SANDPILES, RIVER NETWORKS, AND TYPE-II SUPERCONDUCTORS

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Spatially extended dissipative systems have recently attracted much interest among physicists (see, e.g., [1]). These systems have infinitely many metastable states but, under the action of external forces, they tend to organize themselves into a marginally stable "critical state" and are then able to demonstrate almost instantaneous long-range interactions. The evolution of dissipative systems is often accompanied by sudden collapses, like sandpile avalanches, and hysteresis.

Modifications of a cellular automaton model of sandpile [2], used by many authors for simulating the dissipative systems, enable one to understand qualitatively these systems' behaviour. Such models, however, are crude. Our work concerns the derivation and study of continuous models for sandpiles, river networks, and type-II superconductors. Although these are dissipative systems of a different nature, their continuous models are equivalent to similar evolutionary variational or quasivariational inequalities. The origin of this similarity is that these models are quasistationary models of equilibrium.

To derive such a model, one need to specify only the direction of system's evolution, and which changes of external conditions make the state of the system unstable. The rate with which a dissipative system driven by the external forces rearranges itself is determined implicitly by some conservation law coupled with a condition of equilibrium. This rate appears in the model as a Lagrange multiplier related to the equilibrium constraint. Since the multiplier depends on the system's state and the varying external conditions in a nonlocal way, the model is able to account for the long-range interactions typical of dissipative systems in a critical state. Although the conservation laws and conditions of equilibrium may vary, the multiplicity of metastable states is usually a consequence of the existence of a unilateral equilibrium constraint. This makes variational inequalities a suitable tool for modelling such dissipative systems.

MODEL OF SANDPILE GROWTH

Let a cohesionless ideal granular material having an angle of repose α be poured out onto a rough rigid surface $y = h_0(x)$, where y is vertical and $x \in \Omega \subset R^2$. We find the shape of a growing pile, $y = h(x,t)$.

The flow of granular material down the slope of the pile is usually confined to a thin surface layer separated from the motionless bulk. Hence, if the bulk density of material in the pile is constant, we can write the mass conservation law in the form

$$\partial h/\partial t + \nabla \cdot \boldsymbol{q} = w, \tag{1}$$

where $\boldsymbol{q}(x,t)$ is the horizontal projection of the material flux in the surface layer and $w(x,t)$ is the intensity of the source of material being poured onto the pile. We neglect the inertia and suppose that the surface flow is directed towards the steepest descent, $\boldsymbol{q} = -m\nabla h$, where

$$m(x,t) \geq 0 \tag{2}$$

is an *unknown* scalar function. The conservation law assumes the form

$$\partial h/\partial t - \nabla \cdot (m\nabla h) = w. \tag{3}$$

The free surface initially coincides with the support,

$$h\,|_{t=0} = h_0(x), \tag{4}$$

it never lies below the support surface,

$$h(x,t) \geq h_0(x), \tag{5}$$

and wherever the free surface is above, its slope cannot exceed the angle of repose of the granular material,

$$h(x,t) > h_0(x) \Longrightarrow |\nabla h(x,t)| \leq \gamma, \tag{6}$$

where $\gamma = \tan(\alpha)$. No surface flow occurs over the parts of the pile surface inclined less than at the angle of repose:

$$|\nabla h(x,t)| < \gamma \Longrightarrow m(x,t) = 0. \tag{7}$$

We assume further that the granular material is allowed to leave the system freely through the part Γ_1 of the domain boundary $\partial\Omega$, while the other part, Γ_2, is an impermeable wall:

$$h|_{\Gamma_1} = h_0|_{\Gamma_1} \qquad m\frac{\partial h}{\partial n}\bigg|_{\Gamma_2} = 0. \tag{8}$$

The model of pile growth (2)-(8), proposed in [3], contains two unknowns: the free surface h and an auxiliary function m. The latter function turns out to be a Lagrange multiplier related to the constraint upon the incline of pile surface and can be excluded by transforming (2)-(8) to a more convenient variational formulation,

$$\left\{ \begin{array}{c} h \in K(h): \; (\partial h/\partial t - w, \varphi - h) \geq 0, \forall \varphi \in K(h), \\ h|_{t=0} = h_0, \end{array} \right. \tag{9}$$

where

$$K(h) = \left\{ \varphi \in L^\infty(0,T; W^{1,\infty}(\Omega)) \;\middle|\; \begin{array}{c} |\nabla\varphi| \leq M(h) \text{ a.e.} \\ \varphi|_{\Gamma_1} = h_0|_{\Gamma_1} \end{array} \right\},$$

$$M(h)(x,t) = \begin{cases} \gamma & \text{if } h(x,t) > h_0(x), \\ \max(\gamma, |\nabla h_0(x)|) & \text{otherwise.} \end{cases}$$

Suppose there exists a function $\varphi_0(x)$ such that

$$\varphi_0|_{\Gamma_1} = h_0|_{\Gamma_1}, \quad |\nabla\varphi_0| < \gamma \text{ a.e. in } \Omega. \tag{10}$$

Then, as was shown in [3,4], the function h satisfies the quasivariational inequality (9) if and only if there exists a Lagrange multiplier $m \in (L^\infty(\Omega \times (0,T)))'$ such that the pair $\{h, m\}$ is a weak solution to (2)-(8). At present, we are unable to prove the existence of a solution to (9). However, if $|\nabla h_0| \leq \gamma$ a.e., the inequality becomes variational, and the existence and uniqueness of a solution are proved [4]. The numerical solution of variational problem and a generalization to the case of polydisperse granular material have been considered in [5], while an approach to simulation of sandpile avalanches can be found in [6]. Mathematically, the avalanches correspond to solutions with the jumps caused by sudden variations of admissible set K. Such discontinuous solutions have been studied by Moreau [7].

