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Variational inequalities in critical-state problems

Leonid Prigozhin*

Department of Solar Energy and Environmental Physics, The Jacob Blaustein Institute for Desert Research, Ben Gurion University of the Negev, Sede Boqer Campus, 84990 Israel

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Abstract

Similar evolutionary variational inequalities appear as convenient formulations for continuous quasistationary models for sandpile growth, formation of a network of lakes and rivers, magnetization of type-II superconductors, and elastoplastic deformations. We outline the main steps of such models derivation and try to clarify the origin of this similarity. New dual variational formulations, analogous to mixed variational inequalities in plasticity, are derived for sandpiles and superconductors.
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1. Introduction

Spatially extended dissipative systems have recently attracted much interest among physicists. These systems may have infinitely many metastable states but, driven by the external forces, often organize themselves into a marginally stable "critical state" and are then able to demonstrate almost instantaneous long-range interactions between their distantly separated parts.

Modifications of a crude cellular automata model of sandpile [1] have been used by many authors for simulating such systems behavior and it was sometimes doubted whether the models based on differential equations can in principle be employed: the relaxation in continuous models is expected to be a smooth process evolving in time, while, e.g., cellular automata models are able to mimic sand avalanches as sudden catastrophic events. Nevertheless,

* Tel.: +972 86477806; fax: +972 86477650.
E-mail address: leonid@cs.bgu.ac.il.
URL: http://www.cs.bgu.ac.il/∼leonid.

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continuous models allowing for long-range interactions, hysteresis, metastability, and avalanches have been derived for sandpiles [2–4], river networks [2], type-II superconductors [5–7]. Although these are dissipative systems of a different nature, their continuous models are equivalent to very similar variational or quasivariational evolutionary inequalities, the formulations convenient for both the numerical simulation [8] and theoretical study [9]. Much earlier, similar variational formulations have been derived for various plasticity problems (see [9]).

The aim of this work is to outline the main steps of such models derivation and to clarify the origin of this similarity. We present the simplest version of each model and try to avoid mathematical details but give references to rigorous proofs, numerical simulations, and possible extensions of the considered models. We also derive new dual variational formulations in terms of the conjugate variables for sandpiles and superconductors. The dual problems are similar to mixed variational inequalities in plasticity [10]. Well-posedness of these new problems is yet to be investigated.

2. Variational formulations

2.1. Sandpiles

Let a cohesionless granular material, characterized by its angle of repose α, be poured out onto a rough rigid surface $y = h_0(x)$, where y is vertical and $x \in \Omega \subset \mathbb{R}^2$. We find the shape of a growing pile, $y = h(x, t)$.

Assuming the flow of granular material down the slope of the pile is confined to a thin boundary layer and the bulk density of material in the pile is constant, we can write the mass conservation law in the form $\partial_t h + \nabla \cdot \boldsymbol{q} = w$, where $\boldsymbol{q}$ is the horizontal projection of the material flux and $w(x, t) \geq 0$ is the intensity of the source of material being poured onto the pile. Neglecting inertia, we suppose that the surface flow is directed towards the steepest descent, $\boldsymbol{q} = -m\nabla h$, where

$$m(x, t) \geq 0 \tag{1}$$

is an *unknown* scalar function. The conservation law assumes the form

$$\partial_t h - \nabla \cdot (m\nabla h) = w. \tag{2}$$

The free surface initially coincides with the support,

$$h(x, 0) = h_0(x), \tag{3}$$

and cannot lie below it,

$$h(x, t) \geq h_0(x). \tag{4}$$

Wherever the granular material covers the support, the surface slope cannot exceed the material repose angle α,

$$h(x, t) > h_0(x) \longrightarrow |\nabla h(x, t)| \leq \gamma, \tag{5}$$

where $\gamma = \tan(\alpha)$. No surface flow occurs over the parts of the pile surface inclined less than at the angle of repose:

$$|\nabla h(x, t)| < \gamma \longrightarrow m(x, t) = 0. \tag{6}$$

We assume for simplicity that there is a vertical wall at the boundary Γ of domain Ω, hence

$$m\partial_n h = 0 \text{ on } \Gamma. \tag{7}$$

The formulated model for pile growth contains two unknowns, the free surface h and an auxiliary function m, and it is difficult to deal with the equations and inequalities (1)–(7) directly. Fortunately, a more convenient variational formulation can be derived (here we follow [2], see [11] for a rigorous proof). Let us define, for every continuous function ψ, a nonlinear operator

$$B_\psi(\varphi) = \tfrac{1}{2}(|\nabla\varphi|^2 - M(\psi)),$$

where

$$M(\psi)(x,t) = \begin{cases} \gamma^2, & \text{if } \psi(x,t) > h_0(x), \\ \max(\gamma^2, |\nabla h_0(x)|^2), & \text{if } \psi(x,t) \le h_0(x). \end{cases}$$

We define also a family of closed convex sets[1]

$$K(\psi) = \{\varphi(x,t) | B_\psi(\varphi) \le 0\},$$

denote by (u, v) the scalar product of two functions, and consider an evolutionary quasivariational inequality written for the pile surface alone:

$$\text{Find } h \in K(h) \text{ such that } \quad (\partial_t h - w, \varphi - h) \ge 0 \text{ for any } \varphi \in K(h), \quad h(x,0) = h_0(x). \tag{8}$$

Theorem. *A function h(x, t) is a solution of the quasivariational inequality* (8) *if and only if there exists m(x, t) such that the pair* $\{h, m\}$ *satisfies a weak form of the problem* (1)–(7).

Proof. We formally rewrite the inequality (8) as an implicit optimization problem

$$J_h(h) = \min J_h(\varphi), \quad \varphi \in K(h) \tag{9}$$

where $J_h(\varphi) = (\partial_t h - w, \varphi)$ is a linear functional which depends on the solution h. Let us fix the function h in J_h and $K(h)$ and derive the necessary and sufficient condition of optimality for (9) using the Lagrange multipliers technique ([12], ch. 3, th. 5.1). Substituting then the function h into this condition, we obtain a similar condition for the problem with an implicit constraint: h is a solution of the quasivariational inequality (8) if and only if there exists a Lagrange multiplier $m(x,t) \ge 0$ such that the pair $\{h, m\}$ is a saddle point of Lagrangian, i.e.,

$$J_h(h) + (m', B_h(h)) \le J_h(h) + (m, B_h(h)) \le J_h(h') + (m, B_h(h')) \tag{10}$$

for arbitrary h' and $m' \ge 0$. The condition of supplementary slackness,

$$(m, B_h(h)) = 0, \tag{11}$$

is thereby fulfilled.[2]

Let h be a solution of (8). As follows from (10), the functional $(\partial_t h - w, h') + \frac{1}{2}(m, |\nabla h'|^2 - M(h))$ has a minimum at the point $h' = h$. Hence,

$$(\partial_t h - w, \chi) + (m, \nabla h \cdot \nabla \chi) = 0 \tag{12}$$

for any test function χ. This is a weak formulation of Eq. (2) with the boundary condition (7). Since $h \in K(h)$, condition (5) is satisfied, (6) follows from (11), and to show that $\{h, m\}$ satisfies all model relations it remains only

[1] More exactly, $K(\psi) = \{\varphi \in L^\infty(0,T; W^{1,\infty}(\Omega)) | B_\psi(\varphi) \le 0 \text{ a.e.}\}$, see [11].

