Variational Model for Sand Surface Dynamics

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Outline:

- Model for sandpile growth
- Primal and dual variational formulations
- Numerical approximation
- Long-scale limit of BCRE model

Collaborators: J.W. Barrett, B. Zaltzman
Introduction

Mean shape of a growing pile: we assume that
- sand flow is confined to a thin surface layer
- avalanches are small fluctuations of pile surface

Traces of avalanches on the slip face of a small dune

Large conical pile below a point source
pile growth: mass balance

\[ \partial_t h + \nabla \cdot q = w \quad (x \in \Omega \subset \mathbb{R}^2, \ t > 0) \]
\[ h|_{t=0} = h_0, \ h|_{\partial \Omega} = h_0|_{\partial \Omega} \]

where
\[ y = h(x, t) \] – pile surface
\[ w(x, t) \geq 0 \] – source intensity
\[ h_0(x) \] – support surface,
\[ q(x, t) \] – horizontal projection of sand surface flux

A constitutive relation for the flux \( q \) is needed. Suppose the support has no steep slopes:

\[ |\nabla h_0| \leq k = \tan \alpha_r, \]

where \( \alpha_r \) is the sand angle of repose.
**const. relation: surface flux**

- flux $q$ is towards the steepest decent of $h$
- $|\nabla h| \leq k$ – no oversteep slopes
- $|\nabla h| < k \quad \rightarrow \quad q = 0$ – no flux on subcritical slopes
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We can also write $q = -m \nabla h$, where $m \in f(|\nabla h|)$ is an auxiliary unknown (Lagrange multiplier for the constraint $|\nabla h| \leq k$)

$$\partial_t h - \nabla \cdot \{m(|\nabla h|) \nabla h\} = w$$
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Two variational formulations can be derived:
- **primal** for sand surface $h$
- **dual** for sand surface flux $q$
primal variational formulation

Assume, for simplicity, $h_0|_{\partial \Omega} = 0$.
Let $K = \{ \eta \in H^1_0(\Omega) : |\nabla \eta| \leq k \}$.
Then $h(., t) \in K$ and for any $\eta \in K$

$$(\partial_t h - w, \eta - h) = -(m \nabla h, \nabla (\eta - h))$$

$$(m, k^2 - \nabla h \cdot \nabla \eta) \geq 0.$$ 

This eliminates $m$ and also the dual variable $q$.
We arrived at a variational inequality for $h$
(P. 86,94,96; Evans et al. 96,97).
primal formulation: v.i. for $h$

Find $h \in L^\infty(0, T; K) \cap H^1(0, T; L^2(\Omega))$ such that $h|_{t=0} = h_0$ and

$$(\partial_t h - w, \eta - h) \geq 0 \quad \forall \eta \in L^\infty(0, T; K).$$

Let the source $w \in L^2(\Omega_T)$ and support $h_0 \in K$. Then there exists a unique solution $h$.

Furthermore, $h$ solves the v.i. if and only if there exists a Lagrange multiplier $m$ such that \{h, m\} is a solution to a weak formulation of the sand model.
weak formulation of the model

Find \( h \in L^\infty(0, T; K) \bigcap H^1(0, T; L^2(\Omega)) \)
and \( m \in [L^\infty(\Omega_T)]^* \) such that \( h|_{t=0} = h_0 \) and
\[
\int_0^T (\partial_t h - w, \eta) \, dt + \langle m, \nabla h \cdot \nabla \eta \rangle_{L^\infty(\Omega_T)} = 0 \quad \forall \eta \in L^\infty(0, T; W^{1,\infty}(\Omega)),
\]
\[
\langle m, \phi \rangle_{L^\infty(\Omega_T)} \geq 0 \quad \forall \phi \in L^\infty(\Omega_T) \text{ with } \phi \geq 0,
\]
\[
\langle m, |\nabla h|^2 - k^2 \rangle_{L^\infty(\Omega_T)} = 0.
\]
Here \([ \cdot ]^*\) denotes the corresponding dual space and \( \langle \cdot, \cdot \rangle_V \) the duality pairing on \([V]^* \times V\).
Known analytical solutions: build up of piles on flat open platforms ($h_0 = 0$). Agree with exp., Puhl 92.

1. Point source, $w = \delta(x - x_0)$.

The conical pile with critical slopes grows until its base touches the support boundary. Then there appears a runway connecting the cone apex with the boundary and the pile growth stops: all additional sand follows the runway and leaves the system.

