Two continuous models for the dynamics of sandpile surfaces

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We consider a modified Bouchaud–Cates–Ravi Prakash–Edwards model for pile surface dynamics, and show that in the long-scale limit this model converges to a quasistationary model of pile growth in the form of an evolutionary variational inequality.

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I. INTRODUCTION

Recently much interest in the physics of granular media was stimulated, in particular, by two salient features of the granular state: multiplicity of metastable pile shapes and the occurrence of avalanches upon pile surfaces. It was realized that, to account for metastability, the model of pile surface dynamics should not be written as an evolutionary equation for the pile surface alone. An additional unknown characterizing the flow of grains down the pile surface is useful because such flows are not uniquely determined by the external source and local free surface topography.

A large spatiotemporal scale pile growth model involving two coupled dependent variables, and able to account for metastability, was proposed in Refs. [1,2]. This model neglects avalanches as small fluctuations of the pile surface, and describes the evolving mean surface of a pile that grows on an arbitrary support under a given distributed source of bulk material. The model permits an equivalent formulation as an evolutionary variational or quasivariational inequality; such a formulation significantly simplifies both the mathematical study of the problem [3] and its numerical solution [1,4,5]. As shown in Ref. [2], the shapes of real piles on flat open platforms [6] are described by analytical solutions of this inequality. A modification of the model, able to account for avalanches as almost instantaneous slides, was also discussed in Ref. [2]; according to observations made in the same work, such a slide may, indeed, be a possible avalanche scenario (also see Ref. [7]).

Independently, using different arguments, the same pile growth model in the form of a variational inequality was derived by Aronsson, Evans, and Wu [8]. In Ref. [9], Evans *et al.* studied its discontinuous solutions corresponding to avalanches; in Ref. [10] Evans and Rezakhanlou showed that the cellular automata models of sandpiles, presented as intuitively attractive examples in almost all works on self-organized criticality [11], converge in a continuous limit to a similar variational inequality with an anisotropy inherited from the cellular structure of these crude models.

A different continuous model, also involving two coupled dependent variables and describing the granular surface flow and pile surface dynamics, was proposed by Bouchaud, Cates, Ravi Prakash, and Edwards (the BCRE model) [12,13]. Although the choice of the basic variables in this model is equivalent to that in Refs. [1,2], the model is written for free surfaces only slightly deviating from the critical slope and employs different phenomenological constitutive relations. Emphasis is placed on the simulation of fast processes, like amplification and distinction of rolling grains population during an avalanche. The BCRE model was simplified by de Gennes [14], applied to various one-dimensional surface flow problems (see, e.g., Refs. [15,16]), and modified for thick surface granular flows [17]. Further exact solutions to simplified BCRE equations can be constructed by methods proposed in Refs. [18,19]. Using the BCRE model, Bouchaud and Cates [20] explained another type of avalanche (in a thin granular layer on an inclined plane; see Ref. [21]).

Our aim here is to investigate a relation between the two models mentioned above. After briefly outlining the variational and BCRE models, we propose a full-dimensional generalization of the latter, originally formulated by BCRE in the one-dimensional case. To do this, we modify and extend the constitutive relations determining the surface flow velocity and the rolling-to-immobilized-state transition rate: BCRE's assumption that the slope is almost critical everywhere is too restrictive for our purpose. Rescaling the variables, we show that the modified BCRE model contains a small parameter, the ratio of a characteristic rolling grains layer thickness to the pile size, and hence may often be simplified by employing a quasistationary equation for the rolling grains layer. The issue of scaling turns out to be very important in a description of pile growth: another dimensionless parameter in the model thus obtained is the ratio of a typical rolling grain path length to the pile size. For large piles, this coefficient is also small, and we show that in the long-scale limit the modified BCRE model tends toward the variational model [1,2]. For small piles, the corresponding term can be significant. These results make it clear why the shapes of small and large piles differ, and, correspondingly, why different models should be used to simulate, say, the formation of large sand dunes and small Aeolian ripples.

II. VARIATIONAL MODEL OF PILE GROWTH

Let a cohesionless granular material having an angle of repose α_r be tipped out onto a given rough rigid surface

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$y=h_0(x)$, where $x=(x_1,x_2)\in\mathbb{R}^2$. We want to find the shape of a pile thus generated.

