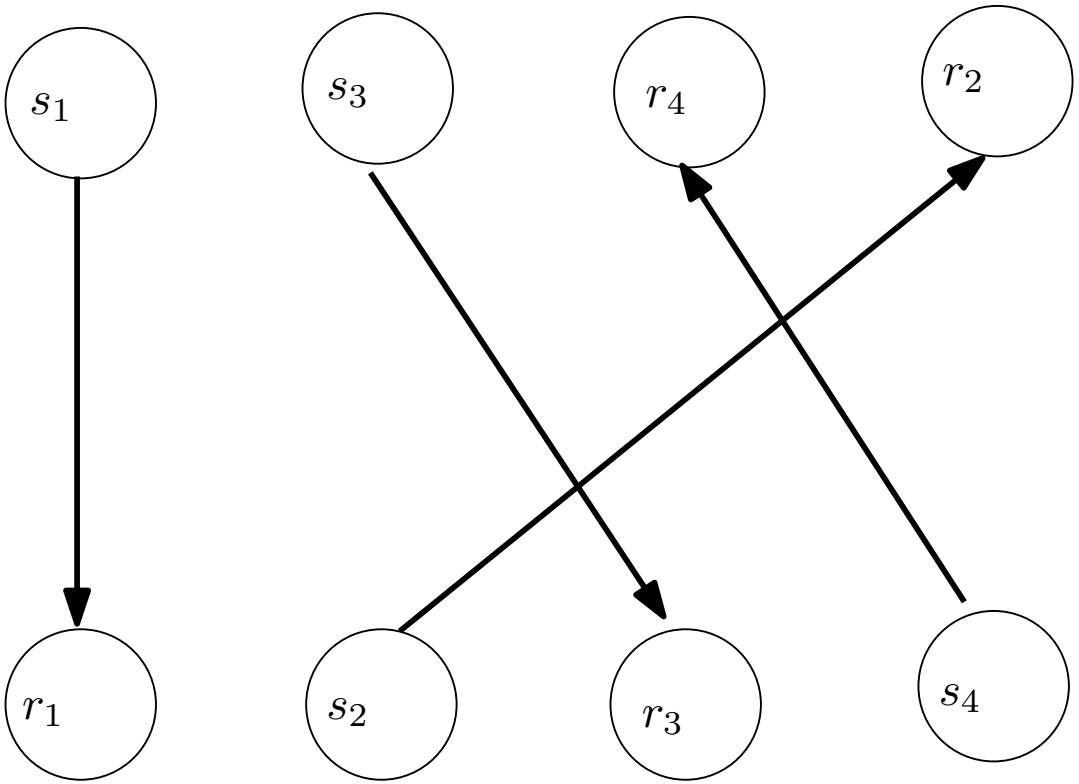
For the following input:
$s_1(0,10), s_2(15,0), s_3(5,16), s_4(52,0)$.
$r_1(0,0), r_2(39,32), r_3(17,0), r_4(40,16)$.
We calculate the affectance:

$$a_w^P(v) = c_v \frac{P_w/d_{wv}^\alpha}{P_v/\ell_v^\alpha} = c_v \frac{P_w}{P_v} \cdot \left(\frac{\ell_v}{d_{wv}}\right)^\alpha,$$

, where $c_v = \beta/(1-\beta N \ell_v^\alpha/P_v)$
With $\beta = 1$, $N = 4$, $\alpha = 2$.
Affectance:

	1	2	3	4
1	1.67	0.01	0.16	0.04
2	47.41	1.11	1000.0	4.44
3	4.61	0.15	1.25	0.41
4	0.42	0.18	0.41	1.25

Example



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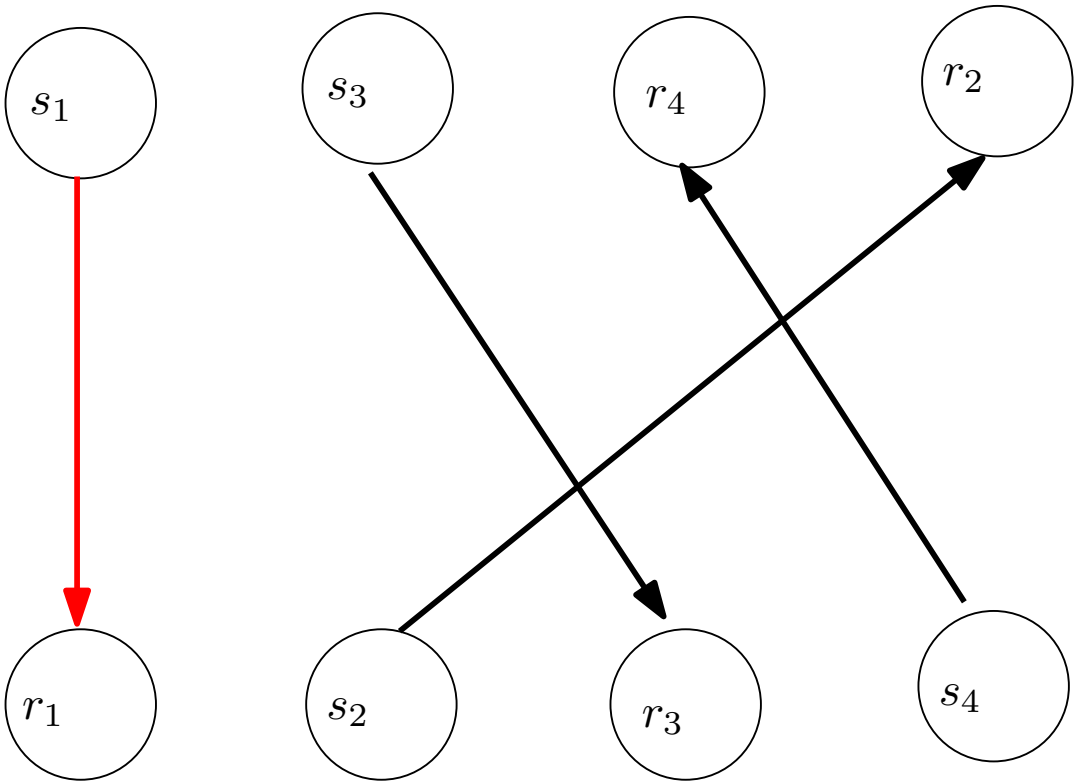

