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The Bean Model in Superconductivity: Variational Formulation and Numerical Solution

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The Bean critical-state model describes the penetration of magnetic field into type-II superconductors. Mathematically, this is a free boundary problem and its solution is of interest in applied superconductivity. We derive a variational formulation for the Bean model and use it to solve two-dimensional and axially symmetric critical-state problems numerically. © 1996 Academic Press, Inc.

1. INTRODUCTION

Into type-II superconductors, the magnetic field penetrates in the form of superconductive electron current vortices. Each of these vortices carries the same amount, one quantum, of magnetic flux; hence the magnetization is proportional to the vortex density. At a close to zero temperature, the distribution of magnetic vortices in type-II superconductors is determined by the balance of electromagnetic driving forces and forces pinning vortices to material inhomogeneities. Whenever the external magnetic field is changed, magnetic vortices start to enter or leave the superconductor through its boundary. There appears a region where driving forces overcome pinning and the system of vortices rearranges itself into another metastable state such that all vortices are pinned again and the equilibrium with the external field at the boundary is re-established. Since the unpinned vortices move rapidly, the system quickly adjusts itself to the changing external conditions, and a quasistationary model of equilibrium can be justified.

These are the basic and somewhat simplified assumptions of the phenomenological critical-state models of hysteretic magnetization in type-II superconductors [1, 2]. In terms of macroscopic quantities, these assumptions may be formulated as follows: current density never exceeds some critical value determined by the density of pinning forces and, as long as this threshold is not reached, the magnetic induction remains unchanged. Jointly with Maxwell's equations, the formulated rules provide a macroscopic model of magnetic field penetration into a type-II superconductor. Solution of the arising mathematical problem is, however, difficult and the known exact analytical solutions are mostly restricted to idealized geometries, such as an infinitely long cylinder in a parallel field, a half-space in a rotating parallel field, an infinitely thin strip or disk in a perpendicular field, etc. [3–8]. Substitutes of exact Bean's model relations were sometimes employed in numerical procedures to make the calculations feasible (see, e.g., [9]).

The main complication is the presence of an unknown (free) boundary dividing the regions of subcritical (usually zero) currents and of critical currents. To solve the problem numerically, several "front-tracking" algorithms have been developed [10–12]. Such algorithms can be very accurate and also efficient for problems with simple free boundaries. However, the implementation of such methods becomes difficult if the free boundary topology is complicated, changes in time, or is not known *a priori*.

In this work we derive a variational formulation for two-dimensional and axially symmetric critical-state problems. This formulation makes unnecessary separate consideration and different treatment of critical and subcritical regions, allows one to avoid front-tracking, and thus facilitates significantly the numerical solution of the free boundary problem. There is, however, another difficulty, typical of problems in electromagnetism: the magnetic field has to be found in an unbounded space. We further reformulate the problem in terms of current density, which is unknown only in the region occupied by a superconductor. This latter variational formulation serves as a basis for an efficient numerical algorithm described in this work. We present the results of numerical simulation and compare them with the known solutions.

A different variational formulation for the generalized Bean model has been recently proposed by Bossavit [13]. This formulation must also be restricted to two-dimensional and axially symmetric problems (because of the assumption that the electric field and current density are parallel; see the next section for a discussion). The difficulty

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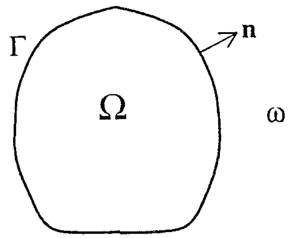

FIGURE 1

of determining the magnetic field in an infinite space has not been resolved in [13], and no numerical results have been reported.

2. MODEL OF THE CRITICAL STATE

Let a superconductor occupy a three-dimensional spatial domain Ω with the boundary Γ, and let ω be the exterior space (Fig. 1).

Maxwell equations with the displacement current omitted read

$$\frac{\partial \mathbf{B}}{\partial t} + \operatorname{curl} \mathbf{E} = \mathbf{0}, \tag{1}$$

$$\mathbf{J} = \operatorname{curl} \mathbf{H}, \tag{2}$$

and we assume $\mathbf{B} = \mu_0 \mathbf{H}$, where μ_0 is the permeability of the vacuum. The current density in the exterior space is assumed to be known:

$$\operatorname{curl} \mathbf{H} = \mathbf{J}_e \quad \text{in } \omega. \tag{3}$$

Here $\mathbf{J}_e(x, t)$ is the density of external current, which should satisfy the condition $\operatorname{div} \mathbf{J}_e = 0$.

In the presence of electrical current flowing through the superconductor, the magnetic vortices respond to the action of the Lorentz force which we average into a body force with density

$$\mathbf{F}_L = \mathbf{J} \wedge \mathbf{B}.$$

If the vortices become unpinned, they move in the direction of this force, and so their velocity $\mathbf{v}$ is parallel to $\mathbf{F}_L$. The movement of vortices induces the electric field

$$\mathbf{E} = \mathbf{B} \wedge \mathbf{v},$$

which is thus parallel to $\mathbf{B} \wedge (\mathbf{J} \wedge \mathbf{B})$. If $\mathbf{B}$ is perpendicular to $\mathbf{J}$, as is always the case for two-dimensional problems and also for some three-dimensional problems, e.g., those with axial symmetry, the vectors of current density and electric field are collinear. Only this case will be considered in our work. Hence

$$\mathbf{E} = \rho \mathbf{J} \quad \text{in } \Omega, \tag{4}$$

where

$$\rho(x, t) \geq 0 \tag{5}$$

is an unknown nonnegative function.

Equation (4) may be regarded as Ohm's law with an effective resistivity ρ. However, since the resistivity is an auxiliary unknown, this relation, for a given current density, fixes only the possible direction of the electric field but not its magnitude.

According to the critical state model, the current density in superconductor cannot exceed some critical value, J_c. In the Bean model of the critical state, J_c is a constant determined by the properties of superconductive material. However, Kim *et al.* [2] found that generally the critical current density depends on the magnetic field and various relations of the type $J_c = J_c(|\mathbf{H}|)$ have been proposed (see, e.g., [14, 15]). The constraint on the current density may be written as

$$|\operatorname{curl} \mathbf{H}| \leq J_c(|\mathbf{H}|) \quad \text{in } \Omega. \tag{6}$$

In the regions, where the current density is less than critical, the vortices are pinned. Hence, there is no dissipation of energy and the current is purely superconductive:

$$|\operatorname{curl} \mathbf{H}| < J_c(|\mathbf{H}|) \Rightarrow \rho = 0. \tag{7}$$

To complete the model, the initial and boundary conditions must be specified. Let

$$\mathbf{B}|_{t=0} = \mathbf{B}_0(x) \tag{8}$$

with $\operatorname{div} \mathbf{B}_0 = 0$ (together with (1), the last condition ensures $\operatorname{div} \mathbf{B} = 0$). On the boundary dividing the two media, the tangential component of electric field $\mathbf{E}$ is continuous,

$$[\mathbf{E}_\tau] = \mathbf{0} \quad \text{on } \Gamma,$$

where $[\cdot]$ denotes the jump across the boundary. We neglect the surface current, which is much less than the total current in most applications of type-II superconductors [16]. The tangential component of magnetic field $\mathbf{H}$ on this boundary is thus assumed to be continuous too:

$$[\mathbf{H}_\tau] = \mathbf{0} \quad \text{on } \Gamma. \tag{9}$$

We also suppose that $|\mathbf{H}| \to 0$ as $|x| \to \infty$.

