Existence and approximation of a mixed formulation for thin film magnetization problems in superconductivity

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We recall a recently introduced mixed formulation of thin film magnetization problems for type-II superconductors written in terms of two variables, the electric field and the magnetization function, see [Electric field formulation for thin film magnetization problems, Supercond. Sci. Technol. 25 (2012) 104002]. A finite element approximation, $(Q^{h,r})$, based on this mixed formulation, involving the lowest-order Raviart–Thomas element for approximating the electric field, was also introduced in [Electric field formulation for thin film magnetization problems, Supercond. Sci. Technol. 25 (2012) 104002]. Here $h, r$ are the spatial and temporal discretization parameters, and $r = \frac{p-1}{p^*} > 1$ with $p-1$ the value of power in the current–voltage relation characterizing the superconducting material. In this paper, we establish well-posedness of $(Q^{h,r})$, and prove convergence of the unique solution of $(Q^{h,r})$ to a solution of the power law model $(Q_r)$, for a fixed $r > 1$, as $h, r \to 0$. In addition, we prove convergence of a solution of $(Q_r)$ to a solution of the critical state model $(Q)$, as $r \to 1$. Hence, we prove existence of solutions to $(Q_r)$, for a fixed $r > 1$, and $(Q)$. Finally, numerical experiments are presented.

Keywords: Superconductivity; thin film; Bean model; variational inequalities; critical state problems; power law; mixed methods; finite elements; existence; convergence analysis.

AMS Subject Classification: 35D30, 35K85, 35R37, 49J40, 49M29, 65M12, 65M60, 82C27, 82D55
1. Introduction

Numerical algorithms, typically employed for solving thin film magnetization problems in type-II superconductivity, are based on formulations written for only one variable, the magnetization function (see, e.g., Refs. 7, 20, 18 and 21). The film current density is calculated as the 2D curl of this function; then the magnetic field can be found using the Biot–Savart law. However, the electric field, needed to calculate the distribution of the energy loss inside the superconductor, remains undetermined for the critical state models and can be difficult to compute accurately for models with, e.g., the power law current–voltage relation characterizing the superconducting material.

The mixed formulation for thin film magnetization problems, recently introduced in Ref. 5, is written for two variables: the magnetization function and the electric field. This formulation enables one to compute accurately all variables of interest. Although the mixed formulation and its discretization were introduced in Ref. 5, no mathematical or numerical analysis of the method was presented there. The aim of the present paper is to prove convergence of the numerical method, and hence prove existence of a solution to this mixed formulation. In this introduction we first give a derivation, simpler to that presented in Ref. 5, of this mixed formulation.

We shall assume throughout that all variables have been nondimensionalized, and the magnetic permeability of the superconductor is equal to that of the vacuum and is scaled to unity. In the infinitely thin approximation, a superconducting film occupies the set \( \{ \Omega \times 0 \} \subset \mathbb{R}^3 \), where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain. We suppose the given external time-dependent uniform magnetic field \( b_e(t) \) is orthogonal to the film plane; that is, \( b_e(t) = (0,0,b_e(t))^{\top} \). Let the sheet current density, \( j(x,t) \), and the component of the electric field tangential to the film, \( e(x,t) \), be two time-dependent vector fields in \( \Omega \) satisfying the critical state model relations:

\[
|j| \leq j_c, \quad |j| < j_c \Rightarrow e = 0, \quad e \neq 0 \Rightarrow e \parallel j,
\]  

(1.1)

where \( j_c \) is the sheet critical current density, which may depend only on \( x = (x_1,x_2)^{\top} \in \Omega \) (the Bean model for an inhomogeneous film, see Ref. 6) or also on the magnetic field (the Kim model, see Ref. 14). By “\( \parallel \)" in (1.1), we mean that the vectors point in the same direction. Hence, we have that

\[
\langle j, \eta - \xi \rangle_{\Omega} \leq \langle j_c, |\eta| \rangle_{\Omega} - \langle j_c, |\xi| \rangle_{\Omega} \quad \forall \eta \in (L^2(\Omega))^2,
\]  

(1.2)

where \( \langle \cdot, \cdot \rangle_{\Omega} \) is the \( L^2(\Omega) \) inner product. Here, for ease of exposition, in this formal introduction we assume that all terms in (1.2) are well-defined. We will return to these regularity issues below. It is convenient to assume that \( \Omega \) is simply connected. If it contains holes, these can simply be filled in with the sheet critical current density in the holes set to be very small.
Mixed formulation for thin film magnetization problems in superconductivity

The normal to the film component of the total magnetic field can be expressed by the Biot–Savart law as

\[ b_3(x, t) = b_3(t) + \frac{1}{4\pi} \text{Curl} \int_\Omega \frac{j(y, t)}{|x - y|} \, dy \quad \forall x \in \Omega, \ t > 0, \]  
(1.3)

where \( \text{Curl} \overline{j} := \partial_{x_1}f_2 - \partial_{x_2}f_1 \). Using Faraday’s law, we obtain that

\[ \partial_t b_3 = -\text{Curl} \overline{j} \quad \text{in } \Omega_T := \Omega \times (0, T). \]  
(1.4)

As \( \text{Div} j(\cdot, t) = 0 \) in \( \Omega \), which is simply connected, we can introduce a stream (magnetization) function \( g(\cdot, t) \), which vanishes on \( \partial \Omega \), such that

\[ \overline{j} = \text{Curl} g := (\partial_{x_2}g, -\partial_{x_1}g)^\top \quad \text{in } \Omega_T. \]  
(1.5)

Substituting (1.5) and (1.4) into the time derivative of (1.3), we obtain that

\[ \frac{1}{4\pi} \text{Curl} \int_\Omega \frac{1}{|x - y|} \text{Curl} \partial_t g(y, t) \, dy + \text{Curl} \overline{g}(x, t) = -\partial_t b_3(t). \]  
(1.6)

In addition, we can rewrite the inequality (1.2) as

\[ (\text{Curl} \overline{g}, \overline{\eta} - \overline{\zeta})_\Omega \leq (j_c, |\overline{\eta}|)_\Omega - (j_c, |\overline{\zeta}|)_\Omega \quad \forall \overline{\eta} \in (L^2(\Omega))^2. \]  
(1.7)

Next we introduce the bilinear form

\[ a(\phi, \psi) := \frac{1}{4\pi} \int_\Omega \int_\Omega \frac{\text{Curl} \overline{\phi}(x) \cdot \text{Curl} \overline{\psi}(y)}{|x - y|} \, dx \, dy \equiv \frac{1}{4\pi} \int_\Omega \int_\Omega \frac{\text{Grad} \overline{\phi}(x) \cdot \text{Grad} \overline{\psi}(y)}{|x - y|} \, dx \, dy. \]  
(1.8)

We note from Lemma 2.1 in Ref. 1 that \( a(\cdot, \cdot) \) is symmetric, continuous and coercive on \( H_{00}^\sharp(\Omega) \times H_{00}^\sharp(\Omega) \), where

\[ H_{00}^\sharp(\Omega) := \left\{ \psi \in H^\sharp(\Omega) : \overline{\psi} := \begin{cases} \psi & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega \end{cases} \in H^\sharp(\mathbb{R}^2) \right\}. \]  
(1.9)

Hence, there exist constants \( C_1, C_2 \in \mathbb{R}_{>0} \) such that

\[ C_1 \| \psi \|_{H_{00}^\sharp(\Omega)} \leq a(\psi, \psi) \leq C_2 \| \psi \|_{H_{00}^\sharp(\Omega)} \quad \forall \psi \in H_{00}^\sharp(\Omega). \]  
(1.10)

We introduce also

\[ K := \{ \psi \in W_0^{1, \infty}(\Omega) : |\text{Grad} \overline{\psi}(x)| \leq j_c(x) \text{ for a.e. } x \in \Omega \}. \]  
(1.11)

On noting (1.8), a weak form of (1.6) and (1.7) can now be written as follows.

Find \( g(\cdot, t) \in H_{00}^\sharp(\Omega) \), with \( \partial_t g(\cdot, t) \in H_{00}^\sharp(\Omega) \), and \( \overline{\eta}(\cdot, t) \in \mathcal{E}_M := \{ \overline{\eta} \in (M(\Omega))^2 : \text{Curl} \overline{\eta} \in (H_{00}^\sharp(\Omega))^2 \} \) such that for a.a. \( t \in (0, T) \)

\[ \begin{align*}
    a(\partial_t g, \psi) + (\text{Curl} \overline{g}, \psi)_\Omega + (d_t b_3, \psi)_\Omega & = 0 \quad \forall \psi \in H_{00}^\sharp(\Omega), \\
    \langle \overline{\eta}, \overline{j}_c \rangle_{\mathcal{E}_M^0} - \langle \overline{\eta}, \overline{j}_c \rangle_{\mathcal{E}_M^0} - (g, \text{Curl} (\overline{\eta} - \overline{\zeta}))_\Omega & \geq 0 \quad \forall \overline{\eta} \in \mathcal{E}_M.
\end{align*} \]  
(1.12a-1.12b)
where \( g(\cdot, 0) = g^0(\cdot) \) with \( g^0 \in K \). Here \( \mathcal{M}(\overline{\Omega}) \equiv (C(\overline{\Omega}))' \) is the Banach space of bounded Radon measures. In addition, for any Banach space \( \mathcal{B} \), \( \langle \cdot, \cdot \rangle_\mathcal{B} \) denotes the duality pairing on \( (\mathcal{B})' \times \mathcal{B} \). Hence, for the first two terms in (1.12b) to be well-defined, we require that \( j_c \in C(\overline{\Omega}) \). As \( g^0 \in K \), this implies that the initial sheet current density \( j_c(\cdot, 0) = \text{Curl} g^0(\cdot) \) is such that \( |j_c(\cdot, 0)| \leq j_c(\cdot) \).

In order to approximate the above using the Raviart–Thomas element for \( e \), we introduce the change of variable

\[
e = R\hat{\eta}, \quad \text{where } R \text{ is the rotation matrix } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

We note that \( |\eta| = |\hat{\eta}| \) and that \( \text{Curl } R \equiv -\text{Div and Curl } = R \text{Grad} \). Of course, one could approximate \( \hat{e} \) directly using the rotated Raviart–Thomas element, i.e. the 2D Nédélec element. However, under the change of variable, the resulting model is related to critical models arising in sandpiles, see Ref. 2. In addition, we believe that \( \text{Grad} \), as opposed to \( \text{Curl} \), makes the analysis in the paper more transparent.

On introducing the Banach spaces

\[
\mathcal{Z}_\mathcal{M} := \{ \hat{\eta} \in (\mathcal{M}(\overline{\Omega}))^2 : \text{Div } \hat{\eta} \in \left( H^1_{\text{div}}(\Omega) \right)' \},
\]

\[
\mathcal{Z}_r := \{ \eta \in (\Omega)^2 : \text{Div } \eta \in \left( H^1_{\text{div}}(\Omega) \right)' \}, \quad r \in [1, \infty];
\]

(1.12a) and (1.12b) can be reformulated as follows.