The real process of pile growth is often intermittent: discharged granular material not only flows continuously over the pile slopes but is also able to build up and then to pour suddenly down the slope in an avalanche. However, the avalanches usually involve only a small amount of particles in a pile and cause small fluctuations of the pile free surface. The model [1,2] neglects these fluctuations, and is a model for the mean surface evolution. Whether the pile evolution is governed by a continuous surface flow or results from many small avalanches, the surface flow is typically confined to a thin boundary layer which is distinctly separated from the motionless bulk [22].

Let us assume for simplicity that the support surface has no steep slopes, i.e.,

$$|\nabla h_0|\leqslant k,$$

where $k=\tan\alpha_r$ (see Refs. [1–3] for the general case). Assuming the bulk density of material in a pile is constant, we can write the conservation law as

$$\partial_t h+\nabla\cdot\boldsymbol{q}=w,$$

where $h(x,t)$ is the free surface, $\boldsymbol{q}(x,t)$ is the horizontal projection of the flux of rolling particles, and $w(x,t)$ the source intensity. We neglect the inertia and suppose that surface flow is directed toward the steepest descent,

$$\boldsymbol{q}=-m\nabla h,$$

where

$$m(x,t)\geqslant 0 \tag{1}$$

is an *unknown* scalar function. The conservation law now takes the form

$$\partial_t h-\nabla\cdot(m\nabla h)=w. \tag{2}$$

It is assumed in this model that the surface slope angle cannot exceed the angle of repose,

$$|\nabla h|\leqslant k, \tag{3}$$

and that no pouring occurs over the parts of the pile surface which are inclined less:

$$|\nabla h(x,t)|<k\Rightarrow m(x,t)=0. \tag{4}$$

To complete the model we have to specify the initial condition,

$$h|_{t=0}=h_0, \tag{5}$$

and a boundary condition. Let the granular material be allowed to leave the system freely through part Γ_1 of the boundary of domain $\Omega\subset\mathbb{R}^2$, and the other part of the boundary, Γ_2, presents an impermeable wall. The boundary conditions are then

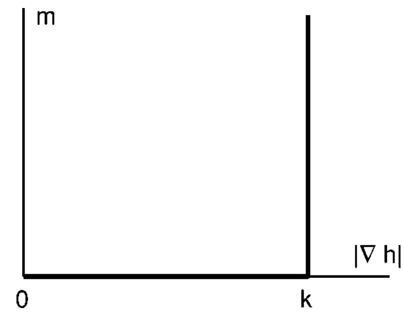

FIG. 1. Multivalued constitutive relation $m=m(|\nabla h|)$.

$$h|_{\Gamma_1}=h_0|_{\Gamma_1},\quad m\partial_n h|_{\Gamma_2}=0. \tag{6}$$

The model (1)–(6) contains two coupled unknowns: the free surface h, and an auxiliary function m determining the rolling grains flux magnitude. Conditions (1), (3), and (4) define m as a *multivalued* function of $|\nabla h|$; see Fig. 1.

The problem (1)–(6) may be considered an anomalous diffusion problem, and solved by approximating this highly nonlinear multivalued relation. However, a better way to solve this problem is based on its following reformulation in the form of an evolutionary variational inequality (see Refs. [23] and [24] for variational inequalities in mechanics and physics and their numerical solution, respectively).

Let us define the set K of possible surfaces as

$$K=\{\varphi(x)\ \mid\ |\boldsymbol{\nabla}\varphi|\leqslant k,\quad \varphi|_{\Gamma_1}=h_0|_{\Gamma_1}\}$$

and the scalar product of two functions as $(\phi,\psi)=\int_\Omega\phi\psi\,dx$. We can now consider the following problem (variational inequality):

$$\text{Find } h(x,t) \text{ such that } h\in K \text{ for all } t>0,$$

$$(\partial_t h-w,\varphi-h)\geqslant 0 \text{ for all } \varphi\in K, \tag{7}$$

$$\text{and } h|_{t=0}=h_0.$$

Theorem. Function $h(x,t)$ is a solution of the variational inequality (7) if and only if there exists $m(x,t)$ such that the pair $\{h,m\}$ is a solution to (1)–(6).

The outline of the proof is given in Ref. [2] [see Ref. [3] for mathematical details and a proof of existence of a unique solution to the variational inequality (7)]. It was also shown that the surface flux magnitude $m(x,t)$ is, in this model, a Lagrange multiplier related to the pointwise constraint (3). The values of such multipliers are not uniquely determined by the local conditions, which is the ''mathematical explanation'' of long-range interactions typical of extended dissipative systems in a critical (marginally stable) state; see Ref. [25].