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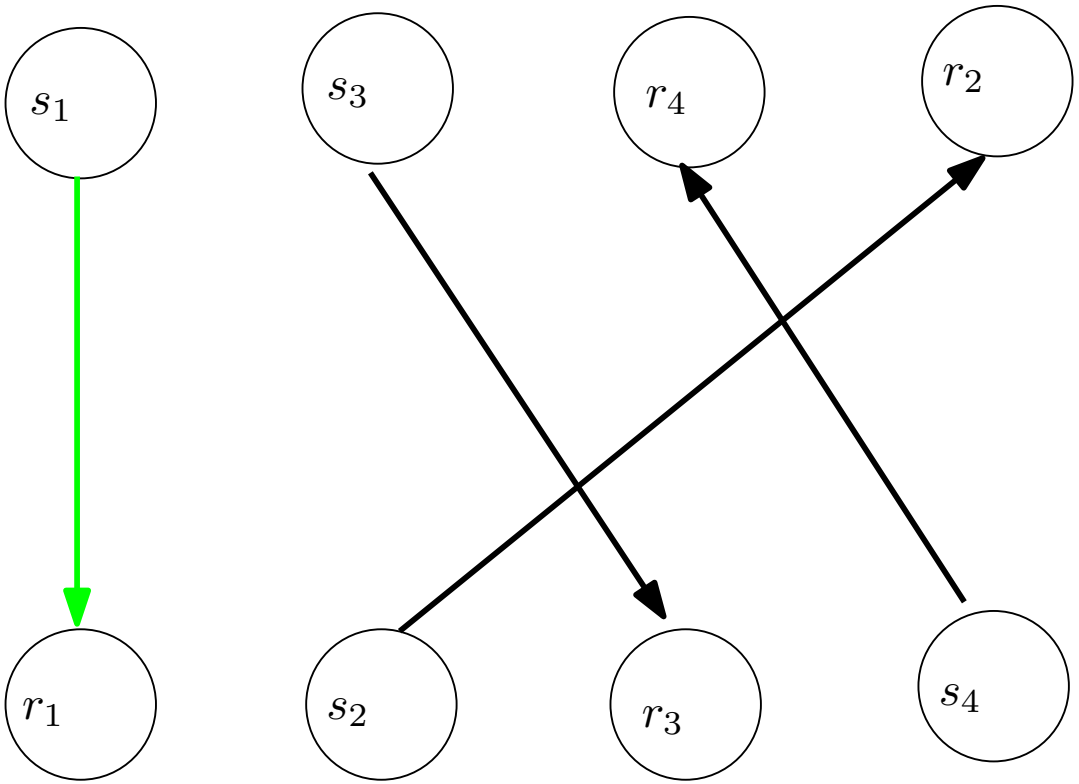

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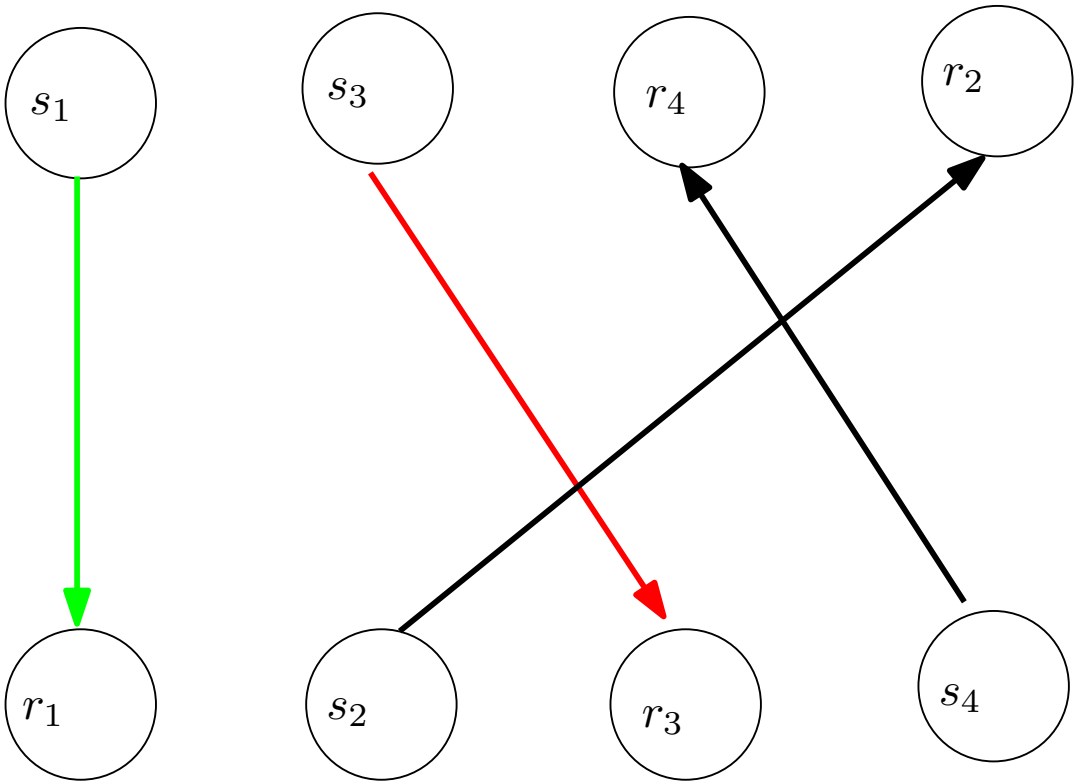