(Q) Find \( g \in H^1(0, T; H^1_{\text{div}}(\Omega)) \) with \( g(\cdot, 0) = g^0(\cdot) \) and \( \eta \in L^2(0, T; \mathcal{Z}_\mathcal{M}) \) such that

\[
\int_0^T (a(\partial_t g, \psi) - \langle \text{Div } \hat{\eta}, \psi \rangle_{H^1_{\text{div}}(\Omega)} + \langle d \hat{e}_c, \psi \rangle_{\Omega}) dt = 0 \\
\quad \forall \psi \in L^2(0, T; H^1_{\text{div}}(\Omega)),
\]

(1.15a)

\[
\int_0^T (\langle |\eta|, j_c \rangle_{C(\overline{\Omega})} - \langle |\hat{\eta}|, j_c \rangle_{C(\overline{\Omega})} + \langle \text{Div } \eta - \hat{\eta}, g \rangle_{H^1_{\text{div}}(\Omega)}) dt \geq 0
\]

\[
\forall \eta \in L^2(0, T; \mathcal{Z}_\mathcal{M}).
\]

(1.15b)

Choosing \( \eta = 2\hat{\eta} \) and \( 0 \) in (1.15b) yields that

\[
\int_0^T (\langle \eta, j_c \rangle_{C(\overline{\Omega})} + \langle \text{Div } \eta, g \rangle_{H^1_{\text{div}}(\Omega)}) dt = 0,
\]

(1.16a)

and hence that

\[
\int_0^T (\langle \eta, j_c \rangle_{C(\overline{\Omega})} + \langle \text{Div } \eta, g \rangle_{H^1_{\text{div}}(\Omega)}) dt \geq 0 \quad \forall \eta \in L^2(0, T; \mathcal{Z}_\mathcal{M}).
\]

(1.16b)

One can deduce from (1.16b) that \( g(\cdot, t) \in K \) for a.a. \( t \in (0, T) \), see Theorem 3.2 below. Similarly, on choosing \( \psi = \phi - g \), where \( \phi \in L^2(0, T; K) \), and noting (1.16a), (1.16b) yields the primal variational inequality associated with the mixed formulation (Q).
(P) Find $g \in L^\infty(0, T; K) \cap H^1(0, T; H^{1\frac{2}{r}}_0(\Omega))$ with $g(\cdot, 0) = g^0(\cdot)$ such that

$$
\int_0^T (a(\partial_t g, \phi - g) + (d_t b_c, \phi - g)\Omega)dt \geq 0 \quad \forall \phi \in L^2(0, T; K). \quad (1.17)
$$

For any fixed $r \in (1, 2]$, approximating $|v|$ by $\frac{1}{r} |v|^r$, we obtain the following regularization of (Q).

(Q_r) Find $g_r \in H^1(0, T; H^{\frac{2}{r}}_0(\Omega))$ with $g_r(\cdot, 0) = g^0(\cdot)$ and $\psi_r \in L^2(0, T; \overline{Z_r})$ such that

$$
\int_0^T (a(\partial_t g_r, \psi) - \langle \text{Div} \psi_r, \psi \rangle_{H^{\frac{2}{r}}_0(\Omega)} + (d_t b_c, \psi)\Omega)dt = 0 \quad \forall \psi \in L^2(0, T; H^{\frac{2}{r}}_0(\Omega)), \quad (1.18a)
$$

$$
\int_0^T ((j_c|\psi_{r}^{\frac{r-2}{r}}\psi_r)\Omega + \langle \text{Div} \psi_r, \psi \rangle_{H^{\frac{2}{r}}_0(\Omega)})dt = 0 \quad \forall \eta \in L^2(0, T; \overline{Z_r}). \quad (1.18b)
$$

Associated with (Q_r) is the corresponding generalized $p$-Laplacian regularization of (P) for $p \in [2, \infty)$, where, here and throughout this paper, $(1/r) + (1/p) = 1$.

(P_p) Find $g_r \in L^p(0, T; W^{1,p}_0(\Omega)) \cap H^1(0, T; H^{\frac{2}{p}}_0(\Omega))$ with $g_r(\cdot, 0) = g^0(\cdot)$ such that

$$
\int_0^T \left( a(\partial_t g_r, \psi) + \left( \frac{|\text{Grad} g_r|^{p-2}}{j_c} \text{Grad} g_r, \text{Grad} \psi \right)_{\Omega} + (d_t b_c, \psi)\Omega \right) dt = 0 \quad \forall \psi \in L^2(0, T; W^{1,p}_0(\Omega)). \quad (1.19)
$$

For $g_r(\cdot, t) \in W^{1,p}_0(\Omega)$, it follows formally from (1.18b) and (1.13) that

$$
\left| j_c \psi_r \right|^{\frac{r-2}{r}} \psi_r = \text{Grad} g_r \Rightarrow \psi_r = \left( \frac{|\text{Grad} g_r|}{j_c} \right)^{\frac{p-2}{p}} \frac{\text{Grad} g_r}{j_c} \quad (1.20a)
$$

$$
\Rightarrow R_{\psi_r} = \left( \frac{|\text{Curl} g_r|}{j_c} \right)^{\frac{p-2}{p}} \frac{\text{Curl} g_r}{j_c}. \quad (1.20b)
$$

Substituting (1.20a) into (1.18a) yields (1.19). As $\psi_r$ and $g_r$ approximate $\psi$ and $g$, respectively, it follows from (1.13) that $R_{\psi_r}$ and $\text{Curl} g_r$ are approximations to $\psi$ and $\text{Curl} g$, respectively. Hence, (1.20b) is the power law approximation of the critical state relations (1.1) used by Brandt.

For later use, we recall the equivalent interpolation definition of $H^{\frac{2}{r}}_0(\Omega)$, see p. 66 in Ref. 16:

$$
H^{\frac{2}{r}}_0(\Omega) \hookrightarrow H^{\frac{2}{r}}_0(\Omega) := [H^1_0(\Omega), L^2(\Omega)]_{\frac{2}{r}} \hookrightarrow L^2(\Omega). \quad (1.21)
$$
We introduce the linear operator $\mathcal{F} : (H^1_0(\Omega))^\prime \to H^1_0(\Omega)$ such that for all $f \in (H^1_0(\Omega))^\prime$, $\mathcal{F}f \in H^1_0(\Omega)$ satisfies

$$a(\mathcal{F}f, \psi) = \langle f, \psi \rangle_{H^1_0(\Omega)} \quad \forall \psi \in H^1_0(\Omega).$$

(1.22)

As $a(\cdot, \cdot)$ is a symmetric, continuous and coercive bilinear form on $H^1_0(\Omega) \times H^1_0(\Omega)$, the well-posedness of $\mathcal{F}$ follows from the Lax–Milgram theorem. We have also from (1.10) that

$$C_1 \|f\|_{H^1_0(\Omega)} \leq \|f\|_{(H^1_0(\Omega))^\prime} \leq C_2 \|f\|_{H^1_0(\Omega)} \quad \forall f \in (H^1_0(\Omega))^\prime.$$  

(1.23)

The outline of this paper as follows. In the next section, we introduce a finite element approximation, $(Q_h^{k,r})$, of the power law model $(Q_r)$ using the lowest-order Raviart–Thomas element for the approximation of $\varphi$, and continuous piecewise linears for the approximation of $g_r$. In addition, we prove well-posedness of this approximation and establish stability bounds independent of the mesh and time step parameters, $h$ and $\tau$, and the regularization parameter $r$. In Sec. 3, we prove convergence of the unique solution of $(Q_h^{k,r})$, for fixed $r > 1$, to a solution, $\{g_r, \varphi_r\}$, of $(Q_r)$, as $h, \tau \to 0$. In addition, we show that this solution of $(Q_r)$ is unique, and that $g_r$ is the unique solution of $(P_p)$. Furthermore, we prove convergence of the unique solution of $(Q_r)$ to a solution, $\{g, \varphi\}$, of the critical state model $(Q)$, as $r \to 1$. In addition, we show that $g$ is unique, and is also the unique solution of $(P)$. Finally in Sec. 4, we present some numerical experiments based on the discretization $(Q_h^{k,r})$.

2. Finite Element Approximation

First, we gather together our basic assumptions on the data and the triangulation.

(A1) $\Omega \subset \mathbb{R}^2$ is polygonal, $b_\epsilon \in H^1(0,T)$, $g^0 \in K$ and $j_\epsilon \in L^\infty(\Omega)$ with $j_\epsilon(\cdot) \geq j_{\epsilon,\min} > 0$ for a.e. $\zeta \in \Omega$.

(A2) Let $\{T^h\}_{h>0}$ be a regular family of triangulations of $\Omega$ into disjoint open triangles $\kappa$ with $h_\kappa := \text{diam}(\kappa)$ and $h := \max_{\kappa \in T^h} h_\kappa$, so that $\overline{\Omega} = \bigcup_{\kappa \in T^h} \overline{\kappa}$. Moreover, $j_\epsilon|_{\kappa}$ can be extended to $j_\epsilon \in C(\overline{\kappa})$ for all $\kappa \in T^h$; that is, $j_\epsilon$ is piecewise continuous and its discontinuities only occur along the internal edges of $T^h$.

We assume that $\Omega$ is polygonal for ease of exposition, in order to avoid perturbation of domain errors in the finite element approximation. We shall also make the following assumptions at later stages in the paper:

(A3) $g^0 \equiv 0$, which implies that $j(\cdot, 0) \equiv 0$.

(A4) $\Omega$ is strictly star-shaped and $j_\epsilon \in C(\overline{\Omega})$. 
Let $\nu_{\partial \kappa}$ be the outward unit normal to $\partial \kappa$, the boundary of $\kappa$. We then introduce the following finite element spaces:

\begin{align}
S^h &:= \{ v^h \in C(\Omega) : v^h|_\kappa = a_\kappa + b_\kappa \cdot x, a_\kappa \in \mathbb{R}^1, b_\kappa \in \mathbb{R}^2 \forall \kappa \in \mathcal{T}^h \}, \\
S_0^h &:= S^h \cap H^1_0(\Omega) \subset H^1_0(\Omega), \\
\mathcal{V}^h &:= \{ \eta^h \in (L^\infty(\Omega))^2 : \eta^h|_\kappa = a_\kappa + b_\kappa \cdot x, a_\kappa \in \mathbb{R}^2, b_\kappa \in \mathbb{R}^1 \forall \kappa \in \mathcal{T}^h \}
\end{align}

(2.1a)

(2.1b)

(2.1c)

Here $\mathcal{V}^h$ is the lowest-order Raviart–Thomas finite element space.