[2] As is explained in [11], to satisfy a constraint qualification hypothesis ([12], ch. 3, (5.24)) we need to define $B_h : L^\infty(0,T; W^{1,\infty}(\Omega)) \to \mathcal{L} = L^\infty((0,T) \times \Omega)$. Hence m belongs to the dual space $\mathcal{L}'$ and is a nonnegative Radon measure.

to check that $h \geq h_0$. Choosing

$$\varphi = \begin{cases} h + (h_0 - h)_+, & \text{for } 0 \leq t \leq t_0, \\ h, & \text{otherwise}, \end{cases}$$

where z_+ means max(z, 0) and taking into account that $\varphi \in K(h)$, $w \geq 0$ we obtain

$$0 \leq (\partial_t h - w, \varphi - h) \leq -\frac{1}{2}\int \{[h_0(x) - h(x, t_0)]_+\}^2 \, \mathrm{d}\Omega,$$

so the inequality (4) is proved.

Let now $\{h, m\}$ be a solution to (1)–(7). By (4), (5) we have $|\nabla h| \leq \gamma$ wherever $h > h_0$ and $h = h_0$ otherwise, hence $h \in K(h)$. To prove that h solves the quasivariational inequality, it is sufficient to show the inequalities (10) hold. It is easy to see that (11) is true, so the left of the inequalities (10) is fulfilled. Using the weak form (12) of Eqs. (2) and (7) we obtain, for $\chi = h' - h$,

$$J_h(h') + (m, B_h(h')) - J_h(h) - (m, B_h(h)) = \tfrac{1}{2}(m, |\nabla\{h' - h\}|^2) \geq 0.$$

Thus the second inequality in (10) holds too, which completes the proof. □

We note that the auxiliary unknown, m, introduced into the pile growth model to fix the possible sand flux direction, turns out to be a Lagrange multiplier related to the equilibrium constraint upon the pile surface incline and is eliminated in transition to the variational formulation. The multiplier depends in a non-local way on the surface and source and that is why instantaneous long range interactions over the critically inclined parts of the surface are possible in this model. Such a situation is typical also of other dissipative systems where the relaxation is fast and the assumption that all the dynamics occur at the border of stability is justified.

If the support has no steep slopes, $|\nabla h_0| \leq \gamma$, the set of admissible functions K becomes fixed (does not depend on the solution) and the inequality (8) becomes simply variational; in this case the existence and uniqueness of a solution have been proved [11]. It remains an interesting open problem to prove existence and uniqueness in the general quasivariational case.

The variational formulation obtained is very convenient for numerical simulation of pile growth, see [13]. There are analytical solutions [2] exactly describing the pile shapes in experiments [14]. Mathematically, the avalanches upon pile surface correspond to solutions with the jumps caused by sudden variations of the admissible functions set K due to local fluctuations of the repose angle, see [2]. Such discontinuous solutions of the variational inequality have been studied in [4]. It has been also shown [15] that the mesoscopic BCRE model [16] for sand surface dynamics converges in the long scale limit to the inequality (8). In a continuous limit, stochastic cellular automata models of sandpiles converge to a similar variational inequality with the anisotropy inherited from the cellular structure of these models [17].

2.2. Lakes and rivers

Let now h_0 be the land surface, w the intensity of precipitation. We assume for simplicity that the water neither evaporates nor penetrates the soil but just flows down the slopes and accumulates into lakes at local depressions of the relief. The level of a lake rises until it reaches the divide of two basins. Then a river running out of the lake appears and transfers all additional water to another lake below.

To model the evolution of this system of lakes and rivers, let us note that the balance equation

$$\partial_t h + \nabla \cdot \boldsymbol{q} = w,$$

in which $\boldsymbol{q}$ is the horizontal projection of water flux, remains valid. The free boundary h in this problem either coincides with h_0 or, where it is higher, is the horizontal surface of a lake:

$$h(x,t) \geq h_0(x), \quad h(x,t) > h_0(x) \longrightarrow \nabla h(x,t) = 0.$$

Over the hill slopes, where $h = h_0$, we assume again that the flux is in the steepest descent direction,

$$h(x,t) = h_0(x) \longrightarrow \boldsymbol{q} = -m\nabla h,$$

where $m(x,t) \geq 0$ is unknown. However, this is not true for the lakes, where $h > h_0$ and $\nabla h = 0$. In fact, although the lake hydrodynamics are not trivial, the flow in the lake does not affect the free surface, and it can be shown [2] that the model relations above lead to the quasivariational inequality (8) with $\gamma = 0$.

Indeed, let the pair $\{h, m\}$ satisfy these relations. Then $h \in K(h)$ and for any function $\varphi \in K(h)$ we obtain

$$(\partial_t h - w, \varphi - h) = -(\nabla \cdot \boldsymbol{q}, \varphi - h) = -\int_0^T \oint_\Gamma q_n(\varphi - h) + \int_0^T \int_\Omega \boldsymbol{q} \cdot \nabla(\varphi - h).$$

The first integral on the right hand side is zero due to the boundary conditions. Gradients ∇h and $\nabla\varphi$ are both zero wherever $h > h_0$. Outside this domain $h = h_0$, $\boldsymbol{q} = -m\nabla h_0$, $m \geq 0$, and $|\nabla\varphi| \leq |\nabla h_0|$. Therefore,

$$\boldsymbol{q} \cdot \nabla(\varphi - h) = -m(\nabla h_0 \cdot \nabla\varphi - |\nabla h_0|^2) \geq 0,$$

the quasivariational inequality holds and determines the free surface evolution.

This is, however, only a part of the solution needed: it is the water flux $q = |\boldsymbol{q}|$, or, equivalently, the auxiliary variable m, which is of interest in most geomorphological and hydrological applications. Provided the free surface h is found, the water flux in the coincidence set $h = h_0$ can be determined, at least in some simple cases [2], from the water balance equation which takes in this set the form

$$-\nabla \cdot \left(q \frac{\nabla h_0}{|\nabla h_0|} \right) = w.$$

Generally, this is a difficult task and a different approach to water flux calculation is desirable. Below, we consider an alternative approach to determining the conjugate variables for variational inequalities.

