<u>דוגמאות (ז. אולחה)</u>

1) $\lim\limits_{n\to\infty}\dfrac{4n^2-2n+3}{n^3+9}=?$

$$\lim_{n\to\infty}\frac{4n^2-2n+3}{n^3+9}=\left[\frac{\infty}{\infty}\right]=\lim_{n\to\infty}\frac{\frac{4n^2}{n^3}-\frac{2n}{n^3}+\frac{3}{n^3}}{\frac{n^3}{n^3}+\frac{9}{n^3}}=\lim_{n\to\infty}\frac{\frac{4}{n}-\frac{2}{n^2}+\frac{3}{n^3}}{1+\frac{9}{n^3}}=\frac{0}{1}=0$$

2) $\lim\limits_{n\to\infty}\left(n\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)\right)=?$

$$\lim_{n\to\infty}\left(n\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)\right)=[\infty\bullet[\infty-\infty]]=\lim_{n\to\infty}\frac{n\,(\sqrt{n^2+1}-\sqrt{n^2-1})(\sqrt{n^2+1}+\sqrt{n^2-1})}{\sqrt{n^2+1}+\sqrt{n^2-1}}=$$

$$\lim_{n\to\infty}\frac{n[(n^2+1)-(n^2-1)]}{\sqrt{n^2+1}+\sqrt{n^2-1}}=\lim_{n\to\infty}\frac{2n}{\sqrt{n^2+1}+\sqrt{n^2-1}}=\left[\frac{\infty}{\infty}\right]=\lim_{n\to\infty}\frac{2}{\sqrt{1+1/n^2}+\sqrt{1-1/n^2}}=\frac{2}{2}=1$$

3) $\lim\limits_{n\to\infty}\sqrt[n]{2+\sqrt[3]{n}}=?$

$$(\forall n>8)\qquad \sqrt[n]{2+2}<\sqrt[n]{2+\sqrt[3]{n}}<\sqrt[n]{\sqrt[3]{n}+\sqrt[3]{n}}\ ,\qquad \lim_{n\to\infty}\sqrt[n]{2}\left(\sqrt[n]{n}\right)^{1/3}=1\cdot 1^{1/3}=1$$

$$\underbrace{\sqrt[n]{4}}_{\downarrow\,1}<\underbrace{\sqrt[n]{2+\sqrt[3]{n}}}_{\Downarrow\,\boxed{1}}<\underbrace{\sqrt[n]{2}\left(\sqrt[n]{n}\right)^{1/3}}_{\downarrow\,1}\quad\Rightarrow\lim_{n\to\infty}\sqrt[n]{2+\sqrt[3]{n}}=1$$

4) $\lim\limits_{n\to\infty}\dfrac{\arctan 5n}{n^2+7}=?$

$$\left.\begin{array}{l}0<\arctan 5n<\dfrac{\pi}{2}\\[2ex] \lim\limits_{n\to\infty}\dfrac{1}{n^2+7}=0\end{array}\right\}\Rightarrow\lim_{n\to\infty}\left((\arctan 5n)\,\frac{1}{n^2+7}\right)=0$$

פתרונות

I.2

a) $1/n < \varepsilon \Rightarrow n > 1/\varepsilon \Rightarrow N = [1/\varepsilon]$

b) $\frac{2n}{n^3+1} < \frac{2n}{n^3} = \frac{2}{n^2} < \varepsilon \Rightarrow n^2 > 2/\varepsilon\ ,\ N = \left[\sqrt{\frac{2}{\varepsilon}}\right]$

c) $1/n! < 1/n < \varepsilon \Rightarrow N = [1/\varepsilon]$

d) $0.999^n < \varepsilon \Rightarrow n \underbrace{\ln 0.999}_{<0} < \ln \varepsilon \Rightarrow n > \frac{\ln \varepsilon}{\ln 0.999}$

I.3 $\frac{\sqrt{n}-1}{2} > B \Rightarrow \sqrt{n} > 2B+1 \Rightarrow N = [(2B+1)^2]$

I.4 $\forall B<0\ \ \exists N \in \mathbb{N}\ \ \forall n>N : n-n^3 < B$

$n-n^3 = -n(n^2-1) < -n < B \Rightarrow n > -B\ ,\ N = [-B]$

$(\forall n>1)$

I.5 a) $(-1, 2, -3, 4, -5, 6, \dots)$ סדרה לא חסומה

$x_{2k} = 2k \to +\infty\ ,\ x_{2k-1} = -(2k-1) \to -\infty$

$\left.\begin{matrix} B>0 & \text{אם} \\ x_n > B & \text{וגם} \end{matrix}\right\} \Rightarrow x_{n+1} < 0 \Rightarrow x_{n+1} < B$

b) $(1^{-1}, 2, 3^{-1}, 4, 5^{-1}, \dots)$

$x_{2k} = 2k \to +\infty\ ,\ x_{2k-1} = \frac{1}{2k-1} \to 0$

ולכן $\lim_{n\to\infty} x_n$ לא ק"ם

III

8) $\frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}} = \frac{(-2/3)^n + 1}{-2\cdot(-2/3)^n + 3} \to \frac{0+1}{0+3} = \frac{1}{3}$

10) $1 + a + a^2 + \dots + a^n \to \frac{1}{1-a}$ אם $|a|<1$ אזי

12) $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \Rightarrow \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}\ ,\ \frac{1}{3\cdot 4} = \frac{1}{3} - \frac{1}{4}, \dots$

13) $\sqrt{2}\cdot\sqrt[4]{2}\cdot\ldots\sqrt[2^n]{2} = 2^{\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^n}}$

24) 1) $x_{n+1} - x_n = \frac{p_{n+1}}{10^{n+1}} \geq 0 \Rightarrow (x_n) \uparrow$

2) $x_n \leq p_0 + \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n} = p_0 + \frac{0.9(1-0.1^n)}{1-0.1} <$

$p_0 \leq x_n < p_0 + 1 \Rightarrow$ (x_n) חסומה

26) 1) $\frac{x_{n+1}}{x_n} = 1 - \frac{1}{2^{n+1}} < 1 \Rightarrow (x_n) \downarrow$

2) $0 < x_n < 1$