Let $\pi^h : C(\Omega) \to S^h$ denote the interpolation operator such that $\pi^h \psi(x_i) = \psi(x_i)$, $i = 1 \to \mathcal{T}$, where $\{x_i\}_{i=1}^\mathcal{T}$ are the vertices of the partitioning $\mathcal{T}^h$. We note for $m = 0$ and 1 that

\begin{align}
|I - \pi^h|\psi|_{W^m,\kappa(\kappa)} &\leq C h_\kappa^{2-m} |\psi|_{W^{2,\kappa(\kappa)}} \quad \forall \kappa \in \mathcal{T}^h, \text{ for any } q \in [1, \infty],
\end{align}

(2.2)

where $I$ is the identity operator. This standard interpolation error for continuous piecewise linears follows from Theorem 3.1.5 in Ref. 8; on noting the embedding $W^{2,\kappa(\kappa)} \to C(\kappa)$, see p. 300 in Ref. 15.

In addition, we introduce the generalized interpolation operator $\mathcal{I}^h : (W^{1,\kappa(\kappa)})^2 \to \mathcal{V}^h$, where $q > 1$, satisfying

\begin{align}
\int_{\partial \kappa} (\eta - \mathcal{I}^h \eta) \cdot \nu_{\partial \kappa, \kappa} d\kappa = 0, \quad i = 1 \to 3, \quad \forall \kappa \in \mathcal{T}^h,
\end{align}

(2.3)

where $\partial \kappa \equiv \bigcup_{i=1}^3 \partial_i \kappa$ and $\nu_{\partial \kappa, \kappa}$ is the corresponding outward unit normal on the edge $\partial \kappa$. Moreover, we have for all $\kappa \in \mathcal{T}^h$ and any $q \in [1, \infty)$ that

\begin{align}
|\eta| - |\mathcal{I}^h \eta|_{L^q(\kappa)} &\leq |\eta - \mathcal{I}^h \eta|_{L^q(\kappa)} \leq C h_\kappa |\eta|_{W^{1,\kappa(\kappa)}}
\end{align}

(2.4)

and

\begin{align}
|\mathcal{I}^h \eta|_{W^{1,\kappa(\kappa)}} &\leq C |\eta|_{W^{1,\kappa(\kappa)}},
\end{align}

(2.4)

e.g., see Lemma 3.1 in Ref. 10 and the proof given there for $q \geq 2$ is also valid for any $q \in (1, \infty]$; and, if $\eta$ is sufficiently smooth,

\begin{align}
|\text{Div}(\eta - \mathcal{I}^h \eta)|_{L^2(\kappa)} &\leq C h_\kappa |\text{Div} \eta|_{H^1(\kappa)},
\end{align}

(2.5)

see, e.g. p. 533 in Ref. 19.

We introduce $(\psi, \chi)^h:=[\sum_{\kappa \in \mathcal{T}^h}(\psi, \chi)^h_\kappa$, and

\begin{align}
(\psi, \chi)^h_\kappa := \frac{1}{|\kappa|} \sum_{i=1}^3 \psi(x_i^\kappa)\chi(x_i^\kappa) = \int_{\kappa} \pi^h(\psi \chi) dx, \quad \forall \psi, \chi \in C(\kappa), \quad \forall \kappa \in \mathcal{T}^h,
\end{align}

(2.6)
where \( \{ x^h_k \}_{k=1}^3 \) are the vertices of \( \kappa \). Hence \( \langle \psi, \chi \rangle^h_\kappa \) averages the integrand \( \psi \chi \) over each triangle \( \kappa \) at its vertices, and is exact if \( \psi \chi \) is piecewise linear over the partitioning \( T^h \). We recall the following well-known results

\[
|\psi^h|^2_{L^2(\Omega)} \leq |\psi|^2_{L^2(\Omega)} := (\psi^h, \psi^h)_\Omega \leq 4|\psi^h|^2_{L^2(\Omega)} \quad \forall \psi^h \in S^h, \tag{2.7a}
\]

see, e.g. Lemma 11 in Ref. 13 and

\[
|\langle \psi^h, \chi^h \rangle_\Omega - (\psi^h, \chi^h)_\Omega | = |(I - \pi^h)(\psi^h \chi^h), 1| \leq |(I - \pi^h)(\psi^h \chi^h)|_{L^1(\Omega)} \\
= \sum_{\kappa \in T^h} \| (I - \pi^h)(\psi^h \chi^h) \|_{L^1(\kappa)} \leq C \sum_{\kappa \in T^h} h^2_\kappa |\psi^h \chi^h|_{W^{1,1}(\kappa)} \\
\leq C \sum_{\kappa \in T^h} h^2_\kappa |\psi^h|_{H^1(\kappa)} |\chi^h|_{H^1(\kappa)} \\
\leq Ch^2 |\psi^h|_{L^2(\Omega)} |\chi^h|_{H^1(\Omega)} \quad \forall \psi^h, \chi^h \in S^h, \tag{2.7b}
\]

where we have noted (2.6), (2.2) with \( q = 1 \) and a local inverse inequality.

For all \( \hat{b}, \hat{c} \in \mathbb{R}^d \) and any \( r > 1 \), we note that the convexity of \( | \cdot |^r \) and

\[
\frac{1}{r} \frac{\partial | \hat{b} |^r}{\partial \hat{b}_i} = | \hat{b} |^{r-2} \hat{b}_i \Rightarrow | \hat{b} |^{r-2} \hat{b} \cdot (\hat{b} - \hat{c}) \geq \frac{1}{r}(| \hat{b} |^r - | \hat{c} |^r), \tag{2.8a}
\]

\[
\Rightarrow (| \hat{b} |^{r-2} | \hat{b} - \hat{c} | - | \hat{b} |^{r-2} \hat{c}) \cdot \hat{b} \geq \frac{r-1}{r}(| \hat{b} |^r - | \hat{c} |^r); \tag{2.8b}
\]

and if \( r \in (1, 2] \)

\[
(| \hat{b} |^{r-2} \hat{b} - | \hat{c} |^{r-2} \hat{c}) \cdot (\hat{b} - \hat{c}) \geq (r-1)(| \hat{b} | + | \hat{c} |)^{r-2} | \hat{b} - \hat{c} |^2, \tag{2.8c}
\]

see (3.2) in Ref. 3.

Similarly to (2.7a), we have from the equivalence of norms on finite-dimensional spaces and the convexity of \( | \cdot |^r \) for any \( r > 1 \) that for any \( \hat{y}^h \in \mathcal{V}^h \)

\[
C(| \hat{y}^h |^r, 1)^h_\kappa \leq \int_{\kappa} | \hat{y}^h |^r d\xi \leq (| \hat{y}^h |^r, 1)^h_\kappa \quad \forall \kappa \in T^h. \tag{2.9}
\]

Furthermore, it follows from (2.6) and (2.4) for any \( r > 1 \) and any \( \kappa \in T^h \) that

\[
\left| \int_{\kappa} (| \hat{y}^h |^r, 1)^h_\kappa \right| \leq r|\kappa||\hat{y}^h|_{L^1(\kappa)}^{r-1} \max_{\hat{y} \in \mathcal{V}^h} |(\hat{y}^h \eta)(x) - (\hat{y}^h \eta)(y)| \leq Crh_\kappa |\kappa| |\hat{y}||_{W^{1,\infty}(\kappa)} \quad \forall \eta \in [W^{1,\infty}(\kappa)]^2. \tag{2.10}
\]

Let \( 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T \) be a partitioning of \([0, T]\) into possibly variable time steps \( \tau_n := t_n - t_{n-1} \), \( n = 1 \rightarrow N \). We set \( \tau := \max_{n=1 \rightarrow N} \tau_n \). Assuming that (A1) and (A2) hold, our approximation of \((Q)\) is as follows.
Given $G^0_r = G^0 = \pi^h y^0 \in S^h_0$, for $n = 1 \to N$, find $G^r_n \in S^h_0$ and $V^r_n \in \mathcal{Y}^h$ such that

$$a(G^r_n, \psi^h) - \tau_n (\text{Div } V^r_n, \psi^h)_{\Omega}$$
$$= a(G^{r-1}_n, \psi^h) - (b_e(t_n) - b_e(t_{n-1}), \psi^h)_{\Omega} \quad \forall \psi^h \in S^h_0,$$  \hspace{1cm} (2.11a)

Furthermore if

$$(j_n \nabla y^r_n) - 2V^r_n, \eta^h)_{\Omega} + (G^r_n, \text{Div } \eta^h)_{\Omega} = 0 \quad \forall \eta^h \in \mathcal{Y}^h.$$ \hspace{1cm} (2.11b)

Obviously $(Q^h, r)$ is implicit, and involves solving a complicated nonlinear algebraic system at each time level. Similarly to (1.22), we introduce $\mathcal{F}^h : (H^1_0(\Omega))^r \to S^h_0$ such that for all $f \in (H^1_0(\Omega))^r$, $\mathcal{F}^h f \in S^h_0$ satisfies

$$a(\mathcal{F}^h f, \psi^h) = (f, \psi^h)_{\Omega} \quad \forall \psi^h \in S^h_0.$$ \hspace{1cm} (2.12)

Once again, the well-posedness of $\mathcal{F}^h$ follows from the Lax–Milgram theorem on noting (2.1b). Finally, we note from (A1) that

$$\sum_{n=1}^{N} \tau_n \left| b_e(t_n) - b_e(t_{n-1}) \right|^2 = \sum_{n=1}^{N} \tau_n \left| \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} d_t b_e dt \right|^2 \leq \int_0^T |d_t b_e|^2 dt \leq C. \hspace{1cm} (2.13)$$

**Theorem 2.1.** Let the assumptions (A1) and (A2) hold. Then for all $r \in [1, 2]$ and for all $\tau_n \in (0, \frac{T}{2})$, there exists a unique solution, $G^r_n \in S^h_0$ and $V^r_n \in \mathcal{Y}^h$, to the $n$th step of $(Q^h, r)$, $n = 1 \to N$. In addition, we have that

$$\max_{n=0}^{N} \frac{\|G^r_n\|_{H^{1/2}_0(\Omega)}}{H^{1/2}_0(\Omega)} + \sum_{n=1}^{N} \frac{\|G^r_n - G^{r-1}_n\|_{H^{1/2}_0(\Omega)}}{H^{1/2}_0(\Omega)} + \sum_{n=1}^{N} \tau_n \|V^r_n\|_{L^2(\Omega)} \leq C. \hspace{1cm} (2.14a)$$

Furthermore if (A3) holds, we have that

$$\sum_{n=1}^{N} \tau_n \left| \frac{G^r_n - G^{r-1}_n}{\tau_n} \right|^2_{H^{1/2}_0(\Omega)} + \sum_{n=1}^{N} \tau_n \|\mathcal{F}^h (\text{Div } V^r_n)\|_{H^{1/2}_0(\Omega)}^2 + \sum_{n=1}^{N} \tau_n \|V^r_n\|_{L^2(\Omega)}^2 \leq C. \hspace{1cm} (2.14b)$$