Note that if $\gamma = 0$ the quasivariational inequality (8) is no more equivalent to the sandpile model (1)–(7) in which $q = 0$ if $\nabla h = 0$. This equivalency breaks down because, if $\gamma = 0$, the constraint qualification hypothesis [12] is not true: there exists no φ such that $B_h(\varphi) < 0$.

2.3. Superconductors

Phenomenologically, the magnetic field penetration into type-II superconductors can be understood as a nonlinear eddy current problem. In accordance with the Faraday law of electromagnetic induction, the eddy currents in a conductor are driven by the electric fields induced by time variations of the magnetic flux. Let the superconductor occupy a domain $\Omega \subset \mathbb{R}^3$ and $\omega = \mathbb{R}^3 \backslash \overline{\Omega}$ be the outer space. We denote by Γ the common boundary of these domains and assume $\boldsymbol{n}$, the unit normal to Γ, is directed outside Ω.

Omitting the displacement current in Maxwell equations and assuming the magnetic permeability of the superconductor is equal to that of vacuum and scaled to be unity, we obtain the following eddy current model,

$$\begin{aligned} &\partial_t \boldsymbol{h} + \nabla \times \boldsymbol{e} = 0, \\ &\nabla \times \boldsymbol{h} = \boldsymbol{j} + \boldsymbol{j}_{\rm e}, \end{aligned} \quad x \in \mathbb{R}^3,\ t > 0, \tag{13}$$

with the initial condition $\boldsymbol{h}|_{t=0} = \boldsymbol{h}_0(x)$ such that $\nabla \cdot \boldsymbol{h}_0 = 0$. Here $\boldsymbol{j}$ is the current in the superconductor ($\boldsymbol{j} = 0$ in ω), and $\boldsymbol{j}_{\rm e}$ is the given external current having a bounded support $\mathrm{supp}\{\boldsymbol{j}\} \subset \omega$ and satisfying $\nabla \cdot \boldsymbol{j}_{\rm e} = 0$. Additionally, in the conductive domain Ω a current–voltage law has to be postulated.

In an ordinary conductor, the vectors of the electric field and current density are related by the linear Ohm law. Type-II superconductors are instead characterized in the Bean critical-state model [18] by a highly nonlinear current–voltage relation which gives rise to a free boundary problem. The problem has been solved mainly under the assumption that the electric field and current density are parallel (see, e.g., [7] and the references therein). Then $\boldsymbol{e} = \rho \boldsymbol{j}$, where the a priory unknown effective resistivity $\rho(x, t) \geq 0$ characterizes the energy losses accompanying movement of magnetic vortices in a superconductor. It is assumed in the Bean model that the current density $\boldsymbol{j}$ cannot exceed some critical value $j_{\rm c}$ and, until this value is reached, the vortices are pinned and the current is purely superconductive:

$$|\boldsymbol{j}(x, t)| \leq j_{\rm c}, \quad |\boldsymbol{j}(x, t)| < j_{\rm c} \longrightarrow \boldsymbol{e}(x, t) = 0. \tag{14}$$

The simplest geometric configuration is that of a long superconductive cylinder having a simply connected cross-section $\Omega \subset \mathbb{R}^2$ and placed into a non-stationary parallel uniform external magnetic field $\boldsymbol{h}_e(t)$. In this case the most convenient variational formulation can be derived for the magnetic field in the superconductor. The current density $\boldsymbol{j}$ induced by the external field variations is parallel to the cross-section plane and produces the magnetic field $\boldsymbol{h}_i(x, t)$, parallel to $\boldsymbol{h}_{\rm e}$ and equal to zero on Γ. The problem is two-dimensional. Denoting by h_i and $h_{\rm e}$ parallel to the cylinder axis components of $\boldsymbol{h}_i$ and $\boldsymbol{h}_{\rm e}$, correspondingly, and using the standard notations $\mathrm{curl}\, \boldsymbol{v} = \partial_{x_1} v_2 - \partial_{x_2} v_1$, $\mathbf{curl}\, v = (\partial_{x_2} v, -\partial_{x_1} v)$, we rewrite the model (13) as

$$\partial_t (h_i + h_{\rm e}) + \mathrm{curl}\, \boldsymbol{e} = 0, \quad \mathbf{curl}\, h_i = \boldsymbol{j}, \quad x \in \Omega,\ t > 0. \tag{15}$$

Since $|\nabla h_i| = |\mathbf{curl}\, h_i| = |\boldsymbol{j}| \leq j_{\rm c}$, $h_i(x, t)$ should belong to the set

$$K = \{\varphi(x, t) | |\nabla \varphi| \leq j_{\rm c}, \varphi|_\Gamma = 0\}. \tag{16}$$

Multiplying the first of Eqs. (15) by $\varphi - h_i$, $\varphi \in K$, integrating, and using the Green formula we obtain $(\partial_t \{h_i + h_{\rm e}\}, \varphi - h_i) = (\boldsymbol{e}, \boldsymbol{j}) - (\boldsymbol{e}, \mathbf{curl}\, \varphi)$. Taking the Bean current–voltage relations into account we get

$$(\boldsymbol{e}, \boldsymbol{j}) = (|\boldsymbol{e}|, |\boldsymbol{j}|) = (|\boldsymbol{e}|, j_{\rm c}) \geq (|\boldsymbol{e}|, |\mathbf{curl}\, \varphi|) \geq (\boldsymbol{e}, \mathbf{curl}\, \varphi)$$

and arrive at the variational inequality for the induced field h_i:

$$\text{Find } h_i \in K \text{ such that,} \quad (\partial_t \{h_i + h_{\rm e}\}, \varphi - h_i) \geq 0, \quad \text{for any } \varphi \in K,\ h_i(x, 0) = h_0(x) - h_e(0). \tag{17}$$

In the general three-dimensional case the h-formulation of the Bean model can also be derived [6] but then the magnetic field should be determined in the whole space. A similar variational formulation in terms of the current density we derive now is probably more convenient.

