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ON THE APPROXIMATION OF THE DYNAMICS OF SANDPILE SURFACES

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Abstract: We consider a continuous model for sandpile surface dynamics and show that in the long-scale limit the finite-difference approximation in time of this model converges to a discretized evolutionary variational inequality with gradient constraint.

I – Two continuous models for pile surface dynamics

Recent much interest to the physics of the granular state was caused, in particular, by two related and only partially understood phenomena, the multiplicity of metastable pile shapes and occurrence of avalanches upon the pile surface. In this work we consider two models proposed several years ago to account for these salient properties of sandpile surface dynamics: the BCRE equations [1, 2] and a model in the form of a variational inequality [3, 4, 5, 6, 7]. It turns out that these two different models are related and describe the surface dynamics on different spacio-temporal scales.

Let us start with the BCRE model (Bouchaud, Cates, Ravi Prakash, and Edwards [1, 2]). The model, formulated by the authors for the one-dimensional case, was written for two basic variables: the pile surface $h(x,t)$ and the effective thickness of thin surface layer of rolling particles $R(x,t)$. Later, de Gennes [8, 9] simplified the BCRE model by assuming that in many situations the diffusion terms may be omitted. The BCRE model and its modifications have been used to simulate different aspects of pile surface dynamics (see, e.g., [8, 9, 10, 11] and the references therein). Analytical solutions to simplified one-dimensional BCRE equations can often be found by the methods proposed in [12, 13].

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Full-dimensional formulation of the BCRE model without diffusion can be written as follows:

$$\partial_t h = \Gamma[h, R], \quad \partial_t R + \nabla \cdot (\mathbf{v} R) = w - \Gamma[h, R] \qquad (x \in \Omega \subset \mathbb{R}^2,\ 0<t<T)\ . \tag{1}$$

Here the term $\Gamma[h, R]$ accounts for the conversion of rolling grains into immobilized ones, $\mathbf{v}$ is the horizontal projection of rolling grains velocity, and $w(x, t) \geq 0$ is the source intensity. The constitutive relations determining $\mathbf{v}$ and Γ can be chosen (see [14, 15]) as

$$\mathbf{v} = -\mu \, \nabla h\,, \qquad \Gamma = \gamma \, R \left(1 - \frac{|\nabla h|^2}{k^2} \right), \tag{2}$$

where k is the internal friction coefficient of the granular material (tangent of the repose angle), the coefficients μ and γ characterize, respectively, the mobility of a dislodged grain on the pile surface and the rate at which such grains are trapped and absorbed into the motionless bulk.

It is convenient to rewrite the equations (1)–(2) in dimensionless form. Let us define three characteristic length scales:

- typical thickness of the rolling grain layer, $L_R = \overline{w}/\gamma$, where $\overline{w}$ is the characteristic (mean) source intensity;
- mean path of a rolling particle before it is trapped strongly depends on the slope steepness but, for a fixed subcritical slope, is proportional to the ratio $L_P = \mu/\gamma$ characterizing the competition between rolling and trapping;
- the pile size L.

Rescaling the variables,

$$x' = \frac{1}{L}\, x\,, \quad h' = \frac{1}{L}\, h\,, \quad R' = \frac{1}{L_R}\, R\,, \quad w' = \frac{1}{\overline{w}}\, w\,, \quad t' = \frac{\overline{w}}{L}\, t\ ,$$

we obtain

$$\partial_t h = R\, \psi(|\nabla h|^2)\,, \qquad \frac{L_R}{L}\, \partial_t R - \frac{L_P}{L}\, \nabla \cdot (R\, \nabla h) = w - R\, \psi(|\nabla h|^2)\ , \tag{3}$$

where

$$\psi(u) = 1 - \frac{u}{k^2}\ . \tag{4}$$

Typically, $L_R \ll L_P < L$. The coefficient L_R/L is very small and it is often possible to omit the corresponding term and use a quasistationary equation for the rolling layer, e.g., if one wishes to describe the slow macroscopic pile surface dynamics and not the details of an avalanche initiation and run off. Such a simplification can be quite helpful (see, e.g., a model for Aeolian sand ripples [14]).