The model (1)–(6) or, equivalently, (7), has analytical solutions (see Ref. [2]) describing the shapes of piles built up on flat platforms in the experiments [6]. The simplest of these solutions is the ideal cone growing below a point source $w=w_0\delta(x)$ on the support $h_0(x)\equiv 0$,

$$h(x,t)=k\max\{r(t)-|x|,0\},$$

where $r(t)=(3w_0t/\pi k)^{1/3}$ is the radius of the cone base. Numerical solutions [1,4] also demonstrate simple geometrical structures that agree with one's sandbox memories. Although the model is simplified in many respects, it allows for the multiplicity of possible pile shapes. The avalanches may be introduced into the model as solution discontinuities (in time) triggered by sudden changes of the admissible set K (see Refs. [9,2]), and are instantaneous events. On the time scale of a slow pile growth the life of an avalanche is, indeed, very short.

III. MODIFIED BCRE EQUATIONS

The BCRE equations [12,13] involve two coupled variables: the pile height, $y=h(x,t)$, and the effective thickness (density) of the rolling grains layer, $R(x,t)$ [$R(x,t)d\Omega$ is the volume that the material, currently rolling above the area $d\Omega$, would occupy in the pile]. The model has been formulated for a two-dimensional pile ($x\in\mathbb{R}^1$); free surface slope deviations from the critical angle were assumed to be small. Original BCRE equations included diffusion terms to account for a nonlocality of grains dislodgement and for fluctuations of rolling grain velocity. Although diffusion plays a crucial role in BCRE's scenario of avalanches [12,13,20], these terms were regularly omitted by other researchers who either assumed that diffusion was insignificant in their problems and simplified the model, or proposed a different avalanche scenario (see, e.g., Refs. [14,16,18,26]). Below, we also omit the diffusion terms at first, but introduce small diffusion at a later stage as a means for model regularization in transition to a large-scale limit. Adding small diffusion (artificial viscosity) is often employed also to smooth the discontinuous solutions; see, e.g., Ref. [18].

Simplified BCRE equations may be written as follows:

$$\partial_t h=\Gamma[h,R],\quad \partial_t R+\partial_x(vR)=w-\Gamma[h,R].$$

Here the term $\Gamma[h,R]$ accounts for the conversion of rolling grains into immobilized grains and vice versa, v is the horizontal projection of rolling grains velocity, and $w(x,t)$ is the source intensity (it is assumed that the tipped grains do not stick to the pile surface but join the rolling grains first).

Limiting their consideration to the slopes that are close to critical, BCRE assumed a constant downslope drift velocity v. The surface flux magnitude $q=vR$ is thus determined solely by the rolling layer thickness R. Since in the previous model $q=mk$ for the critical slopes, m and R play similar roles, and the two choices of basic variables, $\{h,m\}$ and $\{h,R\}$, are essentially equivalent. The exchange term Γ in the BCRE model is linearized in a vicinity of the critical angle α_r and is proportional (for thin surface flows) to R, $\Gamma[h,R]=\gamma R(\alpha_r-\theta)$, where $\theta(x,t)$ is the surface slope angle and γ is a coefficient.

For a three-dimensional pile ($x\in\mathbb{R}^2$), the model equations are similar,

$$\partial_t h=\Gamma[h,R],\quad \partial_t R+\nabla\cdot(\boldsymbol{v}R)=w-\Gamma[h,R],\tag{8}$$

but the constitutive relations determining $\boldsymbol{v}$ and, probably, $\Gamma[h,R]$ should be modified; here we will follow Ref. [27] (also see Ref. [28]). We assume that the rolling particles drift toward the steepest descent of the free surface with a mean velocity v depending on the slope angle (the steeper the slope, the higher is the velocity). On their way downslope, these particles may be trapped and absorbed into the motionless bulk (the steeper the slope, the lower is the trapping rate Γ). If the surface is horizontal, the mean flow velocity is zero and the trapping rate is maximal; for $\theta=\alpha_r$ the rolling particles follow without trapping. Below, we will not consider the overcritical slopes, and also assume that the trapping rate is proportional to the amount of rolling grains R.