The mathematical model obtained contains a system of equations and inequalities which is difficult to attack directly. Furthermore, in accordance with the postulates of the Bean model, the effective resistivity is not defined explicitly but only implicitly determined by (5)–(7). However, it can be shown that this model is well posed mathematically and a much more convenient variational formulation of this model can be derived. Below we present an outline of transition to such formulation; see [17] for the mathematical details.

3. VARIATIONAL FORMULATIONS

3.1. *Magnetic Field Formulation*

Let us define a space of "test" vector functions whose curl vanishes in the exterior ω and whose tangential components are continuous across the boundary Γ,

$$V = \{\boldsymbol{\varphi}(x, t) \mid \operatorname{curl} \boldsymbol{\varphi} = \mathbf{0} \text{ in } \omega,\ [\boldsymbol{\varphi}_\tau] = \mathbf{0} \text{ on } \Gamma\}$$

(the functions and their curls are assumed to be square integrable). Multiplying (1) by an arbitrary function $\boldsymbol{\varphi}$ from V and integrating in time and over ω, we obtain

$$\begin{aligned} 0 &= \int_0^T \int_\omega \left(\frac{\partial \mathbf{B}}{\partial t} + \operatorname{curl} \mathbf{E}\right) \cdot \boldsymbol{\varphi} \\ &= \mu_0 \int_0^T \int_\omega \frac{\partial \mathbf{H}}{\partial t} \cdot \boldsymbol{\varphi} + \int_0^T \int_\omega [\mathbf{E} \cdot \operatorname{curl} \boldsymbol{\varphi} + \operatorname{div}(\mathbf{E} \wedge \boldsymbol{\varphi})] \\ &= \mu_0 \int_0^T \int_\omega \frac{\partial \mathbf{H}}{\partial t} \cdot \boldsymbol{\varphi} - \int_0^T \oint_{\Gamma_+} (\mathbf{E} \wedge \boldsymbol{\varphi}) \cdot \mathbf{n}, \end{aligned}$$

since $\operatorname{curl} \boldsymbol{\varphi} = \mathbf{0}$ in ω. (The normal $\mathbf{n}$ is directed towards the domain ω.) Similarly, in Ω (1), (2), and (4) yield

$$\begin{aligned} 0 = \mu_0 \int_0^T \int_\Omega \frac{\partial \mathbf{H}}{\partial t} \cdot \boldsymbol{\varphi} + \int_0^T \int_\Omega \rho \operatorname{curl} \mathbf{H} \cdot \operatorname{curl} \boldsymbol{\varphi} \\ + \int_0^T \oint_{\Gamma_-} (\mathbf{E} \wedge \boldsymbol{\varphi}) \cdot \mathbf{n}. \end{aligned}$$

Here Γ_+ and Γ_- mean that the boundary values are taken from the sides of ω and Ω correspondingly.

Adding these two equations and taking into account that the tangential components of $\mathbf{E}$ and $\boldsymbol{\varphi}$ are continuous on Γ, we obtain the variational relation

$$\mu_0 \int_0^T \int_{R^3} \frac{\partial \mathbf{H}}{\partial t} \cdot \boldsymbol{\varphi} + \int_0^T \int_\Omega \rho \operatorname{curl} \mathbf{H} \cdot \operatorname{curl} \boldsymbol{\varphi} = 0 \qquad (10)$$

which is valid for all test functions $\boldsymbol{\varphi}$ from V (R^3 denotes the entire space).

The model now consists of Eqs. (3), (5)–(10) and the boundary condition at infinity. The electric field has been excluded and there remain only two unknowns: the magnetic field and the effective resistivity. We will now also exclude from the model the resistivity, an auxiliary variable introduced to specify the possible direction of electric field.

Let us define the external magnetic field $\mathbf{H}_e$ as a quasistationary magnetic field induced by the external current in the absence of the superconductor, i.e., as a solution of the problem

$$\begin{gathered} \operatorname{curl} \mathbf{H}_e = \mathbf{J}_e, \quad \operatorname{div} \mathbf{H}_e = 0, \\ |\mathbf{H}_e| \to 0 \quad \text{as } |x| \to \infty. \end{gathered} \qquad (11)$$

Since $\operatorname{div} \mathbf{J}_e = 0$, the problem has a unique solution and it is the curl of convolution [18],

$$\mathbf{H}_e = \operatorname{curl} (\mathscr{G} * \mathbf{J}_e),$$

where $\mathscr{G} = 1/4\pi\,|x|$ is the Green function of Laplace equation ($\mathscr{G} = (1/2\pi) \ln (1/|x|)$ for two-dimensional problems). We can now introduce a new variable, $\mathbf{h} = \mathbf{H} - \mathbf{H}_e$, satisfying

$$\begin{aligned} \operatorname{curl} \mathbf{h} &= \mathbf{0} && \text{in } \omega, && (12) \\ |\operatorname{curl} \mathbf{h}| &\le J_c(|\mathbf{h} + \mathbf{H}_e|) && \text{in } \Omega, && (13) \\ [\mathbf{h}_\tau] &= \mathbf{0} && \text{on } \Gamma. && (14) \end{aligned}$$

Let us define the set of functions

$$\mathscr{K}(\mathbf{h}) = \{\boldsymbol{\varphi} \in V \mid |\operatorname{curl} \boldsymbol{\varphi}| \le J_c(|\mathbf{h} + \mathbf{H}_e|) \text{ in } \Omega\}.$$

This set depends on $\mathbf{h}$ and, due to (12)–(14), $\mathbf{h}$ itself belongs to $\mathscr{K}(\mathbf{h})$. Furthermore, since $\operatorname{curl} \mathbf{h} = \operatorname{curl} \mathbf{H}$ in Ω, the relations (6), (7) yield

$$\int_0^T \int_\Omega \rho |\operatorname{curl} \mathbf{h}|^2 = \int_0^T \int_\Omega \rho J_c^2(|\mathbf{h} + \mathbf{H}_e|).$$