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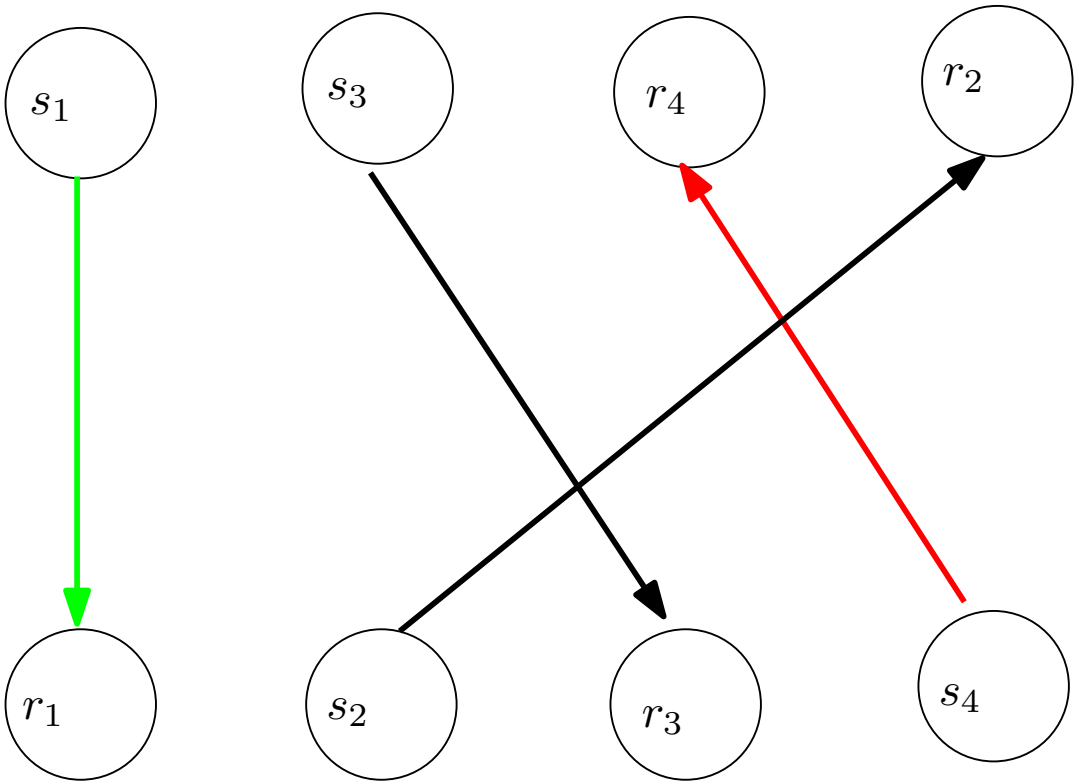

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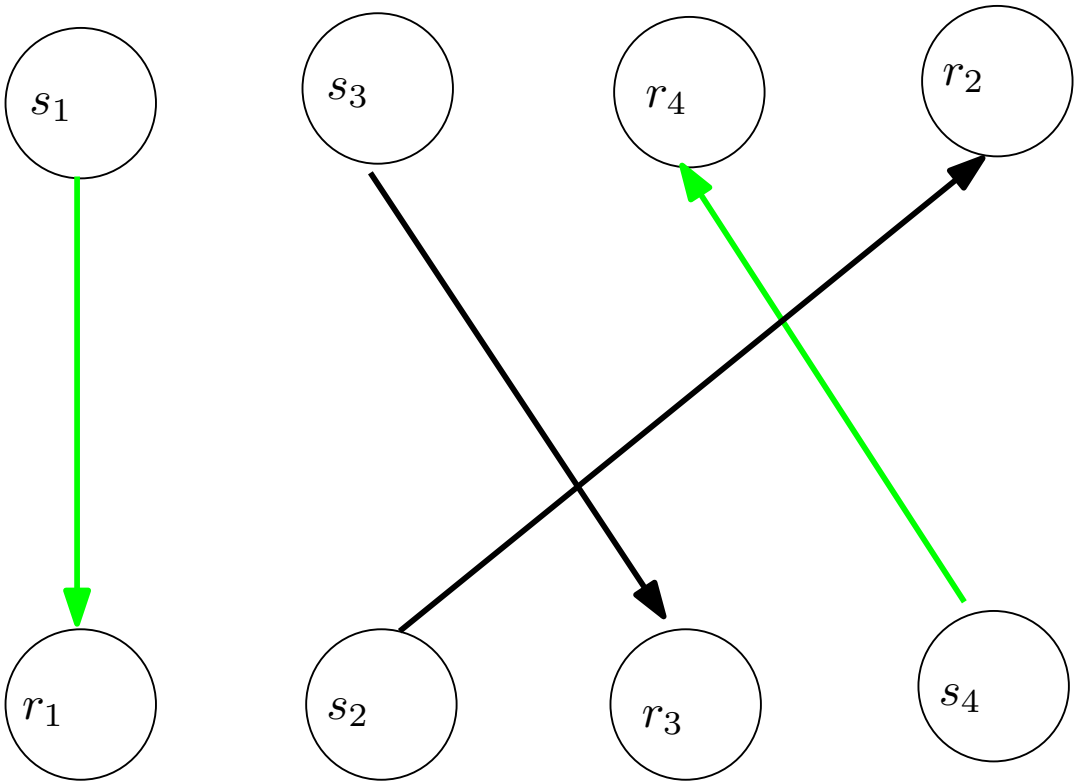