Both $h$ and $q$ are known; $q \sim 1/r$ before the runway appears, then a measure along the runway.
Known analytical solutions: build up of piles on flat open platforms ($h_0 = 0$). Agree with exp., Puhl 92.

1. Point source, $w = \delta(x - x_0)$.

Conical pile + runway.

2. Distributed positive source, $w(x) > 0$:

The final stationary shape of the pile is $h(x) = k_{\text{dist}}(x, \partial \Omega)$. 
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3. General distributed source: $w \geq 0$:
   The stationary solution is given by an integral representation formula for $m$;
   this determines a unique flux $q$.
   $h$ is unique in spt($m$) where $h = k_{\text{dist}}(x, \partial \Omega)$.

Cannarsa and Cardaliaguet, 04.
approximation of v.i. for \( h \)

**Approaches:**

- Constraint optimization at each time layer
  P. 93, Finzi Vita and Falcone 05,06
- \( p \)-Laplacian equations with \( p \to \infty \)
  Evans et al. 99, Barrett and P. 00

**Results:**
Reasonable convergence for the primal variable, \( h \).
three examples
Disadvantage of the primal formulation: difficult to find the dual variable,

\[ q = -m(|\nabla h|) \nabla h, \]

because \( m \) is a multivalued function.

Typical also of other critical state problems (superconductivity, plasticity)
dual formulation: v.i. for $q$

Let $\{h, q\}$ satisfy the constitutive relation. For any test field $\psi$, 

$$
\nabla h \cdot (\psi - q) \geq -|\nabla h||\psi| - \nabla h \cdot q \\
= -|\nabla h||\psi| + k|q| \geq -k|\psi| + k|q|.
$$

Hence

$$
(\nabla h, \psi - q) \geq k \int_{\Omega} |q| - k \int_{\Omega} |\psi|,
$$

Since $h|_{\partial\Omega} = h_0|_{\partial\Omega} = 0$,

$$
-(h, \nabla \cdot \{\psi - q\}) \geq k \int_{\Omega} |q| - k \int_{\Omega} |\psi|.
$$
Using mass balance equation to eliminate $h$,

$$h = h_0 + \int_0^t w \, dt - \nabla \cdot \{ \int_0^t q \, dt \},$$

we get

$$\left( \nabla \cdot \left\{ \int_0^t q \, dt \right\} - f, \nabla \cdot \{ \psi - q \} \right)$$

$$+ k \int_\Omega |\psi| - k \int_\Omega |q| \geq 0 \quad \forall \psi$$

where $f = h_0 + \int_0^t w \, dt$. 
variational inequality for $q$

Find $q(., t) \in V$ such that for all $\psi \in V$

$$\left( \nabla \cdot \left\{ \int_0^t q \, dt \right\} - f, \nabla \cdot \{ \psi - q \} \right)$$

$$+ k \int_\Omega |\psi| - k \int_\Omega |q| \geq 0$$

Here $V = \{ \psi \in [\mathcal{M}(\Omega)]^2 : \nabla \cdot \psi \in L^2(\Omega) \}$, $\mathcal{M}(\Omega)$ is the Banach space of bounded Radon measures and $f = h_0 + \int_0^t w \, dt$.

Existence:
for $w \in L^2$; via convergence of solutions to regularized discretized problems, Barrett and P. 06
approximation of v.i. for $q$

Both $q$ and $h$ can be found:

1. **Flux using the dual formulation:**
   - Smoothing the functional using
     $$|q|_\varepsilon = (|q|^2 + \varepsilon^2)^{1/2}$$
   - Discretization of the regularized equation in time and employment of divergence-conforming **Raviart-Thomas fe** of lowest order with vertex sampling on the nonlinear term;
   - Successive over relaxation.

2. **Pile surface via the surface flux:**
   $$h = h_0 + \int_0^t w \, dt - \nabla \cdot \int_0^t q \, dt$$
   is proved to be a (unique) solution of v.i. for $h$. 
point source
distributed source

Triangular support,

\[ w = \begin{cases} 
1 & \text{inside ellipse,} \\
0 & \text{outside} 
\end{cases} \]

The pile grows, then stabilizes when all points in \( \text{spt}(w) \) are connected to the border by a transport ray.
distributed source
impermeable wall and obstacle