Using the last relation and Eq. (10), taking into account that $\boldsymbol{\varphi} - \mathbf{h} \in V$ for any function $\boldsymbol{\varphi}$ from $\mathscr{K}(\mathbf{h})$, we obtain

$$\begin{aligned} \mu_0 \int_0^T \int_{R^3} &\frac{\partial \{\mathbf{h} + \mathbf{H}_e\}}{\partial t} \cdot (\boldsymbol{\varphi} - \mathbf{h}) \\ &= -\int_0^T \int_\Omega \rho \operatorname{curl} \mathbf{h} \cdot \operatorname{curl}(\boldsymbol{\varphi} - \mathbf{h}) \\ &\ge \int_0^T \int_\Omega \rho (J_c^2(|\mathbf{h} + \mathbf{H}_e|) - |\operatorname{curl} \boldsymbol{\varphi}|\,|\operatorname{curl} \mathbf{h}|) \ge 0. \end{aligned}$$

This proves that $\mathbf{h}$ is a solution to the problem

$$\text{find function } \mathbf{h} \text{ such that}$$

$$\begin{gathered} \mathbf{h} \in \mathcal{K}(\mathbf{h}), \\ (\partial\{\mathbf{h} + \mathbf{H}_e\}/\partial t, \boldsymbol{\varphi} - \mathbf{h}) \geq 0 \quad \text{for any } \boldsymbol{\varphi} \in \mathcal{K}(\mathbf{h}), \\ \mathbf{h}|_{t=0} = \mathbf{h}_0, \end{gathered} \tag{15}$$

where $(\mathbf{u}, \mathbf{w}) = \int_0^T \int_{R^3} \mathbf{u} \cdot \mathbf{w}$ is the scalar product of two vector functions and $\mathbf{h}_0 = \mathbf{B}_0/\mu_0 - \mathbf{H}_e|_{t=0}$.

A problem of the form (15) is called a quasivariational inequality. More familiar are variational formulations where the solution is sought as an extremal point of some variational functional. If the problem contains a unilateral constraint, like condition (6) in our case, the extremum is to be found on the set of functions satisfying this constraint. The variational (if the admissible set does not depend on the unknown solution) or quasivariational inequalities express then an optimality condition for the constrained optimization problems [19]. Although for the nonstationary problems, like the critical-state problem under consideration, no appropriate variational functional exists [20], a formulation in the form of a variational or quasivariational inequality may, as we saw, be still available.

The problem (15) is equivalent to the Bean model: as proved in [17], the function $\mathbf{h}(x, t)$ is a solution of the quasivariational inequality (15) if and only if there exists $\rho(x, t)$ such that the pair $\{\mathbf{H}, \rho\}$, where $\mathbf{H} = \mathbf{h} + \mathbf{H}_e$, is a solution of the critical-state problem (3), (5)–(10). It is also shown in [17] that the effective resistivity ρ is a Lagrange multiplier related to the current density constraint in the Bean model.

It may be noted that the variational inequalities arise also in models of some other electromagnetic phenomena [21]. Also, the similarity of magnetic field accumulation in type-II superconductors to the growth of a sandpile has been mentioned by several authors [16, 3]. Indeed, the inequality (15) is similar to a scalar variational inequality arising in the model of pile growth [22].

If, as was assumed by Bean, the critical current density does not depend on the magnetic field, the set of admissible functions is fixed: $\mathcal{K}(\mathbf{h}) \equiv \mathcal{K}$. The inequality (15) becomes a variational inequality:

$$\begin{gathered} \text{find } \mathbf{h} \in \mathcal{K} \text{ such that} \\ (\partial\{\mathbf{h} + \mathbf{H}_e\}/\partial t, \boldsymbol{\varphi} - \mathbf{h}) \geq 0 \quad \text{for any } \boldsymbol{\varphi} \in \mathcal{K}, \\ \mathbf{h}|_{t=0} = \mathbf{h}_0. \end{gathered} \tag{16}$$

The existence and uniqueness of solution to this problem were proved in [17].

Variational reformulations of free boundary problems are very convenient for numerical solution because the free boundary does not appear in such formulations explicitly. The same numerical algorithm can be applied everywhere, e.g., in the critical and subcritical regions of the superconductor, and the front-tracking becomes unnecessary (see [23]). However, the solution of (16) must be calculated in an infinite domain, and this is an additional complication. To avoid this difficulty, we now derive an equivalent variational formulation for the current density.

3.2. *Current Density Formulation*

Let us first note that instead of the set of admissible functions $\mathcal{K}$ in the variational inequality (16) we can use the subset

$$\mathcal{K}_0 = \{\boldsymbol{\varphi} \in \mathcal{K} \mid \operatorname{div} \boldsymbol{\varphi} = 0\}.$$

Indeed, if $\mathbf{h}$ is a solution to (16), there exists a function ρ such that the pair $\{\mathbf{H}, \rho\}$, where $\mathbf{H} = \mathbf{h} + \mathbf{H}_e$, is a solution of the critical state problem. Since $\operatorname{div} \mathbf{H} = 0$, we have $\operatorname{div} \mathbf{h} = 0$ and $\mathbf{h}$ belongs to $\mathcal{K}_0$. Thus $\mathbf{h}$ is a solution to (16) with the set $\mathcal{K}_0$ instead of $\mathcal{K}$. It can be proved that this new variational inequality has also only one solution, and so it is equivalent to (16).

Furthermore, for any function $\boldsymbol{\varphi}$ from $\mathcal{K}_0$, the function $\operatorname{curl} \boldsymbol{\varphi}$ belongs to the set

$$\mathcal{K}_1 = \left\{ \boldsymbol{\psi}(x, t) \;\middle|\; \begin{array}{ll} |\boldsymbol{\psi}| \leq J_c & \text{in } \Omega, \\ \boldsymbol{\psi} = \mathbf{0} & \text{in } \omega, \\ \operatorname{div} \boldsymbol{\psi} = 0 & \text{in } R^3 \end{array} \right\},$$

which can be regarded a set of possible current densities in the superconductor. On the other hand, for any function $\boldsymbol{\psi}$ which belongs to $\mathcal{K}_1$, there exists a unique solution to the problem

$$\begin{gathered} \operatorname{curl} \boldsymbol{\varphi} = \boldsymbol{\psi}, \quad \operatorname{div} \boldsymbol{\varphi} = 0, \\ |\boldsymbol{\varphi}| \to 0 \quad \text{as } |x| \to \infty. \end{gathered}$$

This solution, which we denote by $\mathcal{R}\boldsymbol{\psi}$, belongs to $\mathcal{K}_0$ and may be written as a curl of a convolution,

$$\mathcal{R}\boldsymbol{\psi} = \operatorname{curl}(\mathcal{G} * \boldsymbol{\psi}).$$