Provided that no current is fed into a superconductor by electric contacts, $\nabla \cdot \boldsymbol{j} = 0$ in Ω, $\boldsymbol{j}_n = 0$ on Γ. Let us define the set of admissible current densities in Ω,

$$\boldsymbol{j} \in K = \left\{ \boldsymbol{\varphi}(x, t) \left| \begin{array}{l} \nabla \cdot \boldsymbol{\varphi} = 0,\ |\boldsymbol{\varphi}| \leq j_{\rm c} \text{ in } \Omega \\ \boldsymbol{\varphi}_n = 0 \text{ on } \Gamma \end{array} \right. \right\},$$

express the electric field via the vector and scalar magnetic potentials [19],

$$\boldsymbol{e} + \partial_t \boldsymbol{A} + \nabla\psi = 0,$$

and exclude the scalar potential by multiplying this equation by $\boldsymbol{\varphi} - \boldsymbol{j}$ and integrating: $(\boldsymbol{e} + \partial_t \boldsymbol{A}, \boldsymbol{\varphi} - \boldsymbol{j}) = 0$ for any $\boldsymbol{\varphi} \in K$. Just as above, if $\boldsymbol{e}$ and $\boldsymbol{j}$ are parallel and satisfy the Bean model relations (14), the inequality $(\boldsymbol{e}, \boldsymbol{\varphi} - \boldsymbol{j}) \le 0$ holds for every admissible test function $\boldsymbol{\varphi}$. Hence, $(\partial_t \boldsymbol{A}, \boldsymbol{\varphi} - \boldsymbol{j}) \ge 0$.

Up to the gradient of a scalar function, determined by the gauge and eliminated by scalar product with the test functions, the vector potential is a convolution of the Green function of Laplace equation, $G = 1/4\pi|x|$, and the total current:

$$\boldsymbol{A} = G * \{\boldsymbol{j} + \boldsymbol{j}_{\mathrm{e}}\}.$$

We arrived at the evolutionary variational inequality with an “implicit” derivative in time:

$$\text{Find } \boldsymbol{j} \in K \text{ such that,} \quad (G * \partial_t\{\boldsymbol{j} + \boldsymbol{j}_{\mathrm{e}}\}, \boldsymbol{\varphi} - \boldsymbol{j}) \ge 0, \quad \text{for any } \boldsymbol{\varphi} \in K, \; \boldsymbol{j}|_{t=0} = \boldsymbol{j}_0(x), \tag{18}$$

where $\boldsymbol{j}_0 = \nabla \times \boldsymbol{h}_0|_{\Omega} \in K$ is a given initial current density distribution.

Experiments on hard superconductors are often performed on thin flat samples, and we present also a scalar version of this variational inequality for thin films in a perpendicular uniform external magnetic field. We now assume that it is the sheet current density, obtained by integration of bulk current density across the film thickness and also denoted $\boldsymbol{j}(x, t)$, $x \in \Omega \subset \mathbb{R}^2$, obeys the Bean's current–voltage relations. This current density should also satisfy the conditions

$$\operatorname{div} \boldsymbol{j} = 0 \text{ in } \Omega, \quad \boldsymbol{j}_n = 0 \text{ on } \Gamma, \tag{19}$$

where div is the two-dimensional divergence. For simplicity, we assume the domain Ω is simply connected. Due to conditions (19) there exists a stream function $h(x, t)$ such that $\mathbf{curl}\, h = \boldsymbol{j}$ in Ω and $h = 0$ on Γ. Although h is not the induced magnetic field as in the case of a long cylinder in a parallel field, since $|\mathbf{curl}\, h| = |\nabla h|$ this function belongs to the same set K of admissible functions (16). Let $\boldsymbol{j}'$ be another vector function satisfying (19) and the condition $|\boldsymbol{j}'| \le j_{\mathrm{c}}$ in Ω, and $\varphi \in K$ be the corresponding stream function. The external vector potential $\boldsymbol{A}_{\mathrm{e}}$, corresponding to the uniform perpendicular magnetic field $h_e(t)$, can be chosen parallel to the film; then curl $\boldsymbol{A}_{\mathrm{e}} = h_e(t)$. Substituting into the inequality (18) the curls of h and φ instead of the current and test function, correspondingly, using the Green theorem, and taking into account that $\mathbf{curl}\, u \cdot \mathbf{curl}\, v = \nabla u \cdot \nabla v$ we obtain a scalar variational inequality in terms of the stream function:

$$\text{Find } h \in K \text{ such that,} \quad a(\partial_t h, \varphi - h) + (\partial_t h_{\mathrm{e}}, \varphi - h) \ge 0, \quad \text{for any } \varphi \in K, \; h(x, 0) = h_0(x), \tag{20}$$

where $a(u, v) = \int_{\Omega}\int_{\Omega} \nabla u(x) \cdot \nabla v(x')/\{4\pi|x - x'|\}\, \mathrm{d}x\, \mathrm{d}x'$.

The existence and uniqueness of solutions to (18) and (20) were proved in [6] and [20]; see [21] for the numerical solution of (20). It has been shown in [6] that the effective resistivity ρ, excluded in transition to the variational formulation (17), is a Lagrange multiplier related to the current density constraint. Similar variational formulations may be derived for much more general current–voltage relations (see, e.g., [5,7,22]) and present a very convenient description of hysteretic magnetization typical of hard superconductors. In particular, the critical current density j_{c} depends usually on the magnetic field [24]. Then the set of admissible functions K depends on the unknown solution and the inequalities become quasivariational [6]. The power law $|\boldsymbol{e}| = e_0(|\boldsymbol{j}|/j_{\mathrm{c}})^p$ is often employed instead of the Bean's current voltage relation to account for the creep of magnetic flux [23]; as $p \to \infty$, such model converges to the Bean model [20,25]. Thermal fluctuations in a superconductor may cause avalanches of magnetic vortices resembling sand avalanches [26]; in the Bean model these avalanches correspond to discontinuous solutions of variational inequality (18) with the jumps induced by local fluctuations of the critical current density.

2.4. Elastoplastic solids

The variational inequality formulation for models in perfect elastoplasticity is well known [9]. We briefly present this formulation to underline its similarity to the variational formulations above.

Let an elastoplastic body occupy the domain $\Omega \subset \mathbb{R}^3$ and the conditions of equilibrium,

$$\int_\Omega \boldsymbol{g} + \int_\Gamma \boldsymbol{f} = 0, \quad \int_\Omega \boldsymbol{x} \times \boldsymbol{g} + \int_\Gamma \boldsymbol{x} \times \boldsymbol{f} = 0,$$

hold for the given body force $\boldsymbol{g}$ and surface traction $\boldsymbol{f}$. The stress tensor $\boldsymbol{\sigma}$ should satisfy the local equilibrium conditions (the usual summation convention is implied)

$$\sigma_{ij,j} + g_i = 0 \text{ in } \Omega, \quad \sigma_{ij} n_j = f_i \text{ on } \Gamma. \tag{21}$$

Under the assumption of small strain we have

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{22}$$

where $\boldsymbol{u}$ is the displacement vector field, $u_{i,j} = \partial u_i / \partial x_j$.