Pile on a support partly surrounded by a wall. **Left:** the domain (white region) and the impermeable wall (grey regions); \( w = 1 \) inside the white circle and \( w = 0 \) outside. Computed are **middle:** the final stationary pile surface \( h \); **right:** levels of the corresponding stationary surface flux, \( |q| \).
variational model revisited

Generalizations:

• Supports with steep slopes (quasivariational inequality)
• Lakes and river networks on a given relief (qvi, $k = 0$)
• Model for inhomogeneous materials (segregation in size)
• Avalanches as instantaneous slides
• Critical-state models in type-II superconductivity
• $L^1$ Monge-Kantorovich problems ($\int_\Omega w = 0, \ t \to \infty$)

Limitations:

• Not all avalanches are slides
• Small pile shapes are somewhat different
• Could not be used in a model for aeolian ripples...
**BCRE model**

Derived by BCRE (Bouchaud, Cates, Ravi Prakash, and Edwards 94,95) to describe avalanches upon a close-to-critical slope; Simplified and studied by De Gennes et al. 95,97. BCRE equations are written for two variables, sand surface $h$ and effective thickness of rolling layer $R$:

\[
\partial_t h = \Gamma[h, R],
\]

\[
\partial_t R + \nabla \cdot (\nu R) = w - \Gamma[h, R]
\]

Here $\Gamma[h, R]$ accounts for mass exchange between rolling layer and the pile, $\nu$ - horizontal projection of rolling grains velocity, $w \geq 0$ - source intensity.
Constitutive relations in 2d case and for arbitrary slope angle (P. 99):

\[ \mathbf{v} = -\mu \nabla h, \quad \Gamma[h, R] = \gamma R \left( 1 - \frac{|\nabla h|^2}{k^2} \right). \]

Here \( k = \tan \alpha_r \), \( \mu \) and \( \gamma \) - characterize, resp., the mobility of a dislodged grain on the pile surface and the rate at which these grains are trapped and absorbed into the pile.
generalized BCRE equations

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Slightly different relations: Hadeler and Kuttler 99
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What are the correct relations? Which model to use?
generalized BCRE equations

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What are the correct relations? Which model to use? Relation details are less important than scaling!
three length scales:

- mean thickness of the rolling grains layer, \( L_R \sim \bar{w}/\gamma \) where \( \bar{w} \) is the mean source rate
- mean path of a rolling grain before it is trapped, \( L_P \sim \mu/\gamma \)
- pile size \( L \).

Scaling:

\[
x' = \frac{1}{L} x, \quad h' = \frac{1}{L} h, \quad R' = \frac{1}{L_R} R,
\]

\[
w' = \frac{1}{\bar{w}} w, \quad t' = \frac{\bar{w}}{L} t
\]
dimensionless BCRE model

\[ \partial_t h = R \left( 1 - \frac{|\nabla h|^2}{k^2} \right), \]

\[ \frac{L_R}{L} \partial_t R - \frac{L_P}{L} \nabla \cdot (R \nabla h) = w - R \left( 1 - \frac{|\nabla h|^2}{k^2} \right) \]
### Dimensionless BCRE Model

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\]

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\frac{L_R}{L} \partial_t R - \frac{L_P}{L} \nabla \cdot (R \nabla h) = w - R \left( 1 - \frac{\left| \nabla h \right|^2}{k^2} \right)
\]

Typically, \( L_R \ll L_P < L \). Two cases:

1. \( L_R \ll L_P, \quad L_P \simeq L \). Quasistationary equation for \( R \) (sand ripples, P. 99);

2. \( L_R \ll L_P \ll L \). Long-scale limit of BCRE equations (P. and Zaltzman 01,03).
large piles: $L_R \ll L_P \ll L$

Physically, it is clear that for large piles BCRE model should be similar to the variational model:

Although BCRE model permits grains to roll down any slope, the particles are quickly stopped except for the close-to-critical slopes. Their path is short comparing to the pile size. In the variational model these paths are neglected.

Mathematically, the convergence is not easy to prove. It was proved only for time-discretized versions of the two problems, P. and Zalzman 03.
long scale limit: \( L_R \ll L_P \ll L \)

- denote \( \nu = \frac{L_P}{L} \) and \( m = \nu R \);
- assume \( L_R/L = \nu \lambda(\nu) \), where \( \lambda(\nu) \to 0 \) as \( \nu \to 0 \);
- add small diffusion to regularize the model.