We have already considered such a problem above and defined $\mathbf{H}_e$ as $\mathcal{R}\mathbf{J}_e$. It is easy to see that $\mathbf{h} = \mathcal{R}\mathbf{J}$ and we can rewrite inequality (16) with the admissible set

$\mathscr{K}_0$ as

$$\text{find } \mathbf{J} \in \mathscr{K}_1 \text{ such that}$$
$$(\mathscr{R}\,\partial\{\mathbf{J} + \mathbf{J}_e\}/\partial t, \mathscr{R}\,\boldsymbol{\psi} - \mathscr{R}\,\mathbf{J}) \geq 0 \quad \text{for any } \boldsymbol{\psi} \in \mathscr{K}_1,$$
$$\mathscr{R}\,\mathbf{J}|_{t=0} = \mathbf{h}_0,$$

or, equivalently,

$$\text{find } \mathbf{J} \in \mathscr{K}_1 \text{ such that}$$
$$(\mathscr{R}^*\mathscr{R}\,\partial\{\mathbf{J} + \mathbf{J}_e\}/\partial t, \boldsymbol{\psi} - \mathbf{J}) \geq 0 \quad \text{for any } \boldsymbol{\psi} \in \mathscr{K}_1,$$
$$\mathbf{J}|_{t=0} = \text{curl } \mathbf{h}_0,$$

where $\mathscr{R}^*$ is the operator adjoint to $\mathscr{R}$.

The product $\mathscr{R}^*\mathscr{R}$ can be calculated explicitly as follows. Let $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$ be two functions which disappear at infinity rapidly enough and such that div $\boldsymbol{\Phi}$ = div $\boldsymbol{\Psi} = 0$. Using Green's theorem we obtain

$$\begin{aligned}(\mathscr{R}^*\mathscr{R}\,\boldsymbol{\Phi}, \boldsymbol{\Psi}) &= (\mathscr{R}\,\boldsymbol{\Phi}, \mathscr{R}\,\boldsymbol{\Psi})\\
&= \int_0^T\int_{R^3} \text{curl}(\mathscr{G} * \boldsymbol{\Phi}) \cdot \text{curl}(\mathscr{G} * \boldsymbol{\Psi})\\
&= \int_0^T\int_{R^3} \mathscr{G} * \boldsymbol{\Phi} \cdot \text{curl} \wedge \text{curl}(\mathscr{G} * \boldsymbol{\Psi})\\
&= \int_0^T\int_{R^3} \mathscr{G} * \boldsymbol{\Phi} \cdot [\text{grad div}(\mathscr{G} * \boldsymbol{\Psi}) - \Delta(\mathscr{G} * \boldsymbol{\Psi})]\\
&= \int_0^T\int_{R^3} \mathscr{G} * \boldsymbol{\Phi} \cdot [\mathscr{G} * (\text{grad div } \boldsymbol{\Psi}) - (\Delta\mathscr{G}) * \boldsymbol{\Psi}]\\
&= \int_0^T\int_{R^3} \mathscr{G} * \boldsymbol{\Phi} \cdot \boldsymbol{\Psi},\end{aligned}$$

since div $\boldsymbol{\Psi} = 0$ and $-\Delta\mathscr{G}$ is the delta function. Thus $\mathscr{R}^*\mathscr{R}\,\boldsymbol{\Phi} = \mathscr{G} * \boldsymbol{\Phi}$, which is the magnetic vector potential of current $\boldsymbol{\Phi}$.

We thus arrive at the variational inequality

$$\text{find } \mathbf{J} \in \mathscr{K}_1 \text{ such that}$$
$$(\mathscr{G} * \partial\mathbf{J}/\partial t + \partial\mathbf{A}_e/\partial t, \boldsymbol{\psi} - \mathbf{J}) \geq 0 \quad \text{for any } \boldsymbol{\psi} \in \mathscr{K}_1, \quad (17)$$
$$\mathbf{J}|_{t=0} = \text{curl } \mathbf{h}_0,$$

where

$$\mathbf{A}_e = \mathscr{G} * \mathbf{J}_e \quad (18)$$

is the vector potential of the external current. The inequality (17) contains a nonlocal operator of convolution. However, this is a reasonable price for reducing the problem's formulation to a bounded domain.

Let us now consider the magnetization of superconductors in a temporally varying uniform external magnetic field. Such a uniform field $\mathbf{H}_e(t)$ in the neighborhood of a superconductor can be generated by placing the superconductor inside a long solenoid. For two-dimensional configurations, i.e., a long cylinder in perpendicular field, the uniform external field could be generated by two parallel infinite sheets of current. Since $\mathbf{H}_e = \text{curl } \mathbf{A}_e$, we can easily find the vector potential corresponding to the uniform field $\mathbf{H}_e$ up to a constant of integration. For two-dimensional problems, the potential of the current which induces the field $\mathbf{H}_e = H_{e,1}\mathbf{e}_1 + H_{e,2}\mathbf{e}_2$ is $\mathbf{A}_e = (x_2H_{e,1} - x_1H_{e,2} + C)\mathbf{e}_3$ in the space between the two plates. The potential inside the solenoid, where $\mathbf{H}_e = H_e\mathbf{e}_z$, is $\mathbf{A}_e = (rH_e/2 + C/r)\mathbf{e}_\phi$ in cylindrical coordinates $\{r, \phi, z\}$. A somewhat laborious direct calculation of potentials using (18) shows that the constant of integration C is zero in both cases.

Since $\mathbf{J} = J(x_1, x_2, t)\mathbf{e}_3$ for two-dimensional, and $\mathbf{J} = J(r, z, t)\mathbf{e}_\phi$ for axially symmetric problems, the zero-divergence condition in the definition of $\mathscr{K}_1$ is satisfied automatically. All functions from this set are zero in the exterior ω and may be regarded simply as the functions defined in Ω. For two-dimensional problems, the inequality (17) is scalar and can be written as

$$\text{find } J \in \mathscr{K}_2 \text{ such that}$$
$$(\mathscr{G} * \partial J/\partial t + x_2\partial H_{e,1}/\partial t - x_1\partial H_{e,2}/\partial t, \psi - J) \geq 0 \quad \text{for any } \psi \in \mathscr{K}_2,$$
$$J|_{t=0} = J_0, \quad (19)$$

where J_0 is the only nonzero component of curl $\mathbf{h}_0$ and

$$\mathscr{K}_2 = \{\psi(x, t) \mid -J_c \leq \psi \leq J_c\}$$

is the set of functions defined in Ω.