LAKES AND RIVER NETS

Here we let h_0 be the land surface, w the intensity of precipitation, and assume for simplicity that the water neither evaporates nor penetrates the soil but just flows down the slopes and accumulates into lakes at local depressions of the land surface. The level of a lake rises until it reaches the divide of two basins; then a river running out of the lake appears and transfers all additional water to another lake below.

To describe the evolution of this system of lakes and rivers mathematically, let us note that the balance equation (1), in which $\boldsymbol{q}$ is the horizontal projection of water flux, remains valid. The free boundary h in this problem either coincides with h_0 or, where it is higher, is the horizontal surface of a lake. Over the hill slopes, where $h = h_0$, we again can assume $\boldsymbol{q} = -m\nabla h$, where $m(x,t) \geq 0$ is unknown. However, this is not true for the lakes, where $h > h_0$ and $\nabla h = 0$. In fact, the lake hydrodynamics are not trivial, but the flow in the lake does not affect the free surface, and it can be shown that the relations above lead to the quasivariational inequality (9) with $\gamma = 0$ [6]. Note that in this case the constraint qualification hypothesis (10) is not satisfied. That is why the inequality is now not equivalent to the sandpile model, in which $\boldsymbol{q} = \boldsymbol{0}$ if the free surface incline is zero.

Provided the free surface h is found from (9), the dual variable, the water flux $|\boldsymbol{q}|$, can be calculated in the hills where $h = h_0$ [6]. This is needed, e.g., in hydrology for the automatic drawing of maps of the river networks using the digital elevation data.

CRITICAL STATES IN SUPERCONDUCTIVITY

The magnetic field penetrates into type-II superconductors in the form of superconductive electron current vortices around the extremely thin filaments of normal material. Each vortex carries the same amount (one quantum) of magnetic flux, and so the magnetization depends on the vortex distribution. According to the Bean model [8], the distribution of vortices is determined by the balance between electromagnetic driving forces and forces pinning the vortices to material inhomogeneities. Whenever the balance is violated, the system of vortices rapidly rearranges itself into another metastable state.

Let a superconductor occupy the domain $\Omega \subset R^3$ with the boundary Γ and let $\omega = R^3 \setminus \overline{\Omega}$ be the exterior space. Maxwells equations with the displacement current omitted read

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \wedge \boldsymbol{E} = \boldsymbol{0}, \quad \boldsymbol{J} = \nabla \wedge \boldsymbol{H}.$$

The constitutive relation between the magnetic induction $\boldsymbol{B}$ and magnetic field $\boldsymbol{H}$ is $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, where μ_0 is the permeability of vacuum. In ω we have $\boldsymbol{J} = \boldsymbol{J}_e$, where $\boldsymbol{J}_e(x,t)$ is the given density of external current.

Whenever magnetic vortices in the superconductor become unpinned, they move in the direction of the Lorentz force $\boldsymbol{F}_L = \boldsymbol{J} \wedge \boldsymbol{B}$ and their movement induces the electric field $\boldsymbol{E} = \boldsymbol{B} \wedge \boldsymbol{v}$, which is thus parallel to $\boldsymbol{B} \wedge (\boldsymbol{J} \wedge \boldsymbol{B})$. In two-dimensional problems as well as in three-dimensional with axial symmetry, $\boldsymbol{B}$ is perpendicular to $\boldsymbol{J}$, so the vectors of current density and electric field are collinear. Limiting our consideration to such cases, we can write

$$\boldsymbol{E} = \rho \boldsymbol{J} \quad \text{in } \Omega,$$

where the effective resistivity $\rho(x,t) \geq 0$ is an auxiliary unknown. As postulated in the Bean model, the current density cannot exceed some critical value J_c determined by the pinning, and no movement of vortices occurs until this threshold is reached. Therefore, in Ω

$$|\nabla \wedge \boldsymbol{H}| \leq J_c, \qquad |\nabla \wedge \boldsymbol{H}| < J_c \implies \rho = 0.$$

The critical current may depend on the magnetic field, so in general $J_c = J_c(|\boldsymbol{H}|)$. The initial distribution $\boldsymbol{B}|_{t=0} = \boldsymbol{B}_0(x)$ must satisfy $\nabla \cdot \boldsymbol{B}_0 = 0$; on Γ we assume the continuity of $\boldsymbol{H} \wedge \boldsymbol{n}$ and $\boldsymbol{E} \wedge \boldsymbol{n}$; also $|\boldsymbol{H}| \to 0$ as $|x| \to \infty$.