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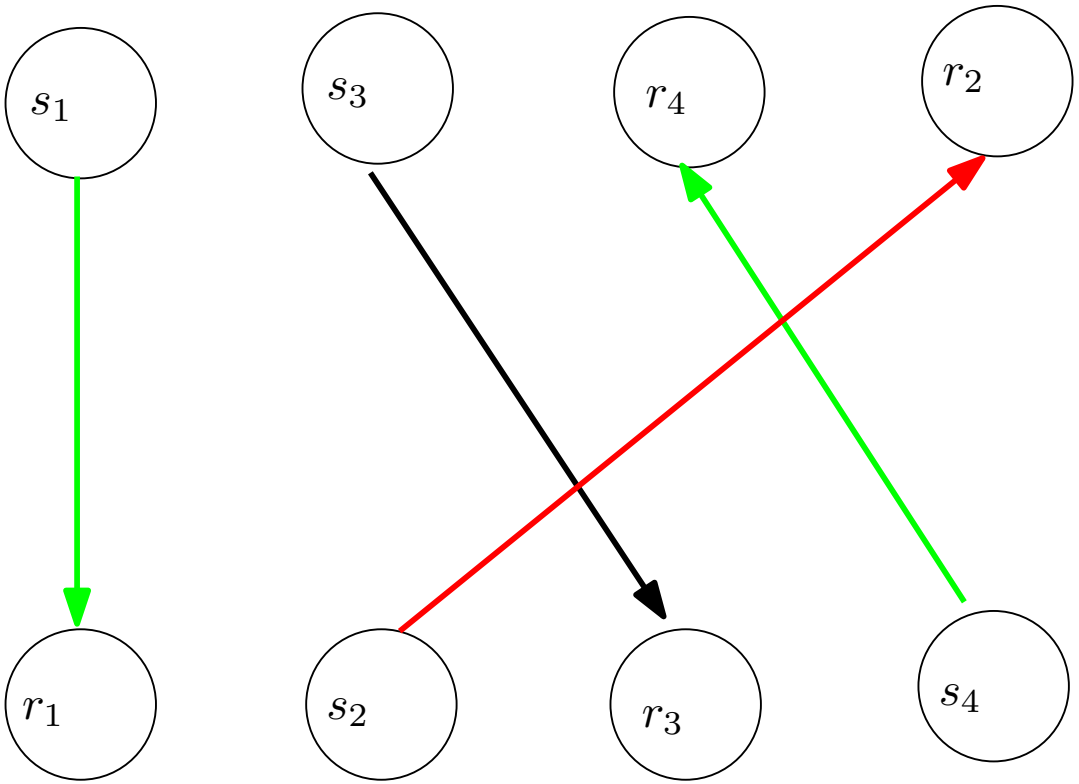

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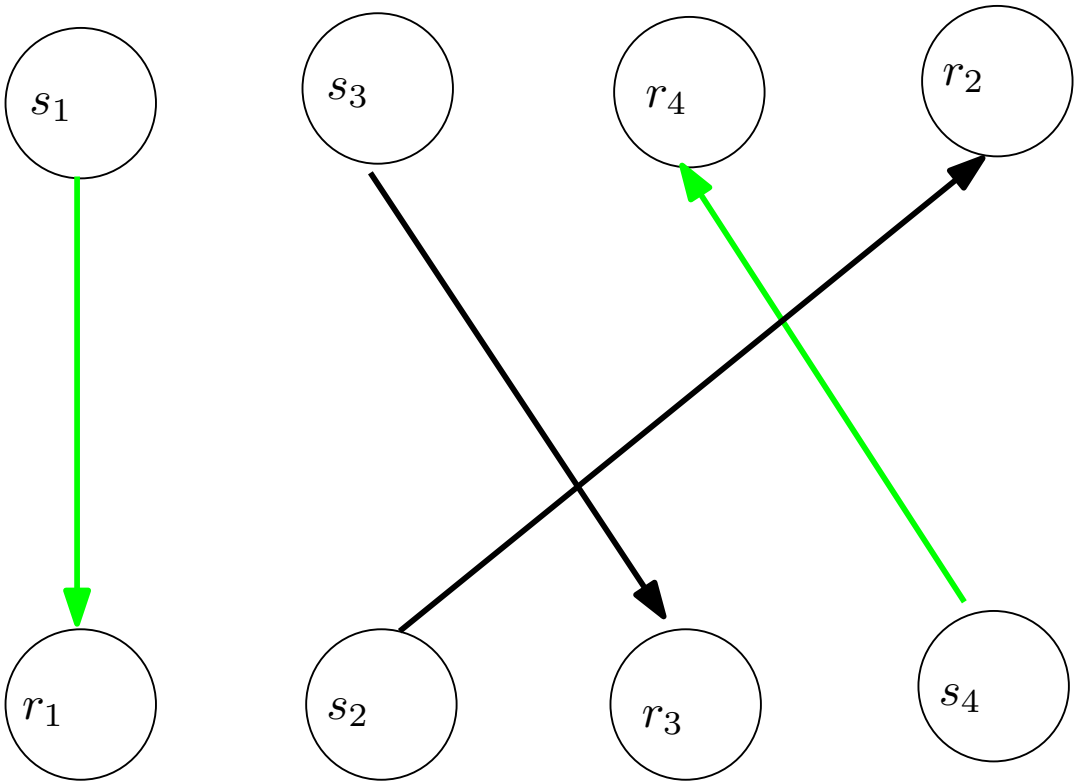


Main Theorem

Theorem

Algorithm **C** *approximates* $\mathcal{P}$*-Capacity within a constant factor for any length-monotone, sub-linear* $\mathcal{P}$*.*

The feasibility of the solution output by **C** is evident by virtue of the last line of the algorithm.

Definition

Definition

Definition A δ-*signal* set is one where the affectance of any link is at most $1/\delta$.

A set is SINR-feasible iff it is a 1-signal set.
Let $OPT_\delta^P = OPT_\delta^P(L)$ be the maximum number of links in a δ-signal set given a power assignment P.