At least partially, this simplified picture can be justified by recent experimental, theoretical, and numerical studies on the motion of a spherical particle on a rough inclined plane [29,30]. For the relevant region of slope angles θ, the energy dissipation due to the multiple shocks experienced by a moving particle is equivalent to the action of a viscous friction force [29]. Because of this such particles reach a constant mean velocity proportional to $\sin\theta$. Sometimes, however, the particles are suddenly trapped in a well, and completely lose their momentum in the direction of motion [30]. Of course, conditions in the collective flow of grains over the pile surface are somewhat different. In particular, the flow velocity may depend on the thickness of rolling grains layer [31], and the exchange rate is not exactly proportional to R [17]. Various improved dependencies can be incorporated into the model. The limiting behavior of the modified BCRE model is, however, robust, and does not depend on details. For clarity of presentation we will consider the long-scale limit of a thin-flow model, with the simplest phenomenological relations determining the flow velocity and the rolling-to-immobilized-state transition.

Since the mean velocity of the surface flow is proportional to $\sin\theta$ [29], its horizontal projection v is proportional to $\sin\theta\cos\theta=\tan\theta/(1+\tan^2\theta)$. Postulating that the flow is in the steepest descent direction, we obtain $\boldsymbol{v}=-\mu\nabla h/(1+|\nabla h|^2)$, where μ is a coefficient. Simplifying this relation, we assume

$$\boldsymbol{v}=-\mu\nabla h.\tag{9}$$

The exchange rate Γ should not depend on the slope orientation, and we assume it to be a smooth decreasing function of $|\nabla h|^2$ that becomes zero for critical slopes. Assuming Γ is proportional to R (thin flows), we arrive at

$$\Gamma[h,R]=\gamma R\left(1-\frac{|\nabla h|^2}{k^2}\right)\tag{10}$$

as the simplest constitutive relation [27]. We will now derive a dimensionless formulation for the modified BCRE model [Eqs. (8)–(10)].

The parameters in this model have the following dimensions: $[\gamma]=\mathcal{T}^{-1}$ and $[\mu]=\mathcal{T}^{-1}$. Let us denote by $\overline{w}$ the characteristic intensity of the external source; $[\overline{w}]=\mathcal{L}T^{-1}$. The three length scales characterizing the pile surface dynamics and surface granular flow may be defined as follows:

(i) the typical thickness of the rolling grains layer, $L_R = \overline{w}/\gamma$; (ii) the mean path of a rolling particle before it is trapped strongly depends on the slope steepness but, for a fixed subcritical slope, is proportional to the ratio $L_P = \mu/\gamma$ characterizing the competition between rolling and trapping; and (iii) the pile size L.

The time $T = L/\overline{w}$ needed for a source with given intensity $\overline{w}$ to produce a pile of size L may be used as a long time scale. Rescaling the variables,

$$x' = \frac{1}{L}x, \quad h' = \frac{1}{L}h, \quad R' = \frac{1}{L_R}R, \quad w' = \frac{1}{\overline{w}}w, \quad t' = \frac{1}{T}t,$$

we arrive at the following dimensionless formulation:

$$\partial_t h = \Gamma[h,R], \tag{11}$$

$$\frac{L_R}{L}\partial_t R - \frac{L_P}{L}\nabla\cdot(R\nabla h) = w - \Gamma[h,R], \tag{12}$$

$$\Gamma[h,R] = R\left(1 - \frac{|\nabla h|^2}{k^2}\right). \tag{13}$$

Typically, $L_R \ll L_P < L$. The first coefficient in Eq. (12) is very small, so it may often be possible to omit the corresponding term and use a quasistationary equation for the rolling layer. Such an approach was already employed in a simulation of the dynamics of sand ripples; see Ref. [27]. The second coefficient L_P/L may be significant for small piles, like sand ripples, but becomes small too for large piles. Further simplification of the model is then appropriate.

IV. LONG-SCALE LIMIT OF BCRE MODEL

Let us denote $\nu = L_P/L$, and study the $\nu \to 0$ behavior of the model [Eqs. (11)–(13)]. This limit corresponds to the case of large piles ($L \gg L_P$). We want to show that in this limit the pile shape evolution is described by the variational inequality (7) which remains invariant under the rescaling employed.

Physically, the situation is clear: although the model [Eqs. (11)–(13)] permits grains to roll down upon any inclined slope, the rolling particles are quickly stopped and their paths are short compared to the pile size for all except the almost critical slopes. This is essentially what is assumed in the model [1,2], which permits rolling upon the critical slopes only. Mathematically, the situation is somewhat more complicated.