The second coefficient, L_P/L, may be significant for small piles, like sand ripples, but becomes small too for large piles. Our aim here is to study the long-scale limit of the BCRE equations and to show that in such a limit solution of these equations h tends to the solution of the variational inequality for h proposed as a macroscopic pile growth model by Prigozhin [3, 4, 5] and, independently, by Aronsson, Evans, and Wu [6, 7]. If the pile is built up on a rigid support $y = h_0(x)$ that has no overcritical slopes, i.e., $|\nabla h_0| \le k$, and the domain Ω is bounded by a vertical wall, the inequality can be written as follows,

$$\begin{aligned} h \in K\colon \quad & (\partial_t h - w,\, \varphi - h) \ge 0, \quad \forall\, \varphi \in K\ , \\ & h|_{t=0} = h_0\ , \end{aligned} \tag{5}$$

where $K = \{\varphi(x) \in H^1(\Omega)\colon\ |\nabla \phi| \le k\}$.

In [3, 4, 5], this inequality has been be obtained as a variational formulation for the following "physical" model written for two dependent variables, h and m,

$$\begin{aligned} & \partial_t h - \nabla\cdot(m\,\nabla h) = w\ , \\ & m(x,t) \ge 0\ , \\ & |\nabla h(x,t)| \le k\ , \\ & |\nabla h(x,t)| < k \implies m(x,t) = 0\ , \\ & h|_{t=0} = h_0\ , \qquad m\,\partial_n h|_\Gamma = 0\ , \end{aligned} \tag{6}$$

where m turns out to be a Lagrange multiplier related to the slope constraint $|\nabla h| \le k$. In this model the material flows upon the pile surface only if the slope is critical: the flux $-m\nabla h$ is zero for all subcritical slopes. Although surface flow upon subcritical slopes is permitted in the BCRE model, for large piles the path of a rolling particle is negligible comparing to the pile size for all except the very close-to-critical slopes. One may, therefore, expect that, although the BCRE model can be more appropriate on a short (mesoscopic) spacio-temporal scale, for large piles the two models become close.

It can be noted also that the variational inequality is much simpler than the BCRE model. Indeed, the variational inequality has a unique solution [5] which even in some two-dimensional cases can be found analytically [4]; in the general case numerical solutions are not difficult to obtain [16]. The BCRE equations are mathematically more complicated and were solved analytically only for the simplest piles with close-to-critical slopes. For large-scale problems these equations become stiff [15] and this complicates their numerical solution.

Although simulations [15] confirm the convergence statement above, we have been able to prove rigorously only a weaker result: solution to a finite-difference approximation in time of a regularized BCRE model converges to the corresponding solution of the discretized variational inequality.

II – The long-scale limit of BCRE model

Let us denote $\nu = L_P/L$ and study the $\nu \to 0$ behavior of the BCRE model. Since $L_R \ll L_P$, we assume L_R/L is $o(\nu)$ and set $L_R/L = \nu\lambda(\nu)$, where λ tends to zero as $\nu \to 0$. Let us also introduce a new variable, $m = \nu R$, and add small diffusion to regularize the model:

$$\partial_t h = \frac{m\,\psi(|\nabla h|^2)}{\nu} + \varepsilon_h \Delta h \ , \tag{7}$$

$$\lambda\,\partial_t m - \nabla\cdot(m\,\nabla h) = w - \frac{m\,\psi(|\nabla h|^2)}{\nu} + \varepsilon_m \Delta m \ . \tag{8}$$

Here $\varepsilon_h(\nu), \varepsilon_m(\nu)$ vanish as $\nu \to 0$. We note that, although small diffusion may be physically meaningful and has been included into the original BCRE formulation, here we introduce it merely as a parabolic regularization of the first order partial differential equations convenient for analyzing the model's behavior at $\nu \to 0$. Let us specify the initial and boundary conditions, e.g.,

$$\begin{aligned} h|_{t=0} = h_0(x) \geq 0\,, \quad & m|_{t=0} = m_0(x) \geq 0 \ , \\ \partial_n h|_\Gamma = 0\,, \quad & \partial_n m|_\Gamma = 0 \ . \end{aligned} \tag{9}$$