Algebric definitions

Definition

For a matrix $\mathbf{M}$ (vector $\mathbf{v}$), let $Se(\mathbf{M})$ ($Se(\mathbf{v})$) denote the sum of entries in the matrix (vector).

Definition

A matrix $\mathbf{M} = (m_{ij})$ is *q-approximately-symmetric* if, for any two indices i and j, $m_{ij} \leq q \cdot m_{ji}$ for some $q \geq 0$.

Claim

Let $\mathbf{M}$ *be a* $n \times n$ *matrix that contains only non-negative entries. Assume* $Se(\mathbf{M}) \leq \gamma n$ *for some* $\gamma \geq 0$*. Then, for any* $\lambda > 1$*, there are at least* $(1 - \frac{1}{\lambda})n$ *rows (columns)* $\mathbf{r}$ *of* $\mathbf{M}$ *for which* $Se(\mathbf{r}) \leq \gamma \cdot \lambda$*.*

Lemma

Let n *be a number and* $\mathbf{p}$ *be a positive* n*-dimensional vector. Let* $\mathbf{M}$ *be a non-negative* q*-approximately-symmetric* n*-by-*n *matrix such that* $\mathbf{Mp} \leq \mathbf{p}$*. Then,* $Se(\mathbf{M}) \leq (q+1)n$*.*

Proposition

Let L be set of links and P be a power assignment for L. Then, for any $\tau \leq \delta$, $OPT_{\delta}^{P}(L) \geq \frac{OPT_{\tau}^{P}(L)}{4\delta/\tau}$.
From: Oblivious Interference Scheduling [2009]

Lemma (3)

Let ℓ_u, ℓ_v *be links in a* q^{α}*-signal set. Then,* $d_{uv} \cdot d_{vu} \geq q^2 \cdot \ell_u \ell_v$.

Lemma

Let S and X be the sets created by the algorithm **C** *on a given link set. Then, $|X| \geq |S|(1-\gamma)$.*

Proof.

Observe that by the algorithm construction, the sum $Se(\mathbf{S}) = \sum_{v,w \in S} a_v(w) = \sum_{v \in S} a_S(v)$ is upper bounded by $\gamma|S|$. Then the Lemma follows from Claim 1 by plugging in $\lambda = \frac{1}{\gamma}$. □

Now we will relate the size of the set S to the optimal solution. We need another definition here:

Definition

For a set S and link ℓ_v, let $S_v^- = \{\ell_w \in S : \ell_w \leq \ell_v\}$ and $S_v^+ = \{\ell_w \in S : \ell_w \geq \ell_v\}$.

Lemma (Red-Blue Lemma)

Let R and B be disjoint sets of links, referred to as red and blue links in a length-monotone, sub-linear power metric. If $|B| > 4|R|$ and B is a 3^α-signal set, then there is a blue link ℓ_b such that $a_{R_b^-}(b) + a_b(R_b^-) \leq 3^\alpha(a_B(b) + a_b(B))$.

Proof for the previous lemma

Lemma

Let R and B be disjoint sets of links, referred to as red and blue links in a length-monotone power metric. If $|B| > 2|R|$ *and B is a* 3^α*-signal set, then there is a set* $B' \subseteq B$ *of size at least* $|B| - 2|R|$ *such that for all* $\ell_b \in B'$, $a_{R_b^-}(b) \leq 3^\alpha a_B(b)$.

Lemma

Let R and B be disjoint sets of links, referred to as red and blue links in a sub-linear power metric. If $|B| > 2|R|$ *and B is a* 3^α*-signal set, then there is a set* $B' \subseteq B$ *of size at least* $|B| - 2|R|$ *such that for all* $\ell_b \in B'$, $a_b(R_b^-) \leq 3^\alpha a_b(B)$.

Proof

We will show proof only for the first lemma, the other is very similar.
For each link $\ell_r \in R$, we will assign a set of "guards" $X_r \subseteq B$.
For different ℓ_r, the X_r's will be of size at most 2. We remove all the guard sets from S and it will satisfied the lemma.

Here's how we choose the guards:

$B' \leftarrow B$.
For each link $\ell_r \in R$
add to X_r the link $\ell_x \in B_r^+$ with the sender nearest to s_r (if one exists)
add the link $\ell_y \in B_r^+$ with the receiver nearest to r_r (if one exists; possibly, $\ell_y = \ell_x$)

Proof (cont.)