A similar scalar formulation of (17) can be obtained for axially symmetric problems. The only nonzero component of the convolution, $(\mathscr{G} * \partial\mathbf{J}/\partial t)_\phi$, can in this case be expressed in terms of complete elliptic integrals of the first and second kind, K and E, and does not depend on ϕ. This leads to the variational inequality

$$\text{find } J \in \mathscr{K}_2 \text{ such that}$$
$$(\mathscr{L}\,\partial J/\partial t + \pi r^2\partial H_e/\partial t, \psi - J) \geq 0 \quad \text{for any } \psi \in \mathscr{K}_2, \quad (20)$$
$$J|_{t=0} = J_0,$$

with the linear operator

$$\mathscr{L}\,\psi(r, z) = \int\int a(r, z; r', z')\psi(r', z')dr'\,dz', \quad (21)$$

where

$$a(r, z; r', z') = (rr')^{1/2} k \left(\frac{2}{k^2} \{K(k) - E(k)\} - K(k) \right),$$

$$k = 2 \left(\frac{rr'}{(r + r')^2 + (z - z')^2} \right)^{1/2}.$$

The current J and all functions $\psi \in \mathscr{K}_2$ are, in this case, defined in the half cross section of the superconductor, which is a body of rotation.

4. NUMERICAL SOLUTION

The first step in the numerical solution of (19) and (20) is the finite difference approximation in time. This leads to stationary variational inequalities at each time layer,

$$\begin{gathered} \text{find } J \in \mathscr{K}_3 \text{ such that} \\ (\mathscr{L}J - f, \psi - J) \geq 0 \quad \text{for any } \psi \in \mathscr{K}_3, \end{gathered} \tag{22}$$

where

$$\mathscr{K}_3 = \{\psi(x) | -J_c \leq \psi \leq J_c\}.$$

For axially symmetric problems, the operator $\mathscr{L}$ is defined by (21), $f = \mathscr{L}\hat{J} - \pi r^2(H_e - \hat{H}_e)$, and "$\hat{\ }$" means that the value is taken from the previous time layer. For two-dimensional problems, $\mathscr{L}$ is the convolution with $\mathscr{G} = (1/2\pi) \ln(1/|x|)$ and $f = \mathscr{L}\hat{J} - x_2(H_{e,1} - \hat{H}_{e,1}) + x_1(H_{e,2} - \hat{H}_{e,2})$.

It can be shown that these stationary variational inequalities are equivalent to constrained optimization problems

$$\min_{J \in \mathscr{K}_3} \left\{ \frac{1}{2} (\mathscr{L}J, J) - (f, J) \right\}. \tag{23}$$

Indeed, let $\tilde{J}$ be a solution to (23) and let ψ be an arbitrary function from $\mathscr{K}_3$. For any $\lambda \in [0, 1]$, the function $J_\lambda = (1 - \lambda)\tilde{J} + \lambda\psi$ also belongs to $\mathscr{K}_3$. Since $\tilde{J}$ is the point of minimum,

$$0 \leq \frac{d}{d\lambda} \left\{ \frac{1}{2} (\mathscr{L}J_\lambda, J_\lambda) - (f, J_\lambda) \right\} \Bigg|_{\lambda=0} = (\mathscr{L}\tilde{J} - f, \psi - \tilde{J}),$$

and $\tilde{J}$ also solves the variational inequality (22). The converse is also true.

To proceed with the solution, we now discretize the problem in space and perform the optimization numerically. Let an infinitely long cylinder with a cross section Ω be placed into a perpendicular magnetic field (Fig. 2). We define a regular finite element net with square elements in a rectangle, covering domain Ω, and denote by I_Ω the set of elements which lie mostly inside this domain. At each time layer, the current density is approximated by a constant, J_i, within each finite element e_i. The discretized optimization problem (23) can be written as

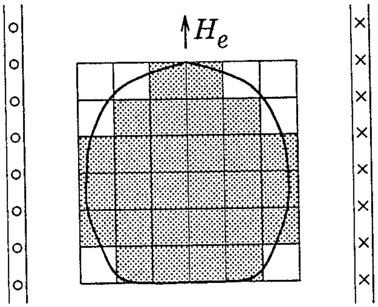

FIG. 2. Long superconductor in perpendicular field and finite element discretization.

$$\min_{\substack{\{J_1, \ldots, J_N\} \\ |J_m| \leq J_c \text{ if } n \in I_\Omega \\ J_m = 0 \text{ if } n \notin I_\Omega}} \left\{ \frac{1}{2} \sum_{i,j=1}^{N} J_i M_{i,j} J_j - \sum_{i=1}^{N} f_i J_i \right\} \tag{24}$$

Here N is the number of finite elements,

$$M_{i,j} = \int_{e_i} \int_{e_j} \mathscr{G}(x - x')\, dx_1 dx_2 dx_1' dx_2',$$

$$f_i = \int_{e_i} f(x) = \sum_{j=1}^{N} M_{i,j} \hat{J}_j - (H_{e,1} - \hat{H}_{e,1}) \int_{e_i} x_2 + (H_{e,2} - \hat{H}_{e,2}) \int_{e_i} x_1.$$

For axially symmetric problems we approximate half the cross section in a similar way and obtain (24) with the coefficients

$$M_{i,j} = \int_{e_i} \int_{e_j} a(r, z; r', z')\, dr dz dr' dz',$$

$$f_i = \sum_{j=1}^{N} M_{i,j} \hat{J}_j - \pi(H_e - \hat{H}_e) \int_{e_i} r^2\, dr dz.$$

Below we compare the numerical and analytical solutions also for infinitely thin strips and disks in perpendicular fields. The cross sections are then intervals; to find the numerical solutions we solve (24) with the appropriate coefficients.

Some of the integrals $M_{i,j}$ are singular and their evaluation needs special consideration (Appendix). Provided the

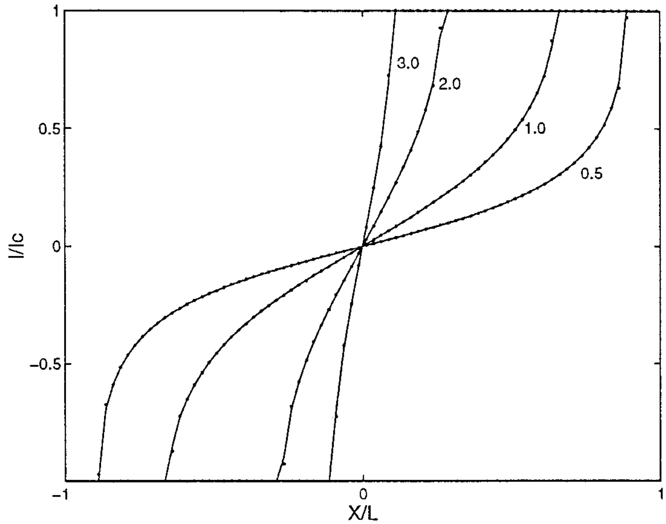

FIG. 3. Thin strip in a perpendicular field. The distribution of current density: ———, analytical; · · ·, numerical solutions. Numbers indicate the ratio H_e/H_c.

coefficients are found, solving the optimization problem is not difficult. We used the method of point underrelaxation with projection [23], which in this case was more efficient than the relaxation or overrelaxation algorithms. The computation of a typical example (see below) needed about 20 min of IBM RS6000/370 time for the finite-element grid 80×80.