Remark. The condition $h_0(x) \geq 0$ can always be achieved for a bounded function $h_0(x)$ by shifting $h \to h - \min_\Omega h_0$; non-negativeness of m is a physical condition. □

Below, we consider the discretized BCRE problem,

$$\frac{h^{j+1} - h^j}{\tau} = \frac{m^{j+1}\,\psi(|\nabla h^{j+1}|^2)}{\nu} + \epsilon_h \Delta h^{j+1} \ , \tag{10}$$

$$\lambda\,\frac{m^{j+1} - m^j}{\tau} - \nabla\cdot(m^{j+1}\,\nabla h^{j+1}) + \frac{m^{j+1}}{\nu}\,\psi(|\nabla h^{j+1}|^2) = w^j + \epsilon_m \Delta m^{j+1}\,, \tag{11}$$

$$\partial_n h^{j+1}|_\Gamma = 0\,, \quad \partial_n m^{j+1}|_\Gamma = 0 \ , \tag{12}$$

$$h^0 = h_0(x)\,, \quad m^0 = m_0(x) \ , \tag{13}$$

where $j = 0..N-1$, $\tau = \frac{T}{N}$, and w^j means $w|_{t=j\tau}$.

Theorem 1. *Let $\Omega \subset R^2$ be a convex bounded region with a piecewise smooth boundary Γ, $0 \le w \in C(0,T;L^\infty(\Omega))$, $0 \le m_0 \in C^\alpha(\overline{\Omega})$, $h_0 \in H^1(\Omega)$, and $|\nabla h_0(x)| \le k$ in Ω. Then there exists a solution (h^j, m^j), $j = 1..N$, to problem (10)–(13), such that $h^j, m^j \in C^{2,\alpha}(\overline{\Omega})$ and the following estimates hold uniformly in $\nu, \lambda, \varepsilon_h, \varepsilon_m > 0$*

$$|\nabla h^j| \le k\,, \quad m^j \ge 0\,, \tag{14}$$

$$0 \le h^j < C_0 = \|w\|_{C(0,T;L^\infty(\Omega))}\,T + \|h_0\|_{C(\overline{\Omega})} + \|m_0\|_{C(\overline{\Omega})} + k\,\mathrm{diam}(\Omega)\,, \tag{15}$$

$$\tau \sum_{j=0}^{N} \int_\Omega \frac{m^j\,\psi(|\nabla h^j|^2)}{\nu}\,dx \;<\; \|w\|_{C(0,T;L^\infty(\Omega))}\,T + \|m_0\|_{L^1(\Omega)}\,. \tag{16}$$

If, furthermore,

$$k^2 - \frac{\lambda\, C_0}{\tau} - \frac{\varepsilon_m\, C_0}{\varepsilon_h\,\tau} \;>\; \delta \tag{17}$$

for some $\delta > 0$, then

$$\int\limits_\Omega m^j\,dx \;<\; \frac{C_1}{\tau} \tag{18}$$

with a positive constant C_1.

Remark. Hereinafter the absolute constants are denoted by C, the constants depending on the model parameters by M. ▫

Proof: Let us redefine the function ψ as

$$\psi(u) \stackrel{\text{def}}{=} \begin{cases} 1 - \dfrac{u}{k^2}, & 0 \le u \le k^2\,, \\ 0, & u > k^2\,, \end{cases} \tag{19}$$

and assume at first that Γ is a twice-differentiable curve, $w \in C(0,T;C^\alpha(\overline{\Omega}))$ and $h_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. To prove the existence of a solution (h^{j+1}, m^{j+1}), $j \ge 0$, we assume that for $l = 1..j$ the solutions (h^l, m^l) exist and satisfy the estimates (14)–(18). Let us define the set $A \subset C^{1,\alpha}(\overline{\Omega})$ as

$$\begin{aligned} A \stackrel{\text{def}}{=} \Big\{h(x)\colon\; & h \in C^{1,\alpha}(\overline{\Omega}),\;\; \|h\|_{C^{1,\alpha}(\overline{\Omega})} \le \widetilde{M},\;\; h \ge 0, \\ & |\nabla h| \le k \;\text{ in } \overline{\Omega},\; \partial_n h|_\Gamma = 0\Big\}\,. \end{aligned} \tag{20}$$