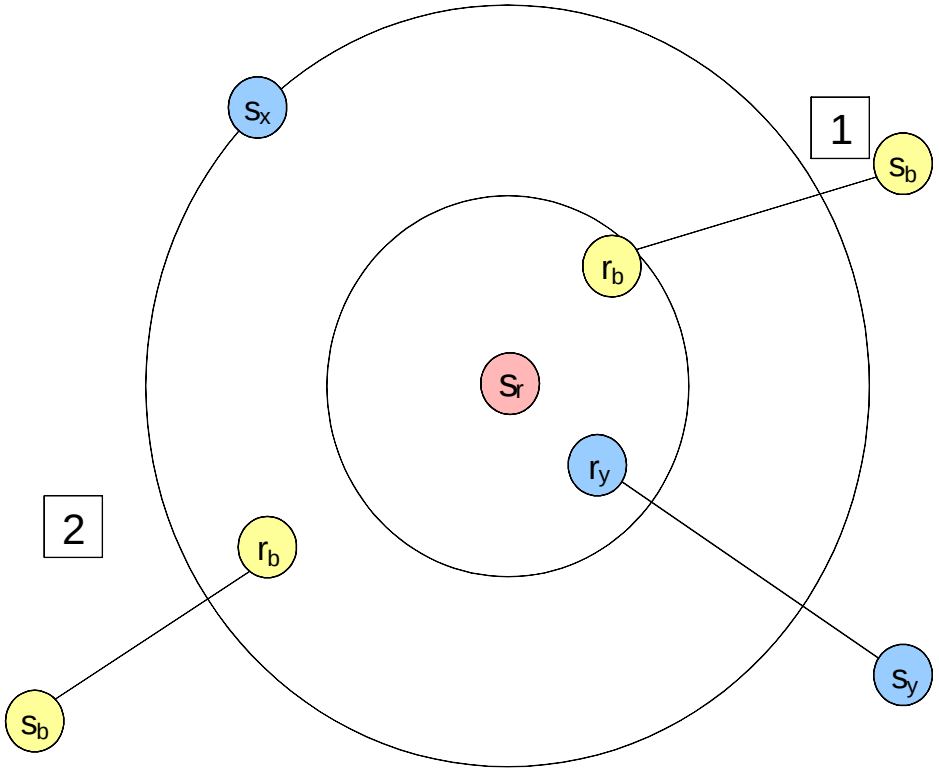


Figure : Example configuration of guards and points. Guard s_x (on left) is the nearest blue sender to red point s_r (in center) among those of links larger than ℓ_r.

Proof (cont.)

Since $|B| > 2|R|$, $|B'| \geq |B| - 2|R| > 0$, by construction. Consider any link $\ell_b \in B'$. In what follows, we will show that the affectance of any link $l_r \in R_b^-$ on ℓ_b is comparable to the affectance of one of the guards of l_r on ℓ_b. Once we recall that the guards are blue, this implies that the overall affectance on ℓ_b from R_b^- is not much worse than that from B. Since $\ell_b \notin X_r$,

X_r is non-empty and contains guards ℓ_x and ℓ_y (potentially $\ell_x = l_y$). Consider any $\ell_r \in R_b^-$. Let d denote $d(s_x, s_r)$ and observe that since $\ell_b \notin X_r$, $d(s_b, s_r) \geq d$.

Proof (cont.)

We claim that $d_{rb} \geq d/2$. Before proving the claim, let us see how this leads to the proof of the Lemma. If the claim is true, then by the triangular inequality,
$d_{xb} = d(s_x, r_b) \leq d(s_x, s_r) + d(s_r, r_b) = d + d_{rb} \leq 3d_{rb}$. Since $\ell_x \geq \ell_r$ and the power metric is length-monotone, $P_x \geq P_r$.
Thus, $\frac{a_x(b)}{a_r(b)} = \frac{P_x}{P_r}\left(\frac{d_{rb}}{d_{xb}}\right)^{\alpha} \geq 3^{-\alpha}$. Summing over all links in B,
$a_B(b) \geq \sum_{\ell_r \in R_b^-} a_{X_r}(b) \geq 3^{-\alpha} \sum_{\ell_r \in R_b^-} a_r(b) = 3^{-\alpha} a_{R_b^-}(b)$.

Proof (cont.)

To prove the claim that $d_{rb} \geq d/2$, let us suppose otherwise for contradiction. Then, by the triangular inequality, $\ell_b = d(s_b, r_b) \geq d(s_b, s_r) - d(r_b, s_r) > d - d/2 = d/2$. Since ℓ_y was chosen into X_r, its receiver is at least as close to s_r as r_b, that is $d(r_y, s_r) \leq d(r_b, s_r) < d/2$, and its sender is also at least far as s_x, or $d(s_y, s_r) \geq d$. So, $\ell_y \geq d(s_y, s_r) - d(s_r, r_y) > d/2$.

Proof (cont.)

Now, $d(r_y, r_b) \leq d(r_y, s_r) + d(s_r, r_b) < d$ and $d_{yb} \cdot d_{by} \leq (\ell_y + d(r_y, r_b)) \cdot (\ell_b + d(r_y, r_b)) < (\ell_y + d) \cdot (\ell_b + d) < 9 \cdot \ell_y \ell_b$. But since B is a 3^α-signal set, this is a contradiction by Lemma 3. Hence, any link ℓ_r in R_b^- satisfies $d(s_r, r_b) = d_{rb} \geq d/2$.

Lemma

Let S be the set found by the algorithm and $\tau = 3^{\alpha+1}/2\gamma$. Then, $|S| \geq OPT_{2\tau}/10$.

Proof.

By Claim 1, there is a set $O \subseteq OPT_{2\tau}$ of size at least $OPT_{2\tau}/2$ such that for all $u \in O$, $a_u(O) \leq \frac{1}{\tau}$. By definition, $a_O(u) \leq \frac{1}{2\tau}$. We claim that $|S| \geq |O|/5$. Suppose otherwise.
Then, $|S| < |O \setminus S|/4$. Applying Lemma 6 with $B = O \setminus S$ and $R = S$, we find that there is a link ℓ_b in $O \setminus S$ that satisfies $a_{S_b^-}(b) + a_b(S_b^-) \leq 3^\alpha(a_B(b) + a_b(B)) \leq 3^\alpha(\frac{1}{\tau} + \frac{1}{2\tau}) = \gamma$. The operation of the algorithm is then such that the algorithm would have added the link ℓ_b to S, which is a contradiction. □

The problem Definition

Given A set of point A in the three-dimentional space, a positive number q.
We want to find a complete range assignment for A of total cost at most q.

This problem is NP-hard, we will show an 2-approximation algorithm for the problem where q is the miniumum cost available.