**Proof.** It follows from (2.12) and (2.11a) that

$$G^r_n - G^{r-1}_n = \mathcal{F}^h (\tau_n \text{Div } V^r_n - (b_e(t_n) - b_e(t_{n-1}))). \hspace{1cm} (2.15)$$

Substituting (2.15) into (2.11b) yields that

$$(j_n \nabla y^r_n) - 2V^r_n, \eta^h)_{\Omega} + \tau_n (\mathcal{F}^h (\text{Div } V^r_n), \text{Div } \eta^h)_{\Omega}$$
$$= (\mathcal{F}^h (b_e(t_n) - b_e(t_{n-1})) - G^{r-1}_n, \text{Div } \eta^h)_{\Omega} \quad \forall \eta^h \in \mathcal{Y}^h,$$ \hspace{1cm} (2.16)
which, on noting (2.12), can be rewritten as

\[
(j_c [V^n - V^n] - \tau_n a(\mathcal{F}^h(Div V^n) , \mathcal{F}^h(Div \eta^h)))
= (\mathcal{F}^h(b_c(t_n) - b_c(t_{n-1})) - G_r^{n-1}, Div \eta^h) \quad \forall \eta^h \in \Omega^h. \tag{2.17}
\]

We note from (1.10) that (2.17) is the Euler–Lagrange equation for the strictly convex minimization problem:

\[
\min_{\eta^h} \mathcal{J}^h(\eta^h) = \frac{1}{2} \sum_{r=0}^{N} (\|\eta^h_r\|_{a}^2 + \tau_n^2 \|\mathcal{F}^h(\text{Div} \eta^h_r)\|_{a}^2) - (\mathcal{F}^h(b_c(t_n) - b_c(t_{n-1})) - G_r^{n-1}, Div \eta^h) \tag{2.18}
\]

Hence there exists a unique solution $\mathbf{V}_n^h \in \Omega^h$ to (2.16); therefore, on noting (2.15), there exists a unique solution $G_r^h \in S^0_\delta$ and $\mathbf{V}_n^h \in \Omega^h$ to (Q$^h$) in (2.11a) and (2.11b).

Choosing $\psi^h = G_r^h$ in (2.11a), $\eta^h = \tau_n \mathbf{V}_n^h$ in (2.11b), adding and noting (1.10), yields for $n = 1 \to N$ and any $\delta > 0$ that

\[
\frac{1}{2} (\|G_r^h\|_{a}^2 + \|G_r^h - G_r^{n-1}\|_{a}^2) + \tau_n (\|V_r^n\|_{a}^2 + \mathcal{F}^h(\text{Div} \eta^h_r)_{1})
= \frac{1}{2} (\|G_r^{n-1}\|_{a}^2 - (b_c(t_n) - b_c(t_{n-1}), G_r^h))_{1}
\leq \frac{1}{2} \left( \|G_r^{n-1}\|_{a}^2 + \delta^{-1} \tau_n \left( \left| \frac{b_c(t_n) - b_c(t_{n-1})}{\tau_n} \right|^2 + \delta \tau_n \|G_r^h\|_{L^2(\Omega)}^2 \right) \right). \tag{2.19}
\]

It follows from (2.19), (1.21) and for $\delta$ sufficiently small, on noting that $(1 - \tau_n)^{-1} \leq (1 + 2\tau_n) \leq e^{2\tau_n}$ as $\tau_n \in (0, \frac{1}{2}]$, for $n = 1 \to N$ that

\[
C \|G_r^h\|_{L^2(\Omega)}^2 \leq \|G_r^h\|_{a}^2 \leq e^{2\tau_n} \left( \|G_r^{n-1}\|_{a}^2 + C \tau_n \left( \left| \frac{b_c(t_n) - b_c(t_{n-1})}{\tau_n} \right|^2 \right) \right)
\leq e^{2\tau_n} \left( \|g_r^h\|_{a}^2 + C \sum_{m=1}^{n} \tau_m \left( \left| \frac{b_c(t_m) - b_c(t_{m-1})}{\tau_m} \right|^2 \right) \right) \leq C, \tag{2.20}
\]

where for the last inequality we have also noted (1.10), (1.21), (1.11), (2.13) and (A1). Summing (2.19) for $n = 1 \to m$, where $m = 1 \to N$, and noting (2.9), (2.20), (1.10) and (A1), yields the bound (2.14a).

Choosing $\psi^h = (G_r^h - G_r^{n-1})/\tau_n$ in (2.11a) and summing from $n = 1 \to m$, for $m = 1 \to N$, yields that

\[
\sum_{n=1}^{m} \tau_n \|G_r^h - G_r^{n-1}\|_{a}^2 = \sum_{n=1}^{m} \tau_n \left( (\text{Div} \mathbf{V}_n^h, G_r^h - G_r^{n-1})_{1} \right)_{\Omega}
- \sum_{n=1}^{m} \tau_n \left( \frac{b_c(t_n) - b_c(t_{n-1})}{\tau_n}, G_r^h - G_r^{n-1} \right)_{1} \tag{2.21}
\]
Subtracting (2.11b) with \( n \) replaced by \( n - 1 \) from (2.11b), choosing \( \psi^h = V^n \) and noting (2.8b), we obtain for \( n = 2 \rightarrow N \) that

\[
(\text{Div} \, V^n, G^n_r - G^{n-1}_r)_{\Omega} = -(|V^n|^{r-2}V^n - |V^{n-1}|^{r-2}V^{n-1}, j_c V^n)_{\Omega} \\
= \leq -r^{-1}(1 - r^{-1})^{1/r} (V^n, j_c)_{\Omega}.
\]  

(222)

Equation (2.11b) for \( n = 1 \), on assuming that (A3) holds, yields that

\[
(\text{Div} \, V^1, G^1_r - G^0_r)_{\Omega} = -(|V^1|^{r-2}V^1, j_c)_{\Omega}.
\]  

(223)

On combining (2.21), (2.22) and (2.23), we obtain for \( m = 1 \rightarrow N \) that

\[
\sum_{n=1}^{m} \tau_n \left\| \frac{G^n_r - G^{n-1}_r}{\tau_n} \right\|^2 + r^{-1}(1 - r^{-1})^{1/r} \leq \sum_{n=1}^{m} \tau_n \left( \frac{b_e(t_n) - b_e(t_{n-1})}{\tau_n}, G^n_r - G^{n-1}_r \right)_{\Omega}.
\]  

(224)

The first bound in (2.14b) then follows from (2.24) on applying a Young’s inequality and noting (2.21), (1.10) and (2.13).

Choosing \( \psi^h = \mathcal{F}^h(\text{Div} \, V^n) \) in (2.11a), noting (2.12) and summing from \( n = 1 \rightarrow N \) yields that

\[
\sum_{n=1}^{N} \tau_n \mathcal{F}^h(\text{Div} \, V^n) \|_{\Omega}^2 = \sum_{n=1}^{N} \tau_n \left( \frac{G^n_r - G^{n-1}_r}{\tau_n}, \mathcal{F}^h(\text{Div} \, V^n) \right)_{\Omega} \\
+ \sum_{n=1}^{N} \tau_n \left( \frac{b_e(t_n) - b_e(t_{n-1})}{\tau_n}, \mathcal{F}^h(\text{Div} \, V^n) \right)_{\Omega}.
\]  

(225)

The second bound in (2.14b) follows immediately from (2.25) on applying a Young’s inequality and noting (2.21), (1.10), (2.13) and the first bound in (2.14b).

Finally, it follows from (2.11b), (2.9), (A1), (2.12), (1.10) and (2.14a) that for \( n = 1 \rightarrow N \)

\[
\eta_{\text{min}} \| \mathcal{F}^h(\text{Div} \, V^n) \|_{L^\ast(\Omega)} \leq \left( \eta_c, |V^n|^h \right)_{\Omega} = -(G^n_r, \text{Div} \, V^n)_{\Omega} = -a(\mathcal{F}^h(\text{Div} \, V^n), G^n_r) \\
\leq \| \mathcal{F}^h(\text{Div} \, V^n) \|_{\Omega} \| G^n_r \|_{\Omega} \leq C \| \mathcal{F}^h(\text{Div} \, V^n) \|_{\Omega}.
\]  

(226)

The final bound in (2.14b) follows immediately from (2.26), (1.10) and the second bound in (2.14b).

3. Convergence

We introduce the following notation for \( t \in (t_{n-1}, t_n], n = 1 \rightarrow N \),

\[
b_c^+(t) := \frac{(t - t_{n-1})}{\tau_n} b_c(t_n) + \frac{(t_n - t)}{\tau_n} b_c(t_{n-1}),
\]

\[
b_c^-(t) := \frac{(t - t_{n-1})}{\tau_n} b_c(t_{n-1}) + \frac{(t_n - t)}{\tau_n} b_c(t_n),
\]

\[
b_c(t) := \frac{(t - t_{n-1})}{\tau_n} b_c(t_{n-1}) + \frac{(t_n - t)}{\tau_n} b_c(t_n),
\]

\[
b_c(t) := \frac{(t - t_{n-1})}{\tau_n} b_c(t_{n-1}) + \frac{(t_n - t)}{\tau_n} b_c(t_n),
\]

\[
\frac{(t - t_{n-1})}{\tau_n} b_c(t_{n-1}) + \frac{(t_n - t)}{\tau_n} b_c(t_n),
\]

\[
\frac{(t - t_{n-1})}{\tau_n} b_c(t_{n-1}) + \frac{(t_n - t)}{\tau_n} b_c(t_n),
\]

\[
\frac{(t - t_{n-1})}{\tau_n} b_c(t_{n-1}) + \frac{(t_n - t)}{\tau_n} b_c(t_n),
\]

\[
\frac{(t - t_{n-1})}{\tau_n} b_c(t_{n-1}) + \frac{(t_n - t)}{\tau_n} b_c(t_n),
\]

\[
\frac{(t - t_{n-1})}{\tau_n} b_c(t_{n-1}) + \frac{(t_n - t)}{\tau_n} b_c(t_n),
\]

\[
\frac{(t - t_{n-1})}{\tau_n} b_c(t_{n-1}) + \frac{(t_n - t)}{\tau_n} b_c(t_n),
\]

\[
\frac{(t - t_{n-1})}{\tau_n} b_c(t_{n-1}) + \frac{(t_n - t)}{\tau_n} b_c(t_n),
\]
Let the assumptions hold for a subsequence of \( \{h_n\} \). In addition, we write \( G_r^\tau(\pm) \) to mean with or without the superscripts \( \pm \). We note from (3.1) and (2.13) that
\[
d_b^+ \to d_b^- \quad \text{strongly in } L^2(0,T) \text{ as } \tau \to 0. \tag{3.2}
\]

Adopting the notation (3.1), (Q\(h^\tau\)), (2.11a) and (2.11b), can be rewritten as follows: Find \( G_r^\tau \in H^1(0,T; S_h^0) \) and \( V_r^\tau(\pm) \in L^2(0,T; V_h) \) such that
\[
\int_0^T (a(\partial_t G_r^\tau, \psi^h) - (\text{Div} V_r^\tau(\pm), \psi^h)_{\Omega}) dt = 0 \quad \forall \psi^h \in L^2(0,T; S_h^0), \tag{3.3a}
\]
\[
\int_0^T ((\mathcal{J}_r V_r^\tau(\pm)), \psi^h)_{\Omega} + (G_r^\tau(\pm), \text{Div} \psi^h)_{\Omega}) dt = 0 \quad \forall \psi^h \in L^2(0,T; V_h), \tag{3.3b}
\]
where \( G_r^\tau(\cdot,0) = G^0(\cdot) \).