Since $L_R \ll L_P$, we assume L_R/L is $o(\nu)$ and set $L_R/L = \nu\lambda(\nu)$, where λ tends to zero as $\nu \to 0$. Let us introduce a new variable, $m = \nu R$, define $\psi(u) = 1 - u^2/k^2$, and rewrite the model [Eqs. (11)–(13)] as

$$\partial_t h = \frac{m\psi(|\nabla h|)}{\nu}, \quad \lambda\partial_t m - \nabla\cdot(m\nabla h) = w - \frac{m\psi(|\nabla h|)}{\nu}.$$

For any $\nu > 0$ this system consists of two coupled hyperbolic equations. The second equation, which can be regarded as an equation for m, contains in its main part the coefficient ∇h which may be discontinuous. The theory for such equations is complicated and not well developed. To circumvent the difficulty, we add small diffusion to both equations and consider the regularized models

$$\partial_t h = \frac{m\psi(|\nabla h|)}{\nu} + \varepsilon_h \Delta h, \tag{14}$$

$$\lambda\partial_t m - \nabla\cdot(m\nabla h) = w - \frac{m\psi(|\nabla h|)}{\nu} + \varepsilon_m \Delta m, \tag{15}$$

where the positive coefficients $\varepsilon_h(\nu)$ and $\varepsilon_m(\nu)$ vanish as ν tends to zero. It should be noted that, although small diffusion may be physically meaningful and has been included into the original BCRE formulation [12,13], here we introduce it merely as a parabolic regularization of hyperbolic equations convenient for analyzing the model's behavior at $\nu \to 0$.

We assume the same initial and boundary conditions, [Eqs. (5) and (6) correspondingly] for the function h. The non-negative values of m both in Ω at $t=0$ and on the boundary of this domain for $t>0$ may be arbitrary: these initial and boundary conditions result only in the appearance of boundary layers in the solution for any finite $\nu > 0$ and are lost in the $\nu \to 0$ limit. Rigorously, convergence of the problem (14), (15) to variational inequality (7) is considered elsewhere [32]. Here we present a simplified scheme of the proof and avoid technicalities.

The main step is, as usual, to obtain uniform in $\nu > 0$ *a priori* estimates on the solutions of Eqs. (14) and (15). First, taking the gradient and multiplying by ∇h, from Eq. (14) we derive a parabolic partial differential equation for $|\nabla h|^2$. Since $\psi(k) = 0$ and $|\nabla h_0| \le k$, we are able, using the maximum principle for this equation, to show that for $\nu > 0$

$$|\nabla h| \le k \quad \text{for all } (x,t); \quad h \text{ is uniformly bounded.} \tag{16}$$

Second, using the non-negativeness of the source function $w(x,t)$, and applying the maximum principle to Eq. (15), we deduce that for each $\nu > 0$,

$$m \ge 0 \quad \text{for all } (x,t). \tag{17}$$

Applying estimates (16) and (17) to Eq. (14) we obtain

$$m\psi(|\nabla h|) = O(\nu). \tag{18}$$

Sending ν to zero in (16), (17), and (18), we establish the fulfillment in this limit of conditions (1), (3), and (4). Finally, adding Eqs. (14) and (15), we obtain

$$\lambda\partial_t m + \partial_t h - \nabla\cdot(m\nabla h) = w + \varepsilon_h\Delta h + \varepsilon_m\Delta m.$$

Since λ, ε_h, and ε_m vanish as $\nu \to 0$, we can show that the corresponding limits of h and m satisfy also the balance equation (2) in some weak (integral) sense. This completes the proof, because the model (1)–(6) is equivalent to the variational inequality (7).