Like the water flux in the lakes from the previous section, the electric field in the exterior space is irrelevant and may be excluded from the critical-state model. To present a variational formulation, let us define $\boldsymbol{h} = \boldsymbol{H} - \widetilde{\boldsymbol{H}}$, where $\widetilde{\boldsymbol{H}}$ is the unique solution of the problem

$$\nabla \wedge \widetilde{\boldsymbol{H}} = \boldsymbol{J}_e, \quad \nabla \cdot \widetilde{\boldsymbol{H}} = 0,$$
$$|\widetilde{\boldsymbol{H}}| \to 0 \text{ as } |x| \to \infty$$

(it is supposed that $\nabla \cdot \boldsymbol{J}_e = 0$ and $\operatorname{supp} \boldsymbol{J}_e \subset \omega$ is bounded.) We set $\widetilde{J}_c(\boldsymbol{h}) = J_c(|\boldsymbol{h} + \widetilde{\boldsymbol{H}}|)$, denote by X the Hilbert space $(L^2(R^3 \times (0,T))^3$, and define a family of closed convex sets

$$K(\boldsymbol{h}) = \left\{ \boldsymbol{\varphi} \in X \;\middle|\; \begin{array}{c} \nabla \wedge \boldsymbol{\varphi} \in X, \\ \nabla \wedge \boldsymbol{\varphi} = \boldsymbol{0} \text{ a.e. in } \omega \times (0,T), \\ |\nabla \wedge \boldsymbol{\varphi}| \leq \widetilde{J}_c(\boldsymbol{h}) \text{ a.e. in } \Omega \times (0,T) \end{array} \right\}$$

It can be shown that the function $\boldsymbol{h}$ is a solution to the quasivariational inequality

$$\left\{ \begin{array}{c} \boldsymbol{h} \in K(\boldsymbol{h}) : \\ \left(\partial \boldsymbol{h}/\partial t + \partial \widetilde{\boldsymbol{H}}/\partial t, \boldsymbol{\varphi} - \boldsymbol{h} \right) \geq 0, \ \forall \boldsymbol{\varphi} \in K(\boldsymbol{h}), \\ \boldsymbol{h}|_{t=0} = \boldsymbol{B}_0/\mu_0 - \widetilde{\boldsymbol{H}}|_{t=0} \end{array} \right. \tag{11}$$

if and only if there exists $\rho \in (L^\infty(\Omega \times (0,T)))'$ such that the pair $\{\boldsymbol{H}, \rho\}$, where $\boldsymbol{H} = \boldsymbol{h} + \widetilde{\boldsymbol{H}}$, is a weak solution to the critical-state problem [9]. The effective resistivity ρ is a Lagrange multiplier related to the current density constraint in the Bean model.

It is often assumed that $J_c = const.$ Then the inequality (11) becomes variational; the existence and uniqueness of its solution are proved in [9]. The numerical solution of critical-state problems was based on an equivalent variational inequality for the current density [10]. As an example, let us consider the magnetization of axially symmetric superconductors placed inside a long solenoid. Initially, the field is zero everywhere. As the uniform external magnetic field H_e, induced by the electric current in the winding of solenoid, starts to grow, a region of circumferential critical current appears in the vicinity of the boundary of superconductor. The zero-field core shrinks and eventually disappears as the external field becomes sufficiently strong (Fig. 1).

It is interesting that the thermal fluctuations cause experimentally observed avalanches of magnetic vortices, similar to sandpile avalanches. To account for these avalanches and simulate the thermally activated creep of magnetic flux in superconductors, it is only needed to introduce into the model the random local fluctuations of critical current density (in preparation).

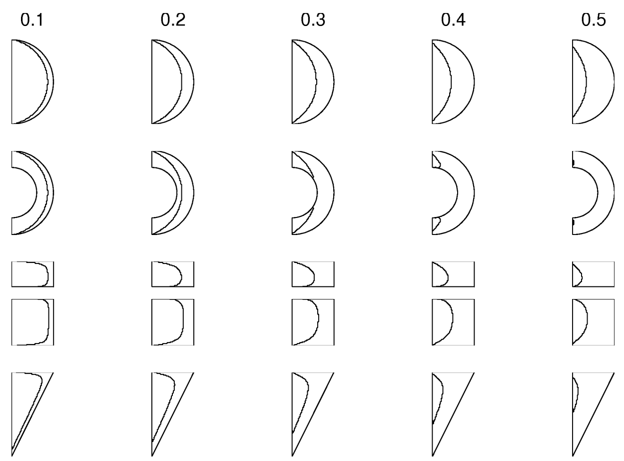

FIG. 1. Magnetization of axially symmetric superconductors in a growing field (ball, hollow ball, two adjacent cylinders, cone; half cross-sections are shown). The zero-field core, which contains no magnetic vortices, shrinks with the growth of external magnetic field. Numbers above the columns indicate the ratio H_e/RJ_c, R is the ball radius.

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