**Theorem 3.1.** Let the assumptions (A1)–(A3) hold. For any fixed \( r \in (1,2] \) and for any \( \tau \in (0, \frac{1}{r}] \), the sequence \( \{G_r^\tau, V_r^\tau(\pm)\}_{h>0, \tau>0} \), where \( \{G_r^\tau, V_r^\tau(\pm)\} \) is the unique solution of (Q\(h^\tau\)), is such that as \( h, \tau \to 0 \)
\[
G_r^\tau(\pm) \to g_r \quad \text{weak-* in } L^\infty(0,T; H^\frac{1}{r}_0(\Omega)), \tag{3.4a}
\]
\[
\partial_t G_r^\tau \to \partial_t g_r \quad \text{weakly in } L^2(0,T; H^\frac{1}{r}_0(\Omega)), \tag{3.4b}
\]
\[
V_r^\tau(\pm) \to v_r \quad \text{weakly in } L^2r(0,T; (L^r(\Omega))^2), \tag{3.4c}
\]
where \( \{g_r, v_r\} \) is the unique solution of (Q\(r\)), (1.18a) and (1.18b). In addition, \( g_r \) is the unique solution of (P\(r\)), (1.19).

**Proof.** It follows immediately from (2.14a), (2.14b) and (3.1) that
\[
\|G_r^\tau(\pm)\|_{L^\infty(0,T; H^\frac{1}{r}_0(\Omega))} + \|\partial_t G_r^\tau\|_{L^2(0,T; H^\frac{1}{r}_0(\Omega))} + \|V_r^\tau(\pm)\|_{L^2r(0,T,L^r(\Omega))} \leq C, \tag{3.5a}
\]
\[
\|G_r^\tau - G_r^\tau(\pm)\|^2_{L^2(0,T; H^\frac{1}{r}_0(\Omega))} \leq \tau^2 \|\partial_t G_r^\tau\|^2_{L^2(0,T; H^\frac{1}{r}_0(\Omega))} \leq C\tau^2. \tag{3.5b}
\]

We then deduce from (3.5a) and (3.5b) that the convergence results (3.4a)–(3.4c) hold for a subsequence of \( \{G_r^\tau, V_r^\tau(\pm)\}_{h>0, \tau>0} \).

For any fixed \( \psi \in C([0,T]; C^\infty_0(\Omega)) \), we choose \( \psi^h = \pi^h \psi \) in (3.3a) and perform integration by parts on the second term. On noting (3.4b), (3.4c), (3.2) and (2.2),
we then pass to the limit $h, \tau \to 0$ in this rewrite of (3.3a) for the above subsequence and obtain that
\[
\int_0^T \left( a(\partial_t g_r, \psi) + (\varphi_r, \text{Grad} \psi) + (d_t b_e, \psi) \right) dt = 0
\]
\[\forall \psi \in C([0, T]; C_0^\infty(\Omega)).\]  \hfill (3.6)

It follows from (3.6), (1.10), (1.21), (A1), (3.4b) and (3.5a) that
\[
\left| \int_0^T (\varphi_r, \text{Grad} \psi) dt \right| \leq \int_0^T \left( \|\partial_t g_r\|_a \|\psi\|_a + |d_t b_e| \|\psi\|_{L^1(\Omega)} \right) dt
\]
\[\leq C \|\psi\|_{L^2(0, T; H^1_0(\Omega))} \quad \forall \psi \in C([0, T]; C_0^\infty(\Omega)). \]  \hfill (3.7)

Noting that $C_0^\infty(\Omega)$ is dense in $H^1_0(\Omega)$, see Theorem 1.4.2.2 in Ref. 12 and hence $C([0, T]; C_0^\infty(\Omega))$ is dense in $L^2(0, T; H^1_0(\Omega))$, we obtain from (3.7) that
\[
\|\text{Div} \underline{w}, \underline{v}\|_{L^2(0, T; (H^1_0(\Omega)))} \leq C, \]  \hfill (3.8)

and therefore from (3.6) the desired result (1.18a).

For any fixed $\eta \in C([0, T]; (C^\infty(\Omega))^2)$, we choose $\eta^h = V_r^{r, +} - L^h \eta$ in (3.3b). On noting (3.3a), (2.8a), (A3), (2.9), (1.10), (3.5b) and (2.10), we deduce that
\[
\int_0^T (G_r^{r, +}, \text{Div} L^h \eta) dt
\]
\[= \int_0^T (G_r^{r, +}, \text{Div} V_r^{r, +}) dt + \int_0^T \left( j_c |V_r^{r, -} - 2V_r^{r, +}, V_r^{r, -} - L^h \eta \right) dt
\]
\[\geq \int_0^T \left( a(\partial_t G_r^{r, +}, G_r^{r, +}) + (d_t b_e, G_r^{r, +}) \right) dt
\]
\[+ \frac{1}{r} \int_0^T \left( |V_r^{r, +}|^r - |L^h \eta|^r, j_c \right) dt
\]
\[\geq \frac{1}{2} \|G_r^{r, +}(., T)\|^2 + \int_0^T \left( a(\partial_t G_r^{r, +}, G_r^{r, +} - G_r^g) + (d_t b_e, G_r^{r, +}) \right) dt
\]
\[+ \frac{1}{r} \int_0^T \left( |V_r^{r, +}|^r - |L^h \eta|^r, j_c \right) + \left( |L^h \eta|^r, j_c \right) - \left( |L^h \eta|^r, j_c \right) dt
\]
\[\geq \frac{1}{2} \|G_r^{r, +}(., T)\|^2 + \int_0^T (d_t b_e, G_r^{r, +}) dt - C\tau
\]
\[+ \int_0^T \left( j_c |L^h \eta|^{r-2} L^h \eta, V_r^{r, +} - L^h \eta \right) dt - Ch\|\eta\|_{L^\infty(0, T; W^{1, \infty}(\Omega)))}^2. \]  \hfill (3.9)

It follows from (3.5a) that
\[
\|G_r^{r, +}\|_{C([0, T]; H^1_0(\Omega))} \leq C. \]  \hfill (3.10)
Hence we deduce from (3.10), on extraction of a possible further subsequence, that
\[ \liminf_{h, \tau \to 0} \|G_r^\tau(\cdot, T)\|^2_a \geq \|g_r(\cdot, T)\|^2_a. \]  
(3.11)

On noting (3.4a), (2.5), (3.11), (3.2), (2.4) and (3.4c), we can pass to the limit
$h, \tau \to 0$ in (3.9) for the above subsequence to obtain
\[ \int_0^T (g_r, \text{Div}(\eta - \nu_r))_{H^1_0(\Omega)} dt \geq \frac{1}{2} \|g_r(\cdot, T)\|^2_a + \int_0^T ((\text{Div}(\eta - \nu_r))_{H^1_0(\Omega)} dt
+ (dt_b, g_r)_{H^1_0(\Omega)} \forall \eta \in C([0, T]; (C^\infty(\Omega))^2). \]  
(3.12)

It follows from (3.12), (1.18a) and (A3) that
\[ \int_0^T (\text{Div}(\eta - \nu_r), g_r)_{H^1_0(\Omega)} dt \geq \int_0^T (j_{\cdot \cdot} |\eta|^{r-2} \eta \cdot \nu_r - \eta)_{H^1_0(\Omega)} dt \forall \eta \in C([0, T]; (C^\infty(\Omega))^2). \]  
(3.13)

In addition, we obtain from (3.3b) with $\eta_h = I^h_\mathcal{L} \eta$, for any fixed $\eta \in C([0, T]; (C^\infty(\Omega))^2)$, on noting (2.9), (A1) and the third bound in (3.5a), that
\[ \int_0^T (G_r^{\cdot, +}, \text{Div}(I^h \eta))_{H^1_0(\Omega)} dt = - \int_0^T (j_{\cdot \cdot} |\eta|^{r-2} \eta \cdot \nu_r - \eta)_{H^1_0(\Omega)} dt \leq C \int_0^T (|\eta|^{r-1} \eta, 1)_{L^2(\Omega)} d\tau \leq C \||\eta|^{r-1}\|_{L^2(0, T; L^\infty(\Omega))} \|I^h \eta\|_{L^\infty(0, T; L^r(\Omega))} \leq C(T) \|I^h \eta\|_{L^\infty(0, T; L^r(\Omega))}. \]  
(3.14)

Passing to the limit $h, \tau \to 0$ in (3.14), on noting (3.4a), (2.5) and (2.4), yields that
\[ \int_0^T (g_r, \text{Div}(\eta - \nu_r))_{H^1_0(\Omega)} dt \leq C(T) \|\eta\|_{L^\infty(0, T; L^r(\Omega))} \forall \eta \in C([0, T]; (C^\infty(\Omega))^2). \]  
(3.15)

On noting the denseness of $C([0, T]; (C^\infty(\Omega))^2)$ in $L^r(0, T; (L^r(\Omega))^2)$ it follows that
\[ g_r \in L^p(0, T; W^{1, p}(\Omega)) \text{ and } \|g_r\|_{L^p(0, T; W^{1, p}(\Omega))} \leq C(T), \]  
(3.16)

and hence we deduce from (3.13) that
\[ \int_0^T (\text{Div}(\eta - \nu_r), g_r)_{H^1_0(\Omega)} dt \geq \int_0^T (j_{\cdot \cdot} |\eta|^{r-2} \eta \cdot \nu_r - \eta)_{H^1_0(\Omega)} dt \forall \eta \in Z. \]  
(3.17)

For any fixed $\zeta \in Z$, choosing $\eta = \nu_r \pm \alpha \zeta$ with $\alpha \in \mathbb{R}_{>0}$ in (3.17) and letting $\alpha \to 0$ yields the desired result (1.18b) on repeating the above for any $\zeta \in Z$. Hence $\{g_r, \nu_r\}$ solves $(Q_r)$, (1.18a) and (1.18b).