5. EXAMPLES

In all examples below we assume the superconductor is initially in the virgin (zero field) state. Let us consider first a long strip in a perpendicular external field. If the strip thickness δ is much smaller than the strip width $2L$, the current density integrated across the strip thickness,

$$I(x_1, t) = \int_0^{\delta} J(x_1, x_2, t)\, dx_2,$$

can be found analytically [5, 6]. For a growing external magnetic field $\mathbf{H}_e = H_e(t)\mathbf{e}_2$,

$$\frac{I}{I_c} = \begin{cases} \dfrac{2}{\pi} \operatorname{arctg}[c x_1/(b^2 - x_1^2)^{1/2}], & |x_1| \le b, \\ x_1/|x_1|, & b < |x_1| \le L, \end{cases}$$

where $I_c = J_c\delta$, $c = \operatorname{tgh}(H_e/H_c)$, $b = L(1 - c^2)^{1/2}$, and $H_c = I_c/\pi$. The numerical solution of (19) approximates the analytical one very well (Fig. 3).

The real strips have a finite thickness, and the magnetic field first penetrates into their surface region. This is the region of shielding current which has the critical density. The zero field core shrinks with the growth of external field and completely disappears when this field becomes sufficiently strong (Fig. 4). Note that the possible direction of current density is known and the magnitude can take only the values $\pm J_c$ or zero. This fact can be used in the numerical procedure; instead of performing the minimization in (24) with high accuracy, one can simply round the values of current density, obtained at some stage of minimization, to the closest of these three possible values. By a different method, a similar result was recently obtained by Brandt [24].

Results of numerically simulating the penetration of perpendicular magnetic field into cylindrical superconductors of various cross sections are shown in Fig. 5.

In [4], Bean presented an asymptotic solution for the magnetization of a superconducting half space in a rotating

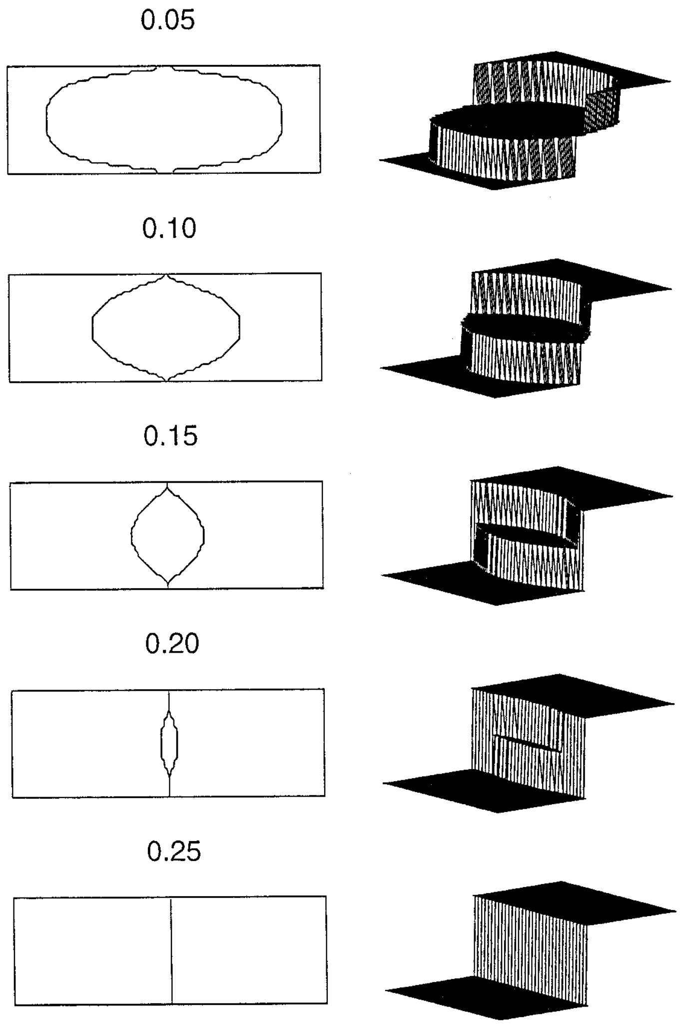

FIG. 4. Thick strip in a perpendicular field. The shrinking zero field core (left) and current density (right). Numbers indicate H_e/LJ_c, where $2L$ is the strip width.

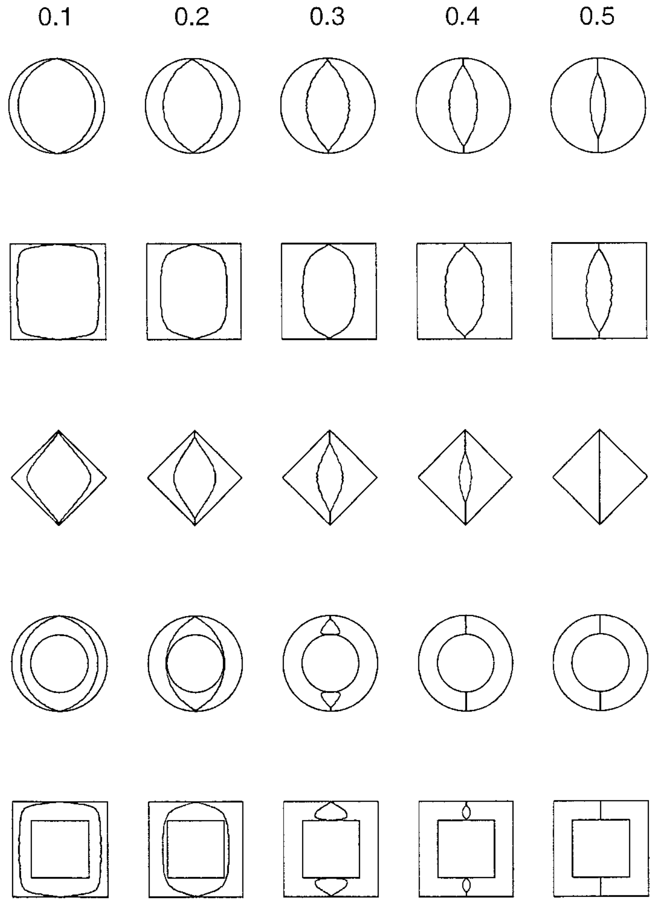

FIG. 5. The penetration of a perpendicular magnetic field into cylindrical superconductors of different cross sections. Numbers indicate H_e/RJ_c, R is the radius of circular cross section.

parallel external field. As shown in his work, a stationary rotating solution develops as $t \to \infty$. We simulated the magnetization of long cylinders with square and circular cross sections in the rotating perpendicular external field (Fig. 6). In these two examples, the external field first grows up to a given magnitude and then starts to rotate. The zero field core may only shrink and, as is apparent from these examples, has almost reached its final shape even before the first turn of external field is completed.