It is assumed that the strain tensor can be presented as a sum of elastic and plastic components, $\boldsymbol{\epsilon} = \boldsymbol{e} + \boldsymbol{p}$, where the elastic component obeys the linear Hooke's law, $e_{ij} = A_{ijkl}\, \sigma_{kl}$. The plastic part is governed by an incremental flow rule, $\boldsymbol{p} = \int_0^t \dot{\boldsymbol{p}} \mathrm{d}t$, in which $\dot{\boldsymbol{p}} = \partial_t \boldsymbol{p}$ is determined as follows. For a prescribed convex yield function $\mathcal{F}(\boldsymbol{\sigma})$, it is postulated that the stress tensor everywhere satisfies

$$\mathcal{F}(\boldsymbol{\sigma}) \le 0 \tag{23}$$

and that

$$\dot{p}_{ij} = \lambda \frac{\partial \mathcal{F}}{\partial \sigma_{ij}},$$

where $\lambda(x, t) \ge 0$ is the deformation rate such that $\mathcal{F}(\boldsymbol{\sigma}(x, t)) < 0 \longrightarrow \lambda(x, t) = 0$. Let the admissible set K be the set of stress tensors satisfying (21) and (23) and $\boldsymbol{\tau} \in K$. Using the strain–displacement relation (22), the Green formula, and the equilibrium conditions (21), one can show [27] that $(\dot{\epsilon}_{ij}, \tau_{ij} - \sigma_{ij}) = 0$, hence

$$(A_{ijkl}\, \dot{\sigma}_{kl}, \tau_{ij} - \sigma_{ij}) + \left(\lambda \frac{\partial \mathcal{F}}{\partial \sigma_{ij}}, \tau_{ij} - \sigma_{ij}\right) = 0.$$

The second term here is nonpositive. Indeed, if $\mathcal{F}(\boldsymbol{\sigma}(x, t)) < 0$ then $\lambda(x, t) = 0$. Otherwise $\mathcal{F}(\boldsymbol{\sigma}(x, t)) = 0$ and, because $\mathcal{F}$ is convex, $\mathcal{F}(\{1 - \theta\}\boldsymbol{\sigma}(x, t) + \theta \boldsymbol{\tau}(x, t)) \le 0$ for any $\theta \in [0, 1]$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \mathcal{F}(\{1 - \theta\}\boldsymbol{\sigma} + \theta \boldsymbol{\tau})\bigg|_{\theta=+0} = \frac{\partial \mathcal{F}(\boldsymbol{\sigma})}{\partial \sigma_{ij}} (\tau_{ij} - \sigma_{ij}) \le 0.$$

We arrived at the variational inequality:

$$\text{Find } \boldsymbol{\sigma} \in K \text{ such that,} \quad (A_{ijkl} \dot{\sigma}_{kl}, \tau_{ij} - \sigma_{ij}) \ge 0, \quad \text{for any } \boldsymbol{\tau} \in K,\ \boldsymbol{\sigma}(x, 0) = \boldsymbol{\sigma}_0(x).$$

3. Dual formulations for conjugate variables

Although a common feature of the considered mathematical models is the presence of conjugate variables, the variational inequalities above are written for only one of them: the free surface in the model for sandpile growth, current density in critical-state superconductivity problems, stress tensor in elastoplasticity problems. The dual variables, i.e, the surface flux, the electric field, and the strain, correspondingly, have been eliminated in transition to the variational inequalities. These inequalities can be solved efficiently, however, knowledge of the primary variables generally does not make determining the dual ones easy because the constitutive relations are multivalued.

In elastoplasticity with hardening, a dual variational formulation for strain has been derived and comprehensively studied by Han and Reddy [10]. Mathematically, this problem takes the form of the so-called mixed variational inequality,

Find $u : [0, T] \to V$ such that for almost all t,

$$a(u(t), v - \partial_t u(t)) - (f(t), v - \partial_t u(t)) + \phi(v) - \phi(\partial_t u(t)) \geq 0, \quad \text{for all } v \in V,\ u(0) = u_0,$$

where V is a Banach space, $a(\cdot, \cdot)$ is a symmetric bilinear form, $f(t) \in V'$ is a linear functional, and $\phi(\cdot)$ is a convex, positively homogeneous, nonnegative functional on V. This problem is not a standard variational inequality although it resembles the parabolic variational inequalities of the second type [8].

Strain formulation for the perfect plasticity problem considered above turns out to be much more complicated because, without hardening, the arising problem is not coercive in the usual Sobolev spaces (coercive in a non-reflexive Banach space) and the solution has to be sought in the space $BD(\Omega)$ of functions of bounded deformation ([28], see [29] for a review of recent results). Physically, the difference is manifested in the ability of perfectly plastic materials to form slip surfaces on which the tangential component of displacement is discontinuous.

Similar difficulties arise in dual variational formulations of other critical-state problems described in the first part of our work and this, indeed, seems to be physically meaningful. Thus, the continuity of only the normal component of surface sand flux follows from the mass conservation law in the pile growth model. Also, according to Maxwell equations, only the tangential component of the electric field has to be continuous. Mathematically, the corresponding problems are not coercive in reflexive Banach spaces.

Below, we derive formally variational formulations in terms of the dual variables for two critical-state problems where the primary formulation is a variational (not quasivariational) inequality. The questions of existence, uniqueness, and numerical approximation of these problems need further investigation; we are going to consider these questions in a separate publication [30].

3.1. Surface flux in the model of pile growth

Determining the flux of granular material pouring down the free surface of a growing pile is necessary, e.g., if the material is polydisperse and it is needed to predict the resulting distribution of different species inside the pile (see [13]). Let us assume the initial support h_0 in the pile growth model has no steep slopes, $|\nabla h_0| \leq k$ in Ω. In this case the model (1)–(7) can be written as

$$\begin{aligned} &\partial_t h + \nabla \cdot \boldsymbol{q} = f, \\ &h|_{t=0} = h_0, \quad \boldsymbol{q}_n|_\Gamma = 0, \end{aligned} \tag{24}$$

where the flux $\boldsymbol{q}$ has the direction of $-\nabla h$ and the following flux-slope relation holds:

$$|\nabla h| \leq k, \quad |\nabla h| < k \to \boldsymbol{q} = 0. \tag{25}$$

As it was shown above, the free surface $h(x, t)$ can be sought as a solution of an evolutionary variational inequality.