We now show that the solution to $(Q_r)$ is unique. If there were two solutions $\{g^i_r, \nu^i_r\}, i = 1, 2$, then setting $\eta^i_r := g^i_r - g^1_r$ and $\nu^i_r := \nu^i_r - \nu^1_r$, taking differences of
(1.18a) and (1.18b) with the different solutions and \( \psi = \tilde{\nabla}_r \chi_{[0,t]} \) and \( \eta = \nabla \chi_{[0,t]} \), where \( \chi_{[0,t]} \) is the characteristic function over the time interval \([0,t]\), and combining yields for all \( t \in [0,T] \) that

\[
\frac{1}{2} \|\tilde{\nabla}_r(\cdot,t)\|^2_a + \int_0^t (|\nabla^2|^r - 2|\nabla^1|^r - 2|\nabla^0|^2)|j, \nabla \psi|_\Omega dt' = 0. \tag{3.18}
\]

It follows from (3.18) and (2.8c) that \( \tilde{\nabla}_r \equiv 0 \) and \( \nabla \equiv 0 \). Hence the solution to (Q_r) is unique, and so the whole sequence \( \{G_r^p, \nabla_r^D, \nabla_r^C\}_{\theta,h,\tau > 0} \) converges in (3.4a)–(3.4c) and (3.11).

We now show that \( g_r \) is the unique solution of (P_r). It follows from (3.16) and (1.18b) that (1.20a) holds a.e. in \( \Omega_T \). Substituting (1.20a) into (1.18a) yields that (1.20a) holds a.e. in \( \Omega_T \). Similarly, for any positive \( \theta > 0 \) and \( \tau > 0 \), it is easy to establish that this solution to (P_r) is unique.

Before proving our results about problems (Q), (1.15a), (1.15b), and (P), (1.17), we note the following density result concerning \( L^2(0,T; \mathbb{Z}_M^1) \).

**Lemma 3.1.** Let \( \Omega \) be strictly star-shaped. For any \( \underline{n} \in L^2(0,T; \mathbb{Z}_M^1) \), there exist \( \underline{n}_i \in (C^\infty(\overline{\Omega_T}))^2 \), \( i \in \mathbb{N} \), such that

\[
\text{Div} \underline{n}_i \rightarrow \text{Div} \underline{n} \quad \text{weakly in} \quad L^2(0,T; (H^1_0(\Omega^2))') \quad \text{as} \quad i \rightarrow \infty \tag{3.19a}
\]

and

\[
\limsup_{i \rightarrow \infty} \int_{\Omega_T} k|\underline{n}_i|^2 d\tau dt \leq \int_0^T \langle |\underline{n}|, k \rangle_{C(\overline{\Omega})} dt \tag{3.19b}
\]

for any positive \( k \in C(\overline{\Omega}) \).

**Proof.** We apply the standard techniques of change of variable and mollification. Without loss of generality, one can assume that \( \Omega \) is strictly star-shaped with respect to \( \bar{\omega} \). For any \( \theta > 1 \), let \( \Omega^\theta := \theta \Omega \) and \( \Omega^\theta_T := \Omega^\theta \times (0,T) \). For any \( \phi \in C(\overline{\Omega^\theta_T}) \), let \( \phi_\theta(\bar{x},t) = \phi(\theta^{-1}x,t) \) for all \( (\bar{x},t) \in \overline{\Omega^\theta_T} \), and so \( \phi_\theta \in C(\overline{\Omega^\theta_T}) \). Then for any \( \underline{\nabla} \in L^2(0,T; \mathbb{Z}_M^1) \), we define \( \underline{\nabla}_\theta \in L^2(0,T; (\mathbb{M}(\Omega^\theta))^2) \) such that

\[
\int_0^T \langle \underline{\nabla}_\theta, \phi_\theta \rangle_{C(\overline{\Omega^\theta_T})}^2 dt = \theta^2 \int_0^T \langle \underline{\nabla}_\theta, \phi \rangle_{C(\overline{\Omega^\theta_T})}^2 dt \quad \forall \phi \in L^2(0,T; (\mathbb{M}(\Omega^\theta))^2), \tag{3.20}
\]

where we have noted that \( |\Omega^\theta| = \theta^2 |\Omega| \). Similarly, for any \( \phi \in L^2(0,T; C^\infty(\Omega)) \), on noting that \( \langle \text{Grad} \phi \rangle_\theta = \theta \text{Grad} \phi_\theta \), we have that

\[
\int_0^T \langle \langle \text{Div} \underline{\nabla} \rangle_\theta, \phi_\theta \rangle_{C^\infty(\Omega^\theta)}^2 dt = \theta^2 \int_0^T \langle \text{Div} \underline{\nabla}, \phi \rangle_{C^\infty(\Omega)}^2 dt = -\theta^2 \int_0^T \langle \underline{\nabla}, \text{Grad} \phi \rangle_{C^\infty(\Omega)}^2 dt
\]
As \( \text{Div } \varphi \in L^2(0,T;(H^1_0(\Omega))') \), it follows that \( \text{Div } \varphi = \theta^{-1}(\text{Div } \varphi)_{\theta} \in L^2(0,T;(H^1_0(\Omega))') \).

Let \( \rho \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}) \), with support in the unit ball \( B((\mathbb{R},0),1) \), be such that
\[
\int_{B((\mathbb{R},0),1)} \rho(\xi,t) d\xi dt = 1 \quad \text{and} \quad \rho(\xi,t) \geq 0 \quad \forall (\xi,t) \in \mathbb{R}^2 \times \mathbb{R}.
\] (3.22)

For \( \varepsilon > 0 \), let \( \mathcal{P}_\varepsilon : L^2(\mathbb{R}:(C_0^\infty(\mathbb{R}^2))') \to C^\infty(\mathbb{R},C^\infty(\mathbb{R}^2)) \) be such that
\[
(\mathcal{P}_\varepsilon(\eta))(\xi,t) = \int_{\mathbb{R}} (\eta(o,s),\rho_\varepsilon(\xi - o,t - s))_{C_0^\infty(\mathbb{R}^2)} ds \quad \forall (\xi,t) \in \mathbb{R}^2 \times \mathbb{R},
\] (3.23)

where \( \rho_\varepsilon(\xi,t) = \varepsilon^{-3} \rho(\varepsilon^{-1} \xi,\varepsilon^{-1} t) \). One can extend this naturally to vectors, and we note that \( \text{Div } \mathcal{P}_\varepsilon(\eta) = \mathcal{P}_\varepsilon(\text{Div } \eta) \) for all \( \eta \in L^2(\mathbb{R}:(C_0^\infty(\mathbb{R}^2))') \). For any \( \theta > 1 \), there exists an \( \varepsilon_0(\theta) \in \mathbb{R}_{>0} \) such that for all \( \varepsilon \in (0,\varepsilon_0(\theta)) \)
\[
\Omega \subset \subset \Omega_\varepsilon^\theta := \{ \xi \in \Omega^\theta : \text{dist}(\xi,\partial \Omega^\theta) > \varepsilon \} \subset \subset \Omega^\theta.
\] (3.24)

We now extend \( \varphi^{\theta} \) by zero to \( \mathcal{E}(\varphi^{\theta}) \in L^2(\mathbb{R}:(\mathcal{M}(\mathbb{R}^2))^2) \). Hence, by the standard properties of mollifiers as \( \text{Div } \mathcal{E}(\varphi^{\theta}) \in L^2(\mathbb{R}:(H^1_0(\Omega^\theta))') \), we have that \( \mathcal{P}_\varepsilon(\mathcal{E}(\varphi^{\theta})) \in (C^\infty(\Omega^\theta))^2 \) such that
\[
\text{Div } \mathcal{P}_\varepsilon(\mathcal{E}(\varphi^{\theta})) = \mathcal{P}_\varepsilon(\text{Div } \mathcal{E}(\varphi^{\theta})) \to \text{Div } \mathcal{E}(\varphi^{\theta})
\] = \( \text{Div } \varphi^{\theta} \) weakly in \( L^2(0,T;(H^1_0(\Omega))') \) as \( \varepsilon \to 0 \). (3.25)

Similarly, one can show that
\[
\text{Div } \mathcal{P}_\varepsilon(\mathcal{E}(\varphi^{\theta})) = \theta^{-1}(\text{Div } \varphi)_{\theta} \to \text{Div } \varphi \quad \text{weakly in } L^2(0,T;(H^1_0(\Omega))') \text{ as } \theta \to 1.
\] (3.26)

Next, we note from (3.23) and (3.22) that for any positive \( k \in C(\Omega) \)
\[
\int_{\Omega_T} k|\mathcal{P}_\varepsilon(\mathcal{E}(\varphi^{\theta}))| d\xi dt
\]
\[
= \int_{\Omega_T} k(\xi) \left| \int_{\mathbb{R}} (\mathcal{E}(\varphi^{\theta})(o,s),\rho_\varepsilon(\xi - o,t - s))_{C_0^\infty(\mathbb{R}^2)} ds \right| d\xi dt
\]
\[
\leq \int_{\mathbb{R}} \left| \mathcal{E}(\varphi^{\theta})(o,s) \right| \int_{\Omega_T} k(\xi) \rho_\varepsilon(\xi - o,t - s) d\xi dt ds
\]
\[
\leq \int_{\Omega_T} k(\xi) \rho_\varepsilon(\xi - o,t - s) d\xi dt ds
\]
\[ = \int_0^T \left[ (k_\theta \varphi_\theta)(\omega, s) \right]_B \rho_\varepsilon(\varphi - \omega, t - s) d\varphi dt \quad C(\bar{\Omega}) \]

\[ + \int_0^T \left[ \varphi_\theta(\omega, s) \right]_B \int_{B((\omega, s), r) \cap \Omega_T} \|k_\varphi(\varphi) - k_\varphi(\omega)\| \rho_\varepsilon(\varphi - \omega, t - s) d\varphi dt \quad C(\bar{\Omega}) \]

\[ \leq \int_0^T \langle \varphi_\theta, k_\varphi \rangle_{C(\bar{\Omega})} \, dt + \int_0^T \langle \varphi_\theta, 1 \rangle_{C(\bar{\Omega})} \, dt \sup_{\varepsilon \in \bar{\Omega}} \sup_{\varepsilon \in B((\omega, s), r) \cap \Omega_T} \|k_\varphi(\varphi) - k_\varphi(\omega)\|. \]  

(3.27)

Therefore Eqs. (3.27) and (3.20) yield that

\[ \limsup_{\varepsilon \to 0} \int_{\Omega_T} k|\mathcal{P}_\varepsilon(E(\varphi_\theta))| \, d\varphi \, dt \]

\[ \leq \int_0^T \langle \varphi_\theta, k_\varphi \rangle_{C(\bar{\Omega})} \, dt + \int_0^T \langle \varphi_\theta, 1 \rangle_{C(\bar{\Omega})} \, dt \sup_{\varepsilon \in \bar{\Omega}} \sup_{\varepsilon \in B((\omega, s), r) \cap \Omega_T} \|k_\varphi(\varphi) - k_\varphi(\omega)\| \]

\[ \leq \theta^2 \int_0^T \langle \varphi, k \rangle_{C(\bar{T})} \, dt + C(\theta - 1) \int_0^T \langle \varphi, 1 \rangle_{C(\bar{T})} \, dt. \]  