The magnetization of a thin disk in a perpendicular field has been recently considered in [7], and the distribution of integrated current density $I(r, t)$ was found analytically. In an increasing magnetic field,

$$\frac{I}{I_c} = \begin{cases} \frac{2}{\pi}\operatorname{arctg}\{(r/R)[(R^2 - b^2)/(b^2 - r^2)]^{1/2}\}, & 0 \le r \le b, \\ 1, & b < r \le R, \end{cases}$$

where $I_c = J_c\delta$, R is the disk radius, $b = R/\cosh(H_e/H_c)$, and $H_c = I_c/2$. The numerical and analytical solutions are presented in Fig. 7.

The problem of magnetic field penetration into a superconducting ball has been solved asymptotically for weak external fields H_e in [3],

$$\varrho(\theta) \approx R - \frac{3H_e}{2J_c}\sin\theta,$$

where θ is the azimuth angle, R is the ball radius, and $\varrho(\theta)$ is the boundary of the zero field core. The asymptotic and numerical solutions are close (Fig. 8). The further magnetization of a superconducting ball and some other bodies of rotation in a growing uniform field has also been simulated numerically (Fig. 9).

6. CONCLUSION

The Bean critical-state model is a quasistationary macroscopic model of equilibrium of the system of magnetic

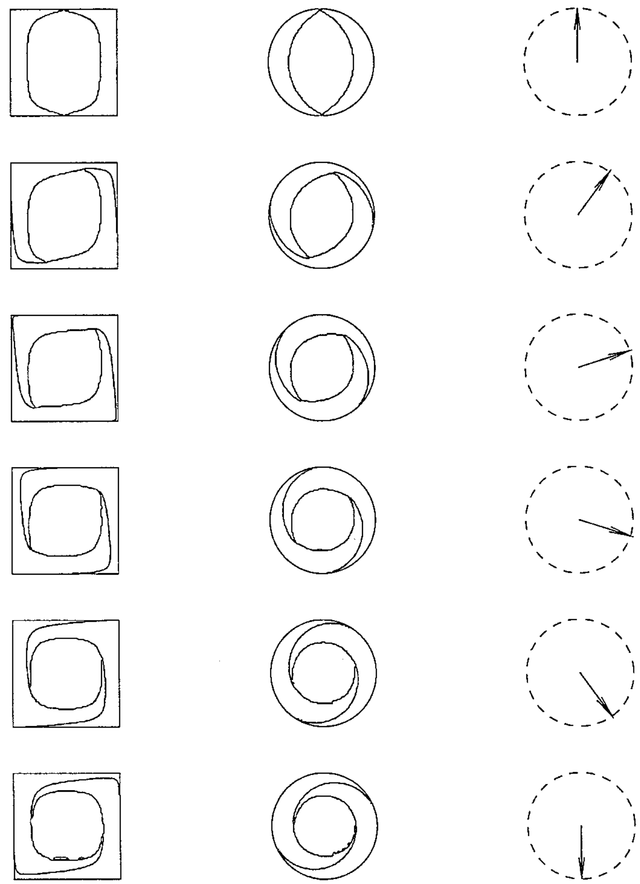

FIG. 6. Cylindrical superconductors in a rotating field: the evolution of zero-field core. The external field (right column) first reaches the value $H_e/RJ_c = 0.2$ and then starts to rotate.

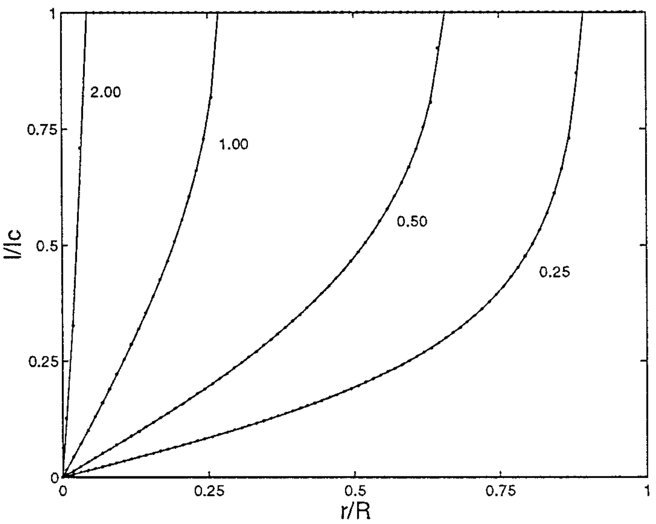

FIG. 7. Thin disk in a perpendicular field. Current density: ———, analytical; · · ·, numerical solutions. Numbers indicate H_e/H_c.

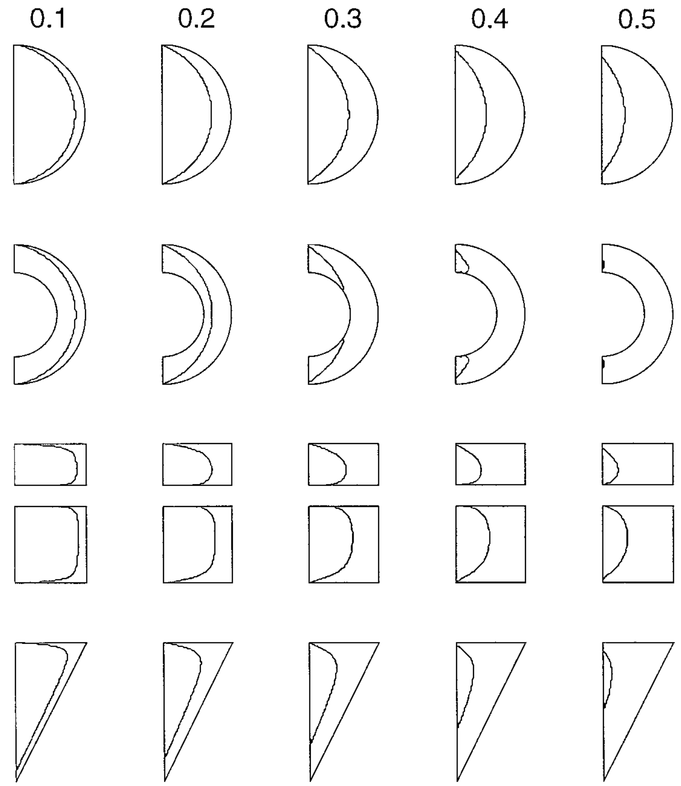

FIG. 9. Magnetization of bodies of rotation: ball, hollow ball, two adjacent cylinders, cone (half cross sections are shown). Numbers are H_e/RJ_c, R is the ball radius.

vortices in type-II superconductors. In this model, the effective resistivity characterizing the energy dissipation due to the movement of vortices is determined by the external conditions and the system's state in a nonlocal way. Physically, this results from the assumption that when the equilibrium is violated the time needed for the system of vortices to reach a new metastable state is negligible on the time scale of observations. Mathematically, the resistivity in the critical-state model is a Lagrange multiplier and it is excluded as the model is formulated as an evolutionary variational or quasivariational inequality describing penetration of the magnetic field.