To derive a variational formulation of this model in terms of the surface flux, let us define

$$V = H_0(\mathrm{div};\Omega) \triangleq \{\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega) | \nabla \cdot \boldsymbol{\varphi} \in L^2(\Omega), \boldsymbol{\varphi}_n|_\Gamma = 0\},$$

assume that $\boldsymbol{q}$ and h satisfy the model relations (24)–(25), and choose an arbitrary test flux $\tilde{\boldsymbol{q}} \in V$. Using the constitutive relations (25) we obtain

$$\nabla h \cdot (\tilde{\boldsymbol{q}} - \boldsymbol{q}) \geq -|\nabla h||\tilde{\boldsymbol{q}}| - \nabla h \cdot \boldsymbol{q} = -|\nabla h||\tilde{\boldsymbol{q}}| + k|\boldsymbol{q}| \geq -k|\tilde{\boldsymbol{q}}| + k|\boldsymbol{q}|.$$

Hence,

$$(\nabla h, \tilde{\boldsymbol{q}} - \boldsymbol{q}) \geq \phi(\boldsymbol{q}) - \phi(\tilde{\boldsymbol{q}}),$$

where $\phi(\boldsymbol{q}) = k \int_\Omega |\boldsymbol{q}|$. Since $(\nabla h, \tilde{\boldsymbol{q}} - \boldsymbol{q}) = -(h, \nabla \cdot \{\tilde{\boldsymbol{q}} - \boldsymbol{q}\})$, we have

$$\phi(\tilde{\boldsymbol{q}}) - \phi(\boldsymbol{q}) - (h, \nabla \cdot \{\tilde{\boldsymbol{q}} - \boldsymbol{q}\}) \geq 0.$$

Let us define $\boldsymbol{u} = \int_0^t \boldsymbol{q}\, \mathrm{d}t$. Then $\partial_t \boldsymbol{u} = \boldsymbol{q}$ and, from (24), $\nabla \cdot \boldsymbol{u} = -h + h_0 + \int_0^t f\, \mathrm{d}t$. We finally arrive at the following variational problem:

Find $\boldsymbol{u} : [0, T] \to V$ such that for any $\tilde{\boldsymbol{q}} \in V$ and almost all t,

$$(\nabla \cdot \boldsymbol{u}, \nabla \cdot \{\tilde{\boldsymbol{q}} - \partial_t \boldsymbol{u}\}) - (\mathcal{F}, \nabla \cdot \{\tilde{\boldsymbol{q}} - \partial_t \boldsymbol{u}\}) + \phi(\tilde{\boldsymbol{q}}) - \phi(\partial_t \boldsymbol{u}) \geq 0, \text{ and } \boldsymbol{u}|_{t=0} = 0, \quad (26)$$

where $\mathcal{F} = h_0 + \int_0^t f\, \mathrm{d}t$. Since the problem is not coercive in V (coercive in a non reflexive Banach space), it may have and may have no solution, which means an appropriate regularization is needed. We do not investigate this issue further in a this work and only note that, after discretization in time, the problem becomes equivalent to a non-smooth optimization problem for each time layer. Indeed, let $\boldsymbol{q} = (\boldsymbol{u}^{n+1} - \boldsymbol{u}^n)/\Delta t$ and $\boldsymbol{u} = \boldsymbol{u}^n + (\Delta t/2)\boldsymbol{q}$ be approximate values at $t = \Delta t(n + 1/2)$. For each time layer we obtain

$$\phi(\tilde{\boldsymbol{q}}) - \phi(\boldsymbol{q}) + \left(\nabla \cdot \left\{\boldsymbol{u}^n + \frac{\Delta t}{2}\boldsymbol{q}\right\}, \nabla \cdot \{\tilde{\boldsymbol{q}} - \boldsymbol{q}\}\right) - (\mathcal{F}^{n+1/2}, \nabla \cdot \{\tilde{\boldsymbol{q}} - \boldsymbol{q}\}) \geq 0,$$

which is equivalent to

$$\boldsymbol{q}^{n+1/2} = \arg\min_{\boldsymbol{q} \in V} \left\{ \frac{\Delta t}{4}(\nabla \cdot \boldsymbol{q}, \nabla \cdot \boldsymbol{q}) + \phi(\boldsymbol{q}) + (\nabla \cdot \boldsymbol{u}^n - \mathcal{F}^{n+1/2}, \nabla \cdot \boldsymbol{q}) \right\} \quad (27)$$

Provided the latter problem has a solution, approximate value of $\boldsymbol{u}$ on the next time layer can be found as $\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \Delta t \boldsymbol{q}^{n+1/2}$.

3.2. Electric field in the Bean model for superconductors

The *h*- and *j*-variational formulations of the Bean model can be used for efficient computation of magnetic fields, current densities and, therefore, forces and their moments in various applications of type-II superconductors. However, even if the current density is known, the electric field is not determined in a unique way by the assumption that the directions of $\boldsymbol{e}$ and $\boldsymbol{j}$ coincide and the current–voltage relation

$$|\boldsymbol{j}| \leq j_\mathrm{c}, \ |\boldsymbol{j}| < j_\mathrm{c} \to \boldsymbol{e} = 0 \quad (28)$$

or another constitutive law described by a monotone multivalued graph is satisfied. That is why calculating the electric field in a superconductor is generally a non-trivial task.[3] In particular, this field is necessary to estimate the local ac loss $\boldsymbol{e}\cdot\boldsymbol{j}=j_c|\boldsymbol{e}|$ that causes heating and thermal instability of superconductors.

For some simple configurations, the electric field in superconductors has been considered in [31,32] and, recently, for a generalized Bean model in [33]. Here we propose a completely different approach based not on the determination of the magnetic field and subsequent integration of Faraday's law along the flux penetration streamlines but on a direct variational reformulation of the Bean model in terms of electric field.