(3.28)

It follows from (3.25), (3.26) and (3.28) that for any \( \eta \in L^2(0, T; \mathbb{Z}_M) \), there exist \( \eta_i \in (C^\infty(\Omega_T))^2, i \in \mathbb{N} \), such that (3.19a) and (3.19b) hold. \( \square \)

**Theorem 3.2.** Let the assumptions (A1)–(A4) hold. There exists a subsequence of \( \{g_\varepsilon, \varphi_\theta\} \in C(1, 2) \) (not indicated), where \( \{g_\varepsilon, \varphi_\theta\} \) is the unique solution of (Q), such that as \( r \to 1 \)

\[ g_\varepsilon \to g \quad \text{weak-* in } L^\infty(0, T; H_{\text{div}}^2(\Omega)), \text{ weakly in } L^2(0, T; H^1(\Omega)), \]  

(3.29a)

\[ \partial_t g_\varepsilon \to \partial_t g \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)), \]  

(3.29b)

\[ \varphi_\varepsilon \to \varphi \quad \text{weakly in } L^2(0, T; (\mathcal{M}(\bar{T}))^2), \]  

(3.29c)

\[ \text{Div} \varphi_\varepsilon \to \text{Div} \varphi \quad \text{weakly in } L^2(0, T; (H_0^1(\Omega))^2), \]  

(3.29d)

where \( \{g, \varphi\} \) solves (Q), (1.15a) and (1.15b). In addition, \( g \) is unique, and the possible non-uniqueness in \( \varphi \) is restricted to the following: If there were two solutions \( \varphi^i, i = 1, 2 \), then

\[ \int_0^T \|\mathcal{F}(\text{Div}(\varphi^2 - \varphi^1))\|^2 \, dt = 0 \quad \text{and} \]

\[ \int_0^T \langle |\varphi^2|, j_c \rangle_{C(\bar{T})} \, dt = \int_0^T \langle |\varphi^1|, j_c \rangle_{C(\bar{T})} \, dt. \]  

(3.30)

Finally, \( g \) is the unique solution of (P), (1.17).
Proof. It follows from (3.5a), (3.4a)–(3.4c) and (3.16) that
\[
\|g_r\|_{L^\infty(0,T;H^1_0(\Omega))} + \|g_r\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t g_r\|_{L^2(0,T;H^2_0(\Omega))} \\
+ \|\nabla g_r\|_{L^2(0,T;L^1(\Omega))} + \|\text{Div } g_r\|_{L^2(0,T;H^1_0(\Omega))} \leq C(T). 
\] (3.31)
The results (3.29a)–(3.29d) with \( g \in L^2(0,T;H^1_0(\Omega)) \) follow immediately from (3.31) for a subsequence \( \{g_r, \nabla g_r\}_{r \in (1,2)} \).

We can pass to the limit \( r \to 1 \) in (1.18a) for the above subsequence to obtain (1.15a).

For any fixed \( \zeta \in (C^\infty(\Omega)) \), we choose \( \eta = \frac{1}{r} \zeta \) in (1.15b). On noting (1.18a), (2.8a) and (A3), we deduce that
\[
\int_0^T (\text{Div } \zeta, g_r)_\Omega dt = \int_0^T (\text{Div } \nabla g_r, g_r)_H^1_0(\Omega) dt + \int_0^T (\zeta, |\nabla g_r|^2 \nabla g_r - \zeta)_\Omega dt \\
\geq \int_0^T (a(\partial_t g_r, g_r) + (d_t b_c, g_r)_\Omega) dt + \frac{1}{r} \int_0^T (|\nabla g_r|^r - |\zeta|^r, j_c)_\Omega dt \\
= \frac{1}{2} \|g_r(\cdot, T)\|_a^2 + \int_0^T (d_t b_c, g_r)_\Omega dt + \frac{1}{r} \int_0^T (|\nabla g_r|^r - |\zeta|^r, j_c)_\Omega dt. 
\] (3.32)

It follows from (3.31) that
\[
\|g_r\|_{C([0,T];H^1_0(\Omega))} \leq C(T). 
\] (3.33)
Hence we deduce from (3.33), on extraction of a possible further subsequence, that
\[
\liminf_{r \to 1} \|g_r(\cdot, T)\|_a^2 \geq \|g(\cdot, T)\|_a^2. 
\] (3.34)

Next, on noting the Young’s inequality
\[
bc \leq \frac{1}{r} b^r + \frac{1}{p} c^p \quad \forall b, c \in \mathbb{R}_{\geq 0}, 
\] (3.35)
we deduce that
\[
\frac{1}{r} \int_0^T (|\nabla g_r|^r, j_c)_\Omega dt \geq \int_0^T (|\nabla g_r|^r, j_c)_\Omega dt - \frac{r}{r} T(j_c, 1)_\Omega. 
\] (3.36)
On noting (3.29a), (3.29c), (3.34) and (3.36), we can pass to the limit \( r \to 1 \) in (3.32) for the above subsequence to obtain
\[
\int_0^T (\text{Div } \zeta, g)_\Omega dt \geq \frac{1}{2} \|g(\cdot, T)\|_a^2 + \int_0^T (d_t b_c, g)_\Omega dt \\
+ \int_0^T (|\nabla g|^r - |\zeta|^r, j_c)_{C^{1,0}(\Omega)} dt \quad \forall \zeta \in (C^\infty(\Omega))^2. 
\] (3.37)
It follows from (3.37), (1.15a), (A3) and the results (3.19a), (3.19b) for \( L^2(0,T;Z) \) that (1.15b) holds. Hence \( \{g, \zeta\} \) solves (Q), (1.15a) and (1.15b).
We now show that $g$ is the unique solution of (P). First, we recall that on choosing $\bar{\eta} = \varphi$ and $2\varphi$ in (1.15b) we obtain (1.16a) and (1.16b). For any $\chi \in C([0, T]; K \cap C_0^\infty(\Omega))$ we have with $\psi = \chi - g$ in (1.15a), on noting (1.16a) and (1.11), that

$$
\int_0^T (a(\partial_t g, \chi - g) + (dB_e, \chi - g))dt = \int_0^T (\text{Div} \chi - g)_{H_0^1(\Omega)} dt
\geq \int_0^T (\|\psi\|_{H^1(\Omega)})^2 dt \geq 0. \quad (3.38)
$$

Noting that $K \cap C_0^\infty(\Omega)$ is dense in $K$, see p. 47 in Ref. 11, we have that (1.17) is satisfied. Next, we need to show that $g \in L^\infty(0, T; K)$. Since (1.16b) is true also for $-\tilde{\eta}$, it follows that

$$
\left| \int_0^T (g, \text{Div} \eta)_{\Omega}dt \right| \leq \int_0^T (\eta, |g|)_{\Omega}dt \quad \forall g \in C([0, T]; C_0^\infty(\Omega))^2). \quad (3.39)
$$

It is easily deduced from (3.39) that $g \in L^\infty(0, T; K)$, and so $g$ solves (P), (1.17); see the argument on p. 698 in Ref. 4. We now show that $g$ is the unique solution of (P). If there were two solutions $g^i$, $i = 1, 2$, then setting $\bar{g} := g^2 - g^1$ choosing $\psi = g^0\chi_{[0, t]}$ in the $g^i$ version of (1.17) for $m \neq i, i = 1, 2$, and combining yields for all $t \in [0, T]$ that

$$
\|\bar{g}(\cdot, t)\|^2 \leq 0. \quad (3.40)
$$

It follows from (3.40) that $\bar{g} \equiv 0$, and hence the solution of (P) is unique. As a solution $\{g, \psi\}$ of (Q) implies that $g$ is the solution of (P), which is unique, it follows that the $g$ solution of (Q) is unique. If there were two solutions $\{g, \psi^i\}$, $i = 1, 2$, of (Q), then taking differences of (1.15a) with the different solutions and choosing $\psi = \mathcal{F}(\text{Div}(\psi^2 - \psi^1))$ yields, on noting (1.22), the first result in (3.30). The second result in (3.30) follows from (1.16a), (1.22) and the first result in (3.30), on recalling the uniqueness of $g$.

4. Numerical Experiments

Our iterative algorithm for solving the nonlinear algebraic system (2.11a) and (2.11b) on each time level is based on approximating the nonlinear term $|V^m|^{-2}V^m$ at the $i$th iteration by

$$
|V^{n,i-1}|^{-2}V^{n,i-1} + |V^{n,i-1}|^{-2}(V^{n,i} - V^{n,i-1}),
$$

where $|s|_\varepsilon := \sqrt{|s|^2 + \varepsilon^2}$ with $\varepsilon > 0$ small (we chose $\varepsilon = 10^{-9}$), see Ref. 5 for the details. We chose as a stopping criteria that the relative change in a discrete $L^1(\Omega)$ norm in both the $G^r$ (based on vertex sampling of $\kappa$) and $V^m$ (based on edge midpoint sampling of $\kappa$) iterates should be less than a specified tolerance tol.

We note that $\tilde{J}(\cdot, t_n)$ can be approximated by $RV_{\rho, n}(\cdot)$, on recalling (1.13), and a piecewise constant approximation to the sheet current density $j(\cdot, t_n)$ is given by...
\[ J^n(\cdot) = \text{curl} G^n_{\psi}(\cdot), \] on recalling (1.5). An approximation to \( j(x, t_n) \) can then be obtained at each node \( x \), as the weighted-by-areas mean of the constant vectors \( \mathbf{J}^n \mid \kappa \) in triangles \( \kappa \) to which \( x \) is a vertex. We refer to Ref. 5 also for a discussion on the possible approaches to computing an approximation to the magnetic field \( b \) based on the main variables, \( G^n_{\psi} \) and \( \mathbf{J}^n \).

The simulations have been performed in Matlab R2012a (64 bit) on a PC with an Intel Core i5-2400 3.10 GHz processor and 8 GB RAM. The Matlab PDE Toolbox was used for the triangulation of \( \Omega \). Although for the convergence analysis in the previous sections, we assumed, for ease of exposition, that \( \Omega \) was polygonal and that the bilinear form \( a(\cdot, \cdot) \) on \( S^h_0 \times S^h_0 \) was calculated exactly; in practice curved domain boundaries were approximated by polygonal ones and \( a(\cdot, \cdot) \) was approximated, see the Appendix in Ref. 5 for details.