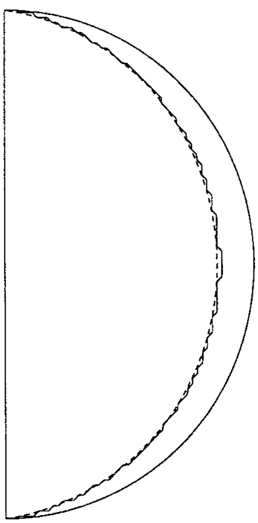

FIG. 8. Ball in a weak external field. The zero-field core boundary: ———, numerical, ---, asymptotic solutions. $H_e/J_cR = 0.1$, half cross section is shown.

In our work such a formulation has been proposed for two-dimensional and axially symmetric configurations. We further reformulated the model and derived an equivalent evolutionary variational inequality in terms of the current density, which is unknown only in a bounded domain. This variational inequality served as a basis for an efficient numerical solution of critical-state problems with the field-independent critical current density. The numerical results were tested by comparison with the known analytical solutions for simple geometries and good accuracy was demonstrated.

Provided the current distribution in the superconductor is found, the magnetic field in any area of interest can be calculated via the Biot–Savart law. This allows us to account also for any given dependence of critical current density on the magnetic field. In this case the inequality is quasivariational and its numerical solution would need an additional loop of iterations.

The main advantage of the proposed approach is its universality; the method allows one to completely avoid the tracking of the free boundary, does not need any *a priori* information on the free boundary behavior, and is therefore applicable without modifications even if the free boundary topology is complicated and changes in time.

APPENDIX: EVALUATION OF INTEGRALS

A1. *Infinitely Thin Strip in a Perpendicular Field*

The finite elements e_i are intervals (x_{i-1}, x_i), $x_i = x_{i-1} + d$. In this case

$$M_{i,j} = \frac{1}{2\pi}\int_{e_i}\int_{e_j} \ln\frac{1}{|x-x'|}\,dx\,dx' = -\frac{d^2}{2\pi}\{\ln(d) + \mathscr{I}_{|i-j|}\},$$

where the integrals $\mathscr{I}_m = \int_0^1\int_0^1 \ln(|x - x' + m|)$ are evaluated analytically: $\mathscr{I}_0 = -\frac{3}{2}$, $\mathscr{I}_1 = -\frac{3}{2} + 2\ln(2)$, and for $m > 1$

$$\mathscr{I}_m = -\tfrac{3}{2} + \tfrac{1}{2}(m+1)^2\ln(m+1) - m^2\ln(m) + \tfrac{1}{2}(m-1)^2\ln(m-1).$$

A2. *Cylinder in a Perpendicular Field*

The cross section of a cylinder is approximated by a regular finite element mesh with the same step d in x_1 and x_2. Denote by e_{mn} the element with the left lower corner at the node with coordinates $(x_{1,m}, x_{2,n})$. Changing the variables to transform each element into the unit square, we obtain

$$M_{i_1 i_2, j_1 j_2} = \int_{e_{i_1 i_2}}\int_{e_{j_1 j_2}} \mathscr{G}(x - x') = -\frac{d^4}{4\pi}\{2\ln(d) + \mathscr{I}_{|i_1-j_1|,|i_2-j_2|}\},$$

where the integrals

$$\mathscr{I}_{m,n} = \int_0^1\int_0^1\int_0^1\int_0^1 \ln((x_1 - x_1' + m)^2 + (x_2 - x_2' + n)^2)$$

can be evaluated analytically (we used the program for symbolic computations Maple to find some of them). However, the resulting expressions are complicated and it is much easier to estimate the nonsingular integrals by means of a simple (with a few knots) quadrature. The values of the singular integrals are given in Table I.

TABLE I

Singular Integrals $\mathscr{I}_{m,n}$

$\mathscr{I}_{0,0}$	$\mathscr{I}_{0,1} = \mathscr{I}_{1,0}$	$\mathscr{I}_{1,1}$
−1.610173	0.0130569	0.6885433

A3. *Axially Symmetric Problems*

Transforming each finite element in the cross section into the unit square by a change of variables, we obtain

$$M_{i_1 i_2, j_1 j_2} = \int_{e_{i_1 i_2}}\int_{e_{j_1 j_2}} a(r, z; r', z') = d^5\,\mathscr{I}_{i_1, j_1, |i_2 - j_2|},$$

where

$$\mathscr{I}_{i,j,m} = \int_0^1\int_0^1\int_0^1\int_0^1 [(r+i)(r'+j)]^{1/2}\, k\left(\frac{2}{k^2}\{K(k) - E(k)\} - K(k)\right),$$

$$k = 2\left[\frac{(r+i)(r'+j)}{(r+r'+i+j)^2 + (z - z' + m)^2}\right]^{1/2}.$$

The integrand is singular at the points where $r + i = r' + j$ and $z - z' + m = 0$ because at these points $k = 1$ and the complete elliptic integral of the first type, $K(k)$, becomes infinite. If there are no such points in the region of integration, a numerical quadrature with a few knots can be used to estimate the integral. Otherwise, one can use the asymptotic expansion [25]

$$K(k) = \ln\frac{4}{k'} + \frac{1}{4}\left(\ln\frac{4}{k'} - 1\right)(k')^2 + \ldots,$$

where $k' = \sqrt{1 - k^2}$, to single out the nonregular part of the integrand. In the neighborhood of a singular point,

$$k\left(\frac{2}{k^2}\{K(k) - E(k)\} - K(k)\right) \sim K(k) \sim \ln\frac{4}{k'} \sim -\frac{1}{2}\ln((r - r' + i - j)^2 + (z - z' + m)^2),$$

and so the singular part of a is

$$a_s = -\tfrac{1}{2}[(r+i)(r'+j)]^{1/2}\ln((r - r' + i - j)^2 + (z - z' + m)^2).$$

The regular part, $a_r = a - a_s$, can be integrated numerically, while the singular part can be first integrated analytically in z and z', which removes the singularity, and then numerically in r and r'. For the infinitely thin disk in a perpendicular field, the singular part of the integral can be evaluated analytically.

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