Let $W=H(\mathrm{curl};\Omega)\triangleq\{\boldsymbol{\varphi}\in \boldsymbol{L}^2(\Omega)|\nabla\times\boldsymbol{\varphi}\in\boldsymbol{L}^2(\Omega)\}$ be the space of electric fields in the superconductive domain $\Omega\subset\mathbb{R}^3$. Assuming directions of $\boldsymbol{j}$ and $\boldsymbol{e}$ coincide and the Bean model relations are satisfied in this domain, for any test field $\tilde{\boldsymbol{e}}\in W$ we obtain

$$(\nabla\times\boldsymbol{h})\cdot(\tilde{\boldsymbol{e}}-\boldsymbol{e})=\boldsymbol{j}\cdot(\tilde{\boldsymbol{e}}-\boldsymbol{e})\le j_c|\tilde{\boldsymbol{e}}|-\boldsymbol{j}\cdot\boldsymbol{e}=j_c|\tilde{\boldsymbol{e}}|-j_c|\boldsymbol{e}|.$$

Integrating over Ω we get $(\nabla\times\boldsymbol{h},\tilde{\boldsymbol{e}}-\boldsymbol{e})\le\phi(\tilde{\boldsymbol{e}})-\phi(\boldsymbol{e})$, where $\phi(\boldsymbol{e})=j_c\int_\Omega|\boldsymbol{e}|$. On the other hand,

$$(\nabla\times\boldsymbol{h},\tilde{\boldsymbol{e}}-\boldsymbol{e})=(\boldsymbol{h},\nabla\times\{\tilde{\boldsymbol{e}}-\boldsymbol{e}\})+\int_\Gamma(\boldsymbol{h}\times\{\tilde{\boldsymbol{e}}-\boldsymbol{e}\})\cdot\boldsymbol{n}.$$

Let $\boldsymbol{u}=-\boldsymbol{A}_0+\int_0^t\boldsymbol{e}\,\mathrm{d}t$, where the vector potential $\boldsymbol{A}_0=G*(\boldsymbol{j}+\boldsymbol{j}_\mathrm{e})|_{t=0}$ satisfies $\boldsymbol{h}_0=\nabla\times\boldsymbol{A}_0$. Then

$$\partial_t\boldsymbol{u}=\boldsymbol{e},\quad \nabla\times\boldsymbol{u}=-\boldsymbol{h}_0+\int_0^t\nabla\times\boldsymbol{e}\,\mathrm{d}t=-\boldsymbol{h}_0-\int_0^t\partial_t\boldsymbol{h}\,\mathrm{d}t=-\boldsymbol{h}.$$

The following inequality is thus obtained: for any $\tilde{\boldsymbol{e}}\in W$

$$(\nabla\times\boldsymbol{u},\nabla\times\{\tilde{\boldsymbol{e}}-\partial_t\boldsymbol{u}\})-\int_\Gamma(\boldsymbol{h}\times\{\tilde{\boldsymbol{e}}-\partial_t\boldsymbol{u}\})\cdot\boldsymbol{n}+\phi(\tilde{\boldsymbol{e}})-\phi(\partial_t\boldsymbol{u})\ge 0. \tag{29}$$

In the general case this is not yet the needed e-formulation because it contains the tangential component of magnetic field on Γ. However, for an infinite cylinder in a parallel uniform external magnetic field $\boldsymbol{h}|_\Gamma=\boldsymbol{h}_e(t)$, where $\boldsymbol{h}_e$ is the field generated by the external current, so

$$(\nabla\times\boldsymbol{u},\nabla\times\{\tilde{\boldsymbol{e}}-\partial_t\boldsymbol{u}\})+\boldsymbol{h}_e(t)\cdot\int_\Gamma\boldsymbol{n}\times(\tilde{\boldsymbol{e}}-\partial_t\boldsymbol{u})+\phi(\tilde{\boldsymbol{e}})-\phi(\partial_t\boldsymbol{u})\ge 0 \tag{30}$$

and it is not difficult to see that, after discretization in time, (30) becomes equivalent to a non-smooth optimization problem similar to (27).

Let us now return to the general case and consider an auxiliary boundary value problem in the exterior domain ω,[4]

$$\begin{aligned}&\partial_t\boldsymbol{h}+\nabla\times\boldsymbol{e}=0,\quad \nabla\times\boldsymbol{h}=\boldsymbol{j}_\mathrm{e},\\&\boldsymbol{h}|_{t=0}=\boldsymbol{h}_0,\quad \boldsymbol{e}_\tau|_\Gamma=\boldsymbol{\mathcal{E}},\end{aligned} \tag{31}$$

where $\boldsymbol{e}_\tau=\boldsymbol{n}\times(\boldsymbol{e}\times\boldsymbol{n})|_\Gamma$ and $\boldsymbol{\mathcal{E}}$ is a tangential field given on Γ. We choose the vector potential of external current as $\boldsymbol{A}_\mathrm{e}=G*\boldsymbol{j}_\mathrm{e}$ and define the field $\boldsymbol{h}_\mathrm{e}=\nabla\times(\boldsymbol{A}_e-\boldsymbol{A}_e|_{t=0})+\boldsymbol{h}_0$. In ω this field satisfies

$$\nabla\times\boldsymbol{h}_\mathrm{e}=\boldsymbol{j}_\mathrm{e},\quad \nabla\cdot\boldsymbol{h}_\mathrm{e}=0,\quad \boldsymbol{h}_\mathrm{e}|_{t=0}=\boldsymbol{h}_0.$$

[3] The case of an infinite cylinder in a perpendicular external magnetic field is an exception.

[4] Since the displacement current is omitted, in our model the electric field in an insulator (the outer space) is not unique. The magnetic field, however, is.

Let us set $\boldsymbol{h} = \boldsymbol{h}_\mathrm{e} + \boldsymbol{H}$, $\boldsymbol{e} = -\partial_t \boldsymbol{A}_\mathrm{e} + \boldsymbol{E}$ and rewrite problem (31) as

$$\begin{aligned} &\partial_t \boldsymbol{H} + \nabla \times \boldsymbol{E} = 0, \quad \nabla \times \boldsymbol{H} = 0, \\ &\boldsymbol{H}|_{t=0} = 0, \quad \boldsymbol{E}_\tau|_\Gamma = \boldsymbol{\mathcal{E}}_1, \end{aligned} \tag{32}$$

where $\boldsymbol{\mathcal{E}}_1 = \boldsymbol{\mathcal{E}} + \partial_t \boldsymbol{A}_{e,\tau}$. Defining $\boldsymbol{U} = \int_0^t \boldsymbol{E}\,\mathrm{d}t$ and integrating in time the equations from (32) containing $\boldsymbol{E}$ yield

$$\boldsymbol{H} + \nabla \times \boldsymbol{U} = 0, \quad \nabla \times \boldsymbol{H} = 0, \quad \boldsymbol{U}_\tau|_\Gamma = \boldsymbol{\mathcal{U}}, \tag{33}$$

where $\boldsymbol{\mathcal{U}} = \int_0^t \boldsymbol{\mathcal{E}}\,\mathrm{d}t + \boldsymbol{A}_{e,\tau}(x,t) - \boldsymbol{A}_{e,\tau}(x,0)$.