Before reporting on our numerical experiments, we make the following formal observation. Choosing the test function \( \psi(\cdot, s) = \phi(\cdot) \in H^{\frac{1}{2}}_0(\Omega) \) for \( s \in (0, t) \) and zero for \( s \in (t, T) \) in (1.15a), and noting from Theorem 3.2 that the unique \( g \) solving (Q) solves (P), we obtain for a.a. \( t \in (0, T) \) that \( g(\cdot, t) \in K \) and \( \psi \in L^2(0, T; \mathbb{H}_M) \) satisfy

\[
\begin{align*}
    a(g(\cdot, t) - g^0(\cdot), \phi(\cdot)) - \left\langle \text{Div} \left( \int_0^t \psi(\cdot, s)ds \right), \phi(\cdot) \right\rangle_{H^{\frac{1}{2}}_0(\Omega)} \\
    + (b_e(t) - b_e(0))(1, \phi)_\Omega = 0 \quad \forall \phi \in H^{\frac{1}{2}}_0(\Omega). \tag{4.1}
\end{align*}
\]

Choosing \( \phi(\cdot) = \psi(\cdot) - g(\cdot, t) \) with \( \psi \in K \) in (4.1) yields on proceeding formally that for a.a. \( t \in (0, T) \)

\[
\begin{align*}
    a(g(\cdot, t) - g^0(\cdot), \psi(\cdot) - g(\cdot, t)) + (b_e(t) - b_e(0))(1, \psi(\cdot) - g(\cdot, t))_\Omega \\
    = - \left( \int_0^t \psi(\cdot, s)ds, \text{Grad}(\psi(\cdot) - g(\cdot, t)) \right)_\Omega \quad \forall \psi \in K. \tag{4.2}
\end{align*}
\]

Now if for a.e. \( (x, t) \in \Omega_T \)

\[
\text{Grad} g(x, t) \cdot \nu(x, s) = j_e(x) |\psi(x, s)| \quad \text{for a.a. } s \in (0, t), \tag{4.3}
\]

then (4.2) yields for a.a. \( t \in (0, T) \) that \( g(\cdot, t) \in K \) satisfies

\[
\begin{align*}
    a(g(\cdot, t) - g^0(\cdot), \psi(\cdot) - g(\cdot, t)) + (b_e(t) - b_e(0))(1, \psi(\cdot) - g(\cdot, t))_\Omega \\
    = \left( \int_0^t |\psi(\cdot, s)|ds, j_e(\cdot) \right)_\Omega - \left( \int_0^t |\psi(\cdot, s)ds, \text{Grad} \psi(\cdot) \right)_\Omega \geq 0 \quad \forall \psi \in K. \tag{4.4}
\end{align*}
\]

That is, for a.a. \( t \in (0, T) \), \( g(\cdot, t) \), and hence \( j = \text{curl} g \), are not only rate independent, but only depend on the initial data \( g^0 \) and the variation \( b_e(t) - b_e(0) \). Therefore, one is justified in using large time steps in obtaining their numerical approximations. One can restate the assumption (4.3), on noting (1.5) and (1.13),
as for a.e. \((\mathbf{x}, t) \in \Omega_T\)
\[
\mathbf{j}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}, s) = j_c(\mathbf{x})|\mathbf{n}(\mathbf{x}, s)| \quad \text{for a.a. } s \in (0, t).
\] (4.5)

We remark, on noting (1.1), that (4.5) holds if for a.e. \(x \in \Omega\)

(a) The direction of \(\mathbf{j}(\mathbf{x}, t)\) does not change for a.a. \(t \in (0, T)\); (4.6a)

(b) If \(|\mathbf{j}(\mathbf{x}, t)|\) reaches \(j_c(\mathbf{x})\) at \(t = t_0(\mathbf{x})\), it remains equal to \(j_c(\mathbf{x})\) for a.a. \(t \in (t_0(\mathbf{x}), T)\). (4.6b)

We conjecture that (4.6a) and (4.6b) are true if \(d_\varepsilon b_\varepsilon\) is of one sign for \(t \in (0, T)\). Although this hypothesis is supported by numerical experiments, its rigorous justification for the thin film magnetization remains an open problem. In models with power current–voltage relation the magnetization function \(g_r\) and current density approximation, \(\text{Curl} g_r\) are not rate independent and large time steps may be applicable only if the power \(p\) is large and so the model is close to the critical state model.

All the film magnetization problems below were solved for the growing external field \(b_\varepsilon(t) = t\), i.e. \(d_\varepsilon b_\varepsilon = 1\) is of one sign, and the initial state \(g^0\) is 0. In addition, unless otherwise stated, \(j_c \equiv 1\).

If \(\Omega\) is the unit circle, the exact Bean model distribution of the sheet current density is known (see Refs. 17 and 9): \(\mathbf{j} = j(\rho, t)\hat{\phi}\), where
\[
 j(\rho, t) = \begin{cases} 
 -1, & \alpha(t) \leq \rho \leq 1, \\
 -\frac{2}{\pi} \arctan \left\{ \rho \sqrt{\frac{1 - \alpha^2(t)}{\alpha^2(t) - \rho^2}} \right\}, & 0 \leq \rho < \alpha(t),
\end{cases} (4.7)
\]
with \(\alpha(t) = 1/\cosh(2b_\varepsilon(t))\) and \(\hat{\phi}\) is the unit azimuthal vector in polar coordinates \((\rho, \phi)\). It is clear from (4.7) and (1.1) that the conjecture (4.5) holds true in this model case. The normal to the film component of the magnetic field can be found by means of a 1D numerical integration using the equation
\[
b_{\varepsilon}(\rho, t) = b_\varepsilon(t) + \frac{1}{2\pi} \int_0^1 Q(\rho, \rho') j(\rho', t) d\rho',
\] (4.8)
where \(Q(\rho, \rho') = (K(s)/(\rho + \rho')) - (E(s)/(\rho - \rho'))\), \(s = 2\sqrt{\rho\rho'}/(\rho + \rho')\) and \(K\) and \(E\) are complete elliptic integrals of the first and second kind. Finally, the electric field \(e = e(\rho, t)\hat{\phi}\), where \(\rho^{-1} \partial_{\rho} (\rho e) = -\partial_t b_\varepsilon\), \(e|_{\rho=0} = 0\). Approximating \(\partial_{\rho} b_\varepsilon\) by \((b_{\varepsilon}(\rho, t_n) - b_{\varepsilon}(\rho, t_{n-1}))/\tau_n\) and integrating numerically, one can find an approximation to the electric field averaged over the time interval \((t_{n-1}, t_n)\). This analytical (semi-analytical for the magnetic and electric fields) solution was used to check our finite element scheme and to demonstrate the established convergence of the power law model to the Bean critical state model as \(p \to \infty\).

For \(p = 100,001\), corresponding to \(r = 1.00001\), a finite element mesh of approximately 12,000 triangles, \(h = 0.03\), and four time steps, \(\tau_n = 0.2, 0.2, 0.2, 0.05\), the solution took 182 min (tol = 5 \cdot 10^{-5}), not including the time for the preliminary
calculation of the coefficients $A_{i,m} = a(\psi_i, \psi_m)$ for each pair of $S_h^0$ basis functions. We note that this matrix $A$ is full, which makes simulations on fine meshes both time and memory consuming. For $T = 0.65$, the relative errors in the discrete $L^1(\Omega)$ norms for the current density and the electric field were $\delta j = 0.5\%$ and $\delta e = 1.7\%$, respectively. Here, and below, we choose a small last time step in order to approximate the electric field better at $T$, as this variable is not rate independent. Taking

![Figure 1](image)

Fig. 1. A rectangular film with a semicircular edge indentation, numerical simulation at $T = 0.45$. Left: $p = 1001$; right: $p = 11$. Top: the modulus of the electric field $\mathbf{e}$ (for both $p$ values the strongest electric field, $|\mathbf{e}| \approx 0.7$, is at the center of the edge indentation). Middle: level contours of $|\mathbf{e}|$ (black lines); the boundary of the electric field penetration domain (red line) represented by the level contour $|\mathbf{e}| = 0.03$. Bottom: the current, $\mathbf{j}$, streamlines (plotted as the level contours of the magnetization function $g$).
only two time steps, $\tau_1 = 0.6$ and $\tau_2 = 0.05$ yielded the approximate solution for the same time interval in a shorter time (52 min) and with about the same accuracy: $\delta j = 0.4\%$ and $\delta e = 1.8\%$. We solved the same problem also for two smaller values of the power $p$ using the same finite element mesh and four time steps,
0.2, 0.2, 0.2, 0.05. For $p = 101$ ($r = 1.01$) the errors were $\delta j = 1.4\%$ and $\delta e = 2\%$ and the computation took 132 min. For $p = 11$ ($r = 1.1$) the solution differs more significantly from that for the critical state model: $\delta j = 12\%$ and $\delta e = 8.7\%$; the computation time was 50 min.

We present two additional examples showing that, although in our proof of convergence to the critical state limit (Q), recall Theorem 3.2, we required assumption (A4), $\Omega$ is strictly star-shaped and $j_c \in C(\overline{\Omega})$, our numerical scheme $(Q_h, \tau)$, (2.11a), (2.11b), still performs well for large $p$ ($r$ slightly greater than unity) if these restrictions are violated.

A rectangular film with a semicircular edge indentation (Fig. 1) is an example of a domain that is not strictly star-shaped. Our mesh contained approximately 10,300 triangles, and the solution was obtained at $T = 0.45$ with tol $= 2 \cdot 10^{-4}$. For the close-to-critical power law model with $p = 1001$ ($r = 1.001$) we employed two discretizations in time: (a) nine equal time steps $\tau = 0.05$ and (b) only two time steps, $\tau_1 = 0.4$, $\tau_2 = 0.05$. In agreement with our time stepping hypothesis, the numerical solutions obtained were very close; for the latter time grid the computation was much faster (33 min compared to approximately 16 hs). For $p = 11$ we used the same mesh and the constant time step $\tau = 0.05$; the computation took 84 min. Qualitatively, the solutions for these two values of the power $p$ are very similar: as the external field grows, the nonzero electric field region propagates inside the film from its boundary; in the shrinking zero electric field core the normal component of the magnetic field is zero as well. The propagation is deeper for the power law model with $p = 11$.

The elliptic film in our last example (Fig. 2) is inhomogeneous: $j_c = 1$ and $j_c = 0.25$ in its left and right halves, respectively. The finite element mesh contained approximately 9200 triangles and was refined near both subdomain boundaries. For $p = 1001$ we used $\tau_1 = 0.45$, $\tau_2 = 0.05$ (very close results were obtained for a larger number of time steps). We note that the electric field is singular in the vicinity of the jump of the critical current density, $j_c$, and the convergence of the iterations was slower (the computation took about 75 min with tol $= 10^{-3}$). For $p = 11$ we used the same mesh and a constant time step $\tau_n = 0.05$; the numerical solution for $T = 0.5$ needed 57 min. For this smaller power value the singularity is weaker and the penetration is deeper; but, qualitatively, the solutions are similar.

References