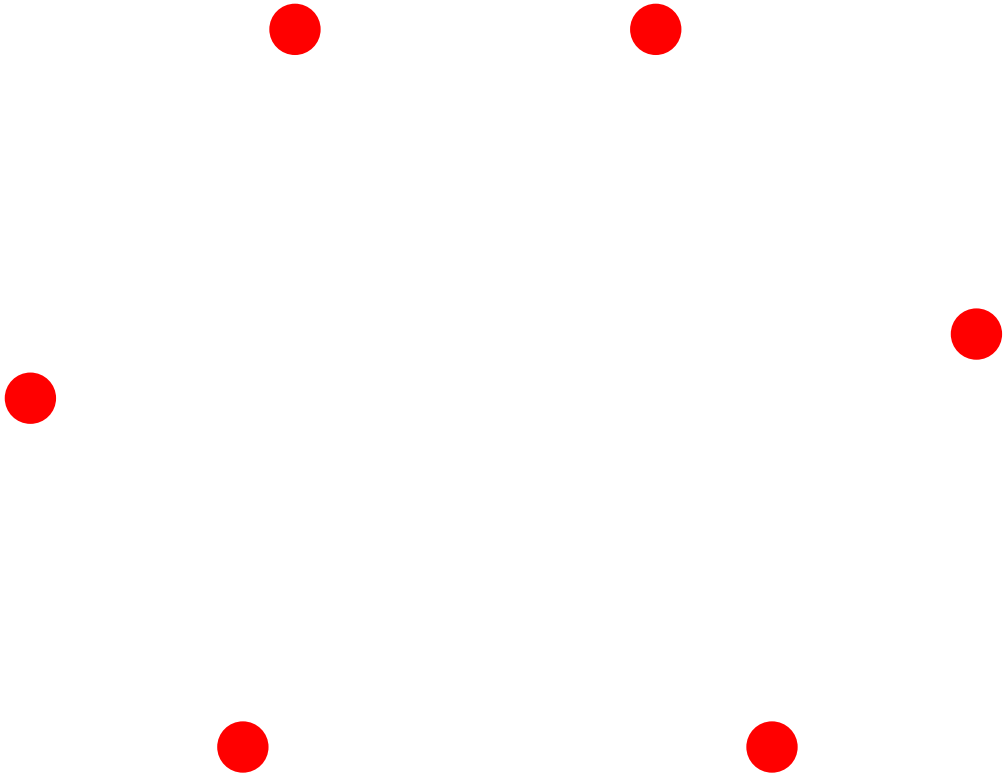
2-Approximation Algorithm

Given a set $V = \{x_1, x_2, \ldots, x_n\}$ of points in 3-space.

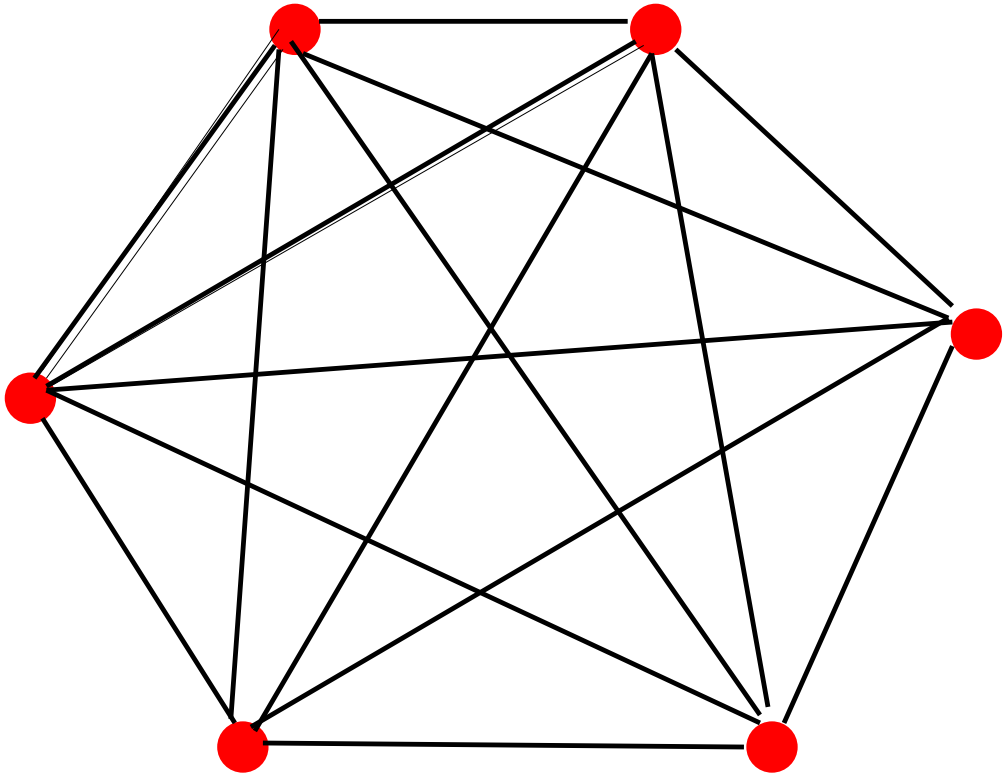
Algorithm approximation(V):

1. Construct an undirected weighted complete graph $G(V)$ with vertices V and where the weight of the edge between x_i and x_j is $d(x_i, x_j)^2$ for all i and j.
2. Find a minimum weight spanning tree T of $G(V)$.
3. For $i = 1, \ldots, n$ assign the range of x_i to be the maximum of $d(x_i, x_j)$ over j such that (x_i, x_j) is an edge in T.