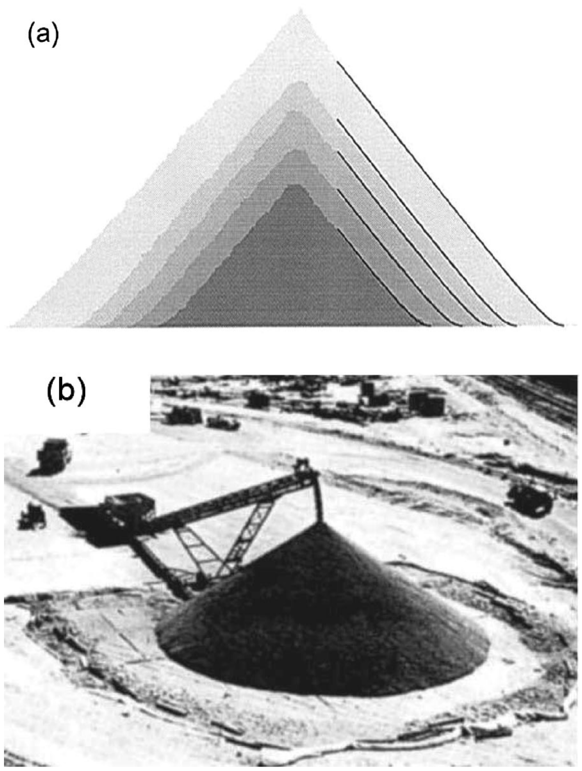

FIG. 2. Piles growing below a pointlike source. (a) Digitized image of a small polenta pile; see Alonso and Herrmann [33]. Different gray levels show the pile at different stages of growth; the curved tails near the pile bottom remain short. (b) Large conical pile; photo by Slater [34].

To illustrate this result we will now compare solutions of the BCRE-type model [Eqs. (14) and (15)], solutions of variational inequality (7), and real shapes of small and large piles. Let us consider first a pile growing under a point source on an infinite horizontal support $h_0=0$. Although in this case the piles are known to be almost perfect cones, sometimes one can note [33] curved tails near the bottom of a small pile [Fig. 2(a)]. As the pile becomes larger, the tail remains of only, say, tens of grain diameters long, so the tail of a large pile is difficult to see [Fig. 2(b)].

The modified BCRE model [Eqs. (14) and (15)] describes this situation quite satisfactorily (Fig. 3). (Here we omitted small diffusion terms added for regularization, and solved the equations numerically for $\varepsilon_h=\varepsilon_m=0, \lambda=0.1\nu$, and two different values of the parameter ν.) Although the tails of small piles ($\nu=0.2$) are clearly seen, tails of the larger piles ($\nu=0.01$) is difficult to detect. We see also that piles, predicted by the BCRE model with small ν and λ, are very close to the growing ideal cone, the analytical solution of the variational inequality (7). It may be noted that for small values of ν and λ the model equations are stiff, and their numerical solution becomes difficult. Thus, even using the implicit finite-difference approximation of Eqs. (14) and (15), we needed 10^5 time steps in the latter example.

Analytical quasistationary solution to the modified BCRE equations (14) and (15) without diffusion can be found in the one-dimensional vessel-filling problem considered previ-

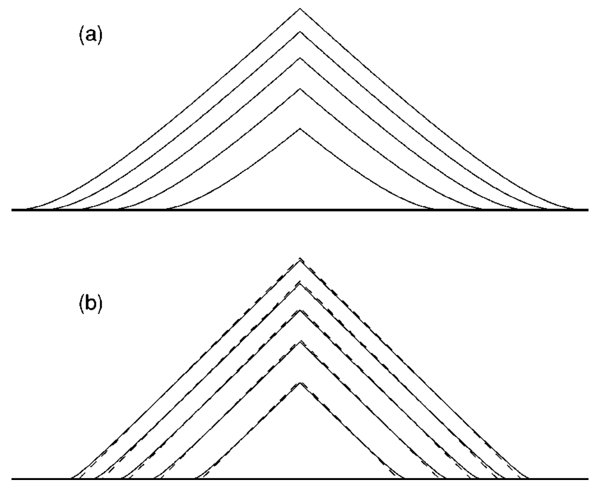

FIG. 3. Piles growing below a pointlike source. Numerical solution of Eqs. (14) and (15) with $\varepsilon_h=\varepsilon_m=0$, $m|_{t=0}=0$, $\lambda=0.1\nu$, and $k=1$. (a) Small piles, $\nu=0.2$. (b) Solid line: large piles, $\nu=0.01$; dashed line: analytical solution of variational inequality (7).