Let us note that we also have $\nabla \cdot \boldsymbol{H} = 0$ a.e. in ω, so it makes sense to seek a solution $\boldsymbol{H}$ in the Sobolev space $\boldsymbol{H}^1(\omega)$. Denote $X = \{\boldsymbol{\psi} \in \boldsymbol{H}^1(\omega) | \nabla \times \boldsymbol{\psi} = 0\}$. The problem (33) admits a weak formulation,

$$\text{Find } \boldsymbol{H} \in X \text{ such that } (\boldsymbol{H}, \boldsymbol{\psi}) = \int_\Gamma (\boldsymbol{\mathcal{U}} \times \boldsymbol{\psi}) \cdot \boldsymbol{n}, \quad \forall \boldsymbol{\psi} \in X, \tag{34}$$

where the normal $\boldsymbol{n}$ is directed inside ω and the integral on the right is understood as the duality pairing on $H^{-1/2} \times H^{1/2}$. Existence of a unique solution to this problem follows from the Lax–Milgram theorem. This defines a linear operator $K : \boldsymbol{\mathcal{U}} \to \boldsymbol{H}_\tau|_\Gamma$ acting from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$. It is not difficult to see that this operator is symmetric in the following sense: for any two functions $\boldsymbol{v}, \boldsymbol{w} \in H^{-1/2}(\Gamma)$ there holds

$$\int_\Gamma (K(\boldsymbol{v}) \times \boldsymbol{w}) \cdot \boldsymbol{n} = \int_\Gamma (K(\boldsymbol{w}) \times \boldsymbol{v}) \cdot \boldsymbol{n}. \tag{35}$$

Indeed, let $\boldsymbol{H}^v$ and $\boldsymbol{H}^w$ be the solutions of (34) with $\boldsymbol{\mathcal{U}} = \boldsymbol{v}$ and $\boldsymbol{\mathcal{U}} = \boldsymbol{w}$, correspondingly. Then

$$\begin{aligned} (\boldsymbol{H}^v, \boldsymbol{H}^w) &= \int_\Gamma (\boldsymbol{v} \times \boldsymbol{H}^w) \cdot \boldsymbol{n} = \int_\Gamma (\boldsymbol{v} \times K(\boldsymbol{w})) \cdot \boldsymbol{n}, \\ (\boldsymbol{H}^w, \boldsymbol{H}^v) &= \int_\Gamma (\boldsymbol{w} \times \boldsymbol{H}^v) \cdot \boldsymbol{n} = \int_\Gamma (\boldsymbol{w} \times K(\boldsymbol{v})) \cdot \boldsymbol{n} \end{aligned}$$

and (35) is proved. We also see that $\int_\Gamma (\boldsymbol{v} \times K(\boldsymbol{v})) \cdot \boldsymbol{n} = (\boldsymbol{H}^v, \boldsymbol{H}^v) \geq 0$ for any $\boldsymbol{v} \in H^{-1/2}(\Gamma)$.

Let us now choose $\boldsymbol{\mathcal{E}}$ to be the continuous tangential component of the electric field on Γ, $\boldsymbol{\mathcal{E}} = \boldsymbol{e}_\tau$. Then

$$\boldsymbol{\mathcal{U}} = \int_0^t \boldsymbol{e}_\tau\,\mathrm{d}t + \boldsymbol{A}_{e,\tau}(x,t) - \boldsymbol{A}_{e,\tau}(x,0) = \boldsymbol{u}_\tau + \boldsymbol{A}_{0,\tau} + \boldsymbol{A}_{e,\tau} - \boldsymbol{A}_{e,\tau}|_{t=0} = \boldsymbol{u}_\tau + G * (\boldsymbol{j}|_{t=0} + \boldsymbol{j}_\mathrm{e})_\tau$$

and the tangential component of the magnetic field can be presented as

$$\boldsymbol{h}_\tau = \boldsymbol{h}_{e,\tau} + \boldsymbol{H}_\tau = K(\boldsymbol{u}_\tau) + \boldsymbol{\mathcal{F}},$$

where $\boldsymbol{\mathcal{F}} = \boldsymbol{h}_{e,\tau} + K(\{G * (\boldsymbol{j}|_{t=0} + \boldsymbol{j}_\mathrm{e})\}_\tau)$.

We can now rewrite (29) as a mixed variational inequality:

Find $\boldsymbol{u} : [0,T] \to W$ such that for any $\tilde{\boldsymbol{e}} \in W$ and almost all t,

$$\begin{aligned} &(\nabla \times \boldsymbol{u}, \nabla \times \{\tilde{\boldsymbol{e}} - \partial_t \boldsymbol{u}\}) - \int_\Gamma (K(\boldsymbol{u}_\tau) \times \{\tilde{\boldsymbol{e}} - \partial_t \boldsymbol{u}\}) \cdot \boldsymbol{n} \\ &- \int_\Gamma (\boldsymbol{\mathcal{F}} \times \{\tilde{\boldsymbol{e}} - \partial_t \boldsymbol{u}\}) \cdot \boldsymbol{n} + \phi(\tilde{\boldsymbol{e}}) - \phi(\partial_t \boldsymbol{u}) \geq 0 \text{ and } \quad \boldsymbol{u}|_{t=0} = -\boldsymbol{A}_0. \end{aligned} \tag{36}$$

Discretizing in time as above and making use of the equality (35) we obtain, for each time layer, a non-smooth optimization problem:

$$e^{n+1/2} = \arg\min_{e \in W} \Phi(e)$$

where

$$\Phi(e) = \frac{\Delta t}{4}(\nabla \times e, \nabla \times e) + \frac{\Delta t}{4}\int_{\Gamma}(e_\tau \times K(e_\tau)) \cdot n + \phi(e)$$
$$-\Delta t \int_{\Gamma}(\{K(e_\tau^n) + \mathcal{F}^{n+1/2}\} \times e_\tau) \cdot n + (\nabla \times u^n, \nabla \times e)$$

and $u^{n+1} = u^n + \Delta t e^{n+1/2}$.

4. Conclusion

To derive quasistationary critical-state models of the spatially extended dissipative systems considered above, we needed to specify only the possible direction of system's evolution, and which changes of external conditions make the state of the system unstable. The rate with which such a system driven by the external forces rearranges itself is determined implicitly by some conservation law coupled with a condition of equilibrium. This rate appears in the model as a Lagrange multiplier related to the equilibrium constraint. Although the conservation laws and conditions of equilibrium may vary, the multiplicity of metastable states, typical of many dissipative systems, is usually a consequence of a unilateral constraint. This makes variational inequalities a suitable tool for modeling these systems.

Typically, some of the physically relevant conjugate variables are eliminated in transition to the variational formulation of critical-state problems. It is, however, possible to derive dual formulations (mixed variational inequalities) in terms of these variables. Although arising mathematical problems need further investigation, we believe these dual formulations will also serve a basis for efficient numerical simulations.

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