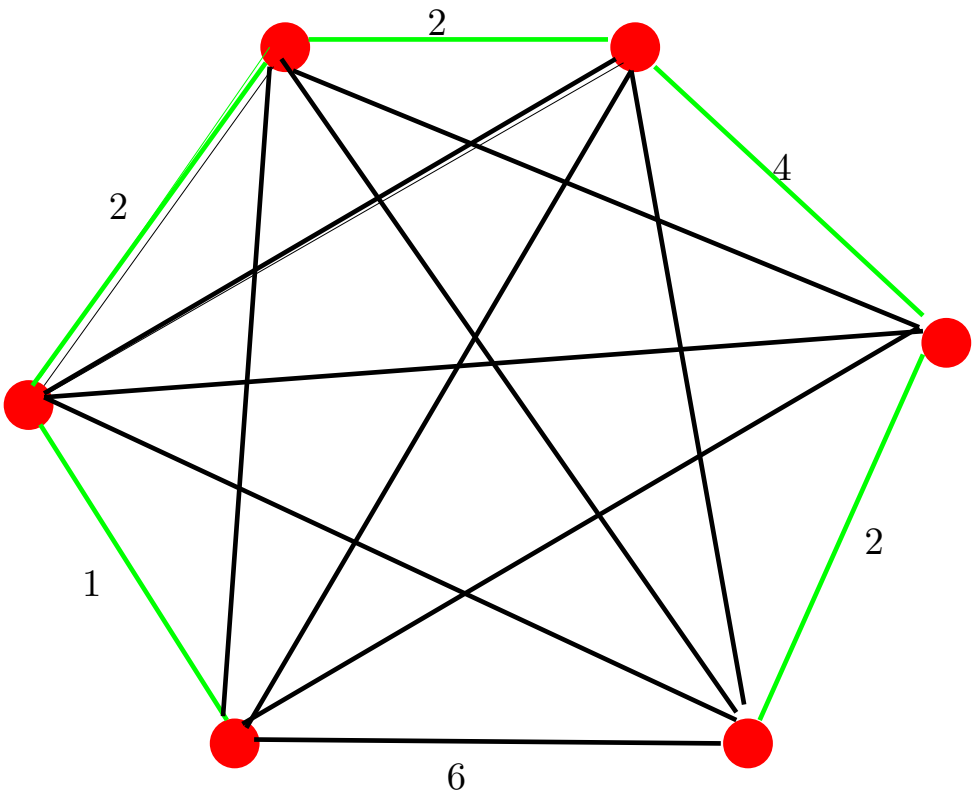
Algorithm Input



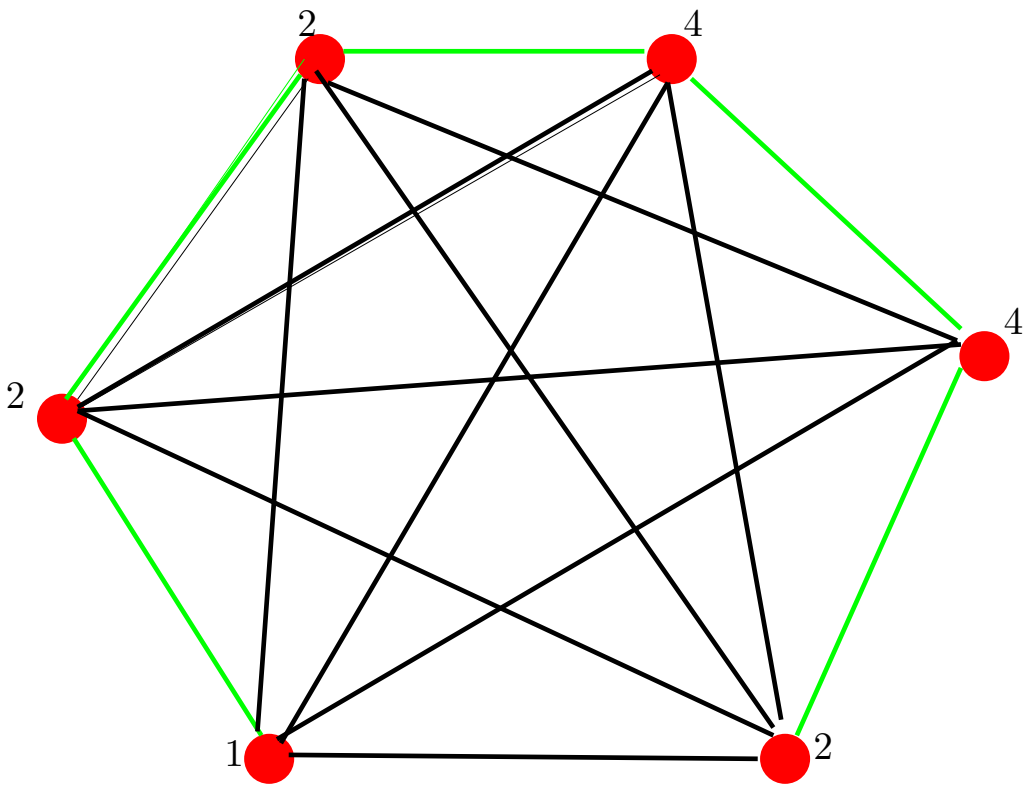
Construct full graph



Find weighted MST



Algorithm Result



Theorem

Let $OPT(V)$ be the minimum cost of a complete range assignment for V and let $APP(V)$ be the cost of the complete range assignment for V found by the alogorithm, Then $APP(V) < 2 * OPT(V)$.

Proof.

Let $MST(V)$ be the cost of the minimum weight spanning tree T of $G(V)$, The theorem follows immediately from the following claims: □

Claim 1

Claim

$OPT(V) > MST(V)$.

Proof.

Given an optimal assignment we can costruct a spanning tree of $G(V)$ by choosing any vertex, constructing a shortest path destination tree with the chosen vertex as the destination. Since each of the $n-1$ vertices other than the root vertex must have a range assigned which establishes the edges of the destination tree, $OPT(V)$ is greater than the weight of the resulting spanning tree which in turn is $\geq MST(V)$.

□

Claim 2

Claim

$APP(V) < 2 * MST(V)$

Proof.

$$APP(V) = \sum_{i=1}^{n} \max_{\{j|\{x_i,x_j\}\in T\}} d(x_i, x_j)^2 \leq \sum_{i=1}^{n} \sum_{\{j|\{x_i,x_j\}\in T\}} d(x_i, x_j)^2 = 2 * MST(V)$$

□