ously in Ref. [16] for the BCRE model with constant drift velocity of particles. Let the domain $\Omega=(0,L)$ be bounded by two vertical walls, and $w=w_0\delta(x-L/2)$. Assuming $\partial_t h \equiv c$ and $\partial_t m \equiv 0$, we obtain

$$c=\frac{m}{\nu}\left(1-\frac{|h_x'|^2}{k^2}\right), \tag{19}$$

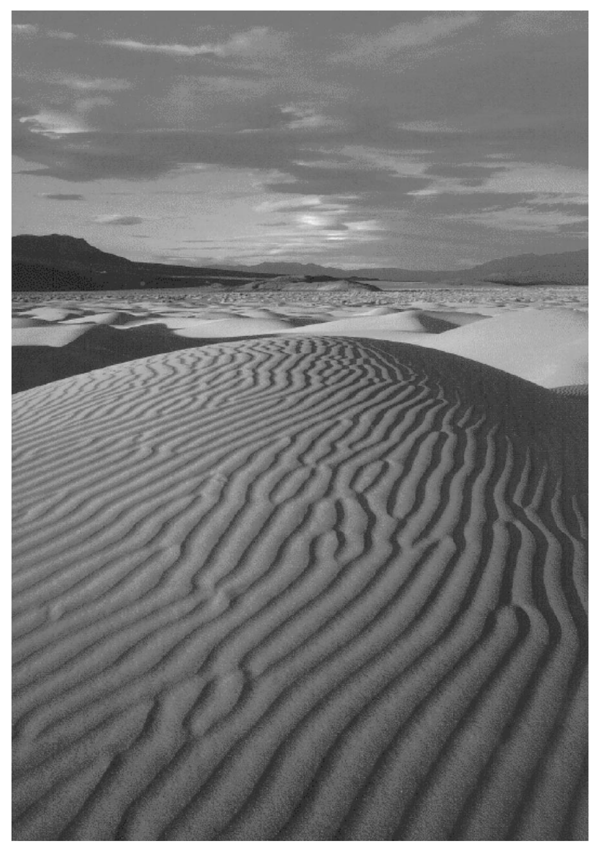

FIG. 4. Geomorphological patterns on two different scales. Photo by © Carr Clifton, reproduced with his kind permission.

and $-(mh'_x)'_x = w - (m/\nu)(1-|h'_x|^2/k^2)$. Adding these equations and integrating in x, we see that $c = w_0/L$ and then $m = x/Lh'_x$ for $0 < x < L/2$ (we can use symmetry). Substituting these expressions into Eq. (19) we find

$$h'_x = \frac{k}{(\nu k/2x) + \sqrt{(\nu^2 k^2/4x^2)+1}}, \quad 0 < x < L/2.$$

Although $h'_x \to 0$ as $x \to 0$ for any $\nu > 0$ (a tail), we also see that as $\nu \to 0$ the slope becomes critical, $h'_x \to k$ for all $0 < x < L/2$, and we arrive at the corresponding solution [2] of the variational inequality (7).

Note that in both examples considered above the model with constant velocity of rolling particles yields solutions with logarithmic singularities that are unphysical and must be ''cut off'' (see, e.g., Ref. [16]). No singularities appear in the modified BCRE model [Eqs. (14) and (15)] where the velocity depends on the slope steepness.

V. CONCLUSION

We considered two different continuous models for pile surface dynamics: the BCRE model [12,13] and the variational model [1,2]. Both models are written for two coupled dependent variables and are able, in principle, to account for multiplicity of metastable pile shapes and surface avalanches. It was found that the models are related and describe the pile surface dynamics on different spatiotemporal scales.

BCRE-type models may be used to simulate the fast processes, such as the initiation, spreading, and settling down of an avalanche. To describe the much slower dynamics of the mean shape of a pile, the model may often be simplified by employing a quasistationary equation for the rolling grains layer. Such a model is able to predict some peculiarities of small pile shapes [33], and was recently employed for simulating the nonlinear dynamics of sand ripples [27].

Unlike the BCRE models, the variational model of pile growth does not permit the discharged grains to roll upon subcritical slopes, and is therefore unable to account for such features of the sand surface as the sand-ripple instability or the surface slope deviation from the critical angle near the bottom of a conical pile. Indeed, these effects are determined by rolling of particles upon the subcritical slopes, and are exhibited on the length scale comparable to the mean path of a particle prior to its being trapped.

On the other hand, sand ripples on the dune surface or tiny tails at the bottom of a pile are seen only from a short distance (Fig. 4). These small details are difficult to distinguish when watching from a longer distance, allowing one to follow the evolution of a large dune or the formation of a large pile. In such situations the BCRE model contains another small parameter. This complicates simulations and makes them inefficient. As shown in our work, in the long-scale limit BCRE-type models converge to the variational model of pile growth. The latter model is more appropriate for simulating the pile surface dynamics on a large spatiotemporal scale.

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