This yields:

\[
\partial_t h = \frac{m\psi(|\nabla h|^2)}{\nu} + \varepsilon_h \Delta h
\]

\[
\lambda \partial_t m - \nabla \cdot (m \nabla h) = w - \frac{m\psi(|\nabla h|^2)}{\nu} + \varepsilon_m \Delta m.
\]

Here \( \psi(u) = 1 - u/k^2 \), \( \varepsilon_h(\nu) \) and \( \varepsilon_m(\nu) \) vanish as \( \nu \to 0 \). Initial and boundary conditions are added; we assume \( m(x, 0) \geq 0 \) and \( |\nabla h(x, 0)| \leq k \).
convergence for time-discretized problems

$$\frac{h^{j+1} - h^j}{\Delta t} = \frac{m^{j+1} \psi(|\nabla h^{j+1}|^2)}{\nu} + \epsilon_h \Delta h^{j+1},$$

$$\lambda \frac{m^{j+1} - m^j}{\Delta t} - \nabla \cdot (m^{j+1} \nabla h^{j+1}) + \frac{m \psi(|\nabla h^{j+1}|^2)}{\nu} = w^j + \epsilon_m \Delta m^{j+1},$$

$$\partial_n h^{j+1} |_{\Gamma} = 0, \quad \partial_n m^{j+1} |_{\Gamma} = 0, \quad h^0 = h_0(x), \quad m^0 = m_0(x),$$

where \( j = 0..N - 1, \Delta t = \frac{T}{N} \). Denote

$$C = \|w\|_{C(0,T;L_{\infty}(\Omega))} T + \|h_0\|_{C(\Omega)} + \|m_0\|_{C(\Omega)} + kdiam(\Omega).$$

**Theorem.** Let functions \( \lambda(\nu), \epsilon_h(\nu), \epsilon_m(\nu) \) tend to zero as \( \nu \to 0 \) so that also \( \epsilon_m(\nu)/\epsilon_h(\nu) \to 0 \) and

$$k^2 - \frac{\lambda(\nu)C}{\Delta t} - \frac{\epsilon_m(\nu)C}{\epsilon_h(\nu)\Delta t} > \delta > 0.$$ 

Then the solutions \( h^{j+1} \) to the discretized BCRE problem converge as \( \nu \to 0 \) to the solutions \( \tilde{h}^{j+1} \) of the following variational inequalities \( (j = 0..N - 1, \tilde{h}^0 = h_0(x)) \):

$$\tilde{h}^{j+1} \in K : \left( \frac{\tilde{h}^{j+1} - \tilde{h}^j}{\Delta t} - w^j, \varphi - \tilde{h}^{j+1} \right) \geq 0, \forall \varphi \in K,$$
large and small conical piles

Growth of a small polenta pile, Alonso and Herrmann, 96. Note the curved tails near the pile bottom. As the pile grows, these tails remain about tens of grain diameters long. Hence the tails of a large pile is difficult to notice.
asymptotics of BCRE equations

\[ \epsilon_h = \epsilon_m = 0, \lambda = 0.1 \nu. \]

(a) small pile, \( \nu = 0.2 \). (b) large pile, \( \nu = 0.01 \); dashed lines – solution of variational inequality. For small \( \nu \) the problem is stiff (\( 10^5 \) time steps).
filling a silo

Sreadily translated solution to BCRE equations:

\[ h'_x = \frac{k}{\nu k^2 + \sqrt{(\nu k^2)^2 + 1}} \]

for \( 0 < x < L/2 \).

- \( h'_x \to 0 \) as \( x \to 0 \) for any \( \nu \): a tail;
- \( h'_x \to k \) as \( \nu \to 0 \) for all \( 0 < x < L/2 \): solution to v.i.
Conclusions

- BCRE and variational models are related and describe the pile surface dynamics on different spatio-temporal scales.

- BCRE-type models may be useful for simulating the fast processes, e.g., the initiation, spreading, and settling down of an avalanche.

- To describe the much slower dynamics of the mean shape of a pile, the model may often be simplified by employing a quasistationary equation for the rolling grains. Such a model may be used for small piles or features (sand ripples, tails near the pile bottom, etc.)
Conclusions

• On the long spatiotemporal scale (large piles, dunes, etc.) the BCRE-type models are inefficient. Their long-scale limit, the variational model of pile growth, is then more appropriate for simulations: although dunes can be covered by ripples, one does not need these details in a model of dune evolution.

• Two variational formulations of this model can be used in numerical simulations; the dual (flux) formulation allows to efficiently determine both the primal and dual variables.
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Thank you!