(The constant $\widetilde{M}$ will be specified below.) For any $h \in A$ we define the function $m(x)$ as a solution of an auxiliary boundary value problem,

$$\lambda\,\frac{m - m^j}{\tau} - \nabla\cdot(m\,\nabla h) \;=\; w^j - \frac{m}{\nu}\,\psi(|\nabla h)|^2) + \epsilon_m\,\Delta m\,, \tag{21}$$

$$\partial_n m|_\Gamma \;=\; 0\,. \tag{22}$$

The existence and uniqueness of a solution to (21)–(22), $m \in C^{2,\alpha}(\overline{\Omega})$, follows directly from the general theory of elliptic partial differential equations (see [17], [18]). It also follows that the operator $P(h) = m$ is a continuous operator from A to $C^{2,\alpha}(\overline{\Omega})$. To estimate m, let us rewrite the equation (21) in the form

$$\epsilon_m \nabla\cdot\left(\exp\left(-\frac{h}{\varepsilon_m}\right)\nabla q\right) - \lambda\,\frac{q\exp\left(-\frac{h}{\varepsilon_m}\right) - m^j}{\tau} - \frac{q\exp\left(-\frac{h}{\varepsilon_m}\right)}{\nu}\,\psi(|\nabla h|^2) = \tag{23}$$
$$= -w^j \,\le\, 0\ ,$$

where $q(x) \overset{def}{=} m(x)\exp(h/\varepsilon_m)$. Applying the maximum principle to this equation we obtain

$$0 \,\le\, m(x) \,\le\, M \,=\, \left(\max(m^j) + \frac{\tau}{\lambda}\max(w)\right)\,\exp\left(\frac{k\,\mathrm{diam}(\Omega)}{\varepsilon_m}\right)\ . \tag{24}$$

Let us now define the function $\widehat{h}(x)$ as a solution to the boundary value problem

$$\frac{\widehat{h} - h^j}{\tau} \,=\, \frac{m}{\nu}\,\psi(|\nabla\widehat{h}|^2) + \varepsilon_h\,\Delta\widehat{h}\ , \tag{25}$$

$$\partial_n\widehat{h}|_\Gamma \,=\, 0\ . \tag{26}$$

Existence of a solution to (25)–(26), $\widehat{h} \in C^2(\overline{\Omega})$, follows from the well-known results [17] on quasi-linear elliptic equations.

Lemma 1. *If the conditions of Theorem 1 hold and ψ is defined by (19) then $|\nabla\widehat{h}| \le k$.*

Proof: To prove Lemma 1 we use standard methods [17], [18]. Applying operator ∇ to equation (25) and multiplying the equation by $\nabla\widehat{h}$ we obtain

$$\begin{aligned}\frac{|\nabla\widehat{h}|^2 - \nabla h^j\cdot\nabla\widehat{h}}{\tau} \,&=\, \frac{1}{\nu}\,\nabla m\cdot\psi(|\nabla\widehat{h}|^2)\,\nabla\widehat{h} \,+\, \frac{m}{\nu}\,\psi'(|\nabla\widehat{h}|^2)\,\nabla(|\nabla\widehat{h}|^2)\cdot\nabla\widehat{h}\\ &+\,\frac{\varepsilon_h}{2}\,\Delta(|\nabla\widehat{h}|^2) \,-\, \frac{\varepsilon_h}{2}(\widehat{h}^2_{x_1x_1} + 2\,\widehat{h}^2_{x_1x_2} + \widehat{h}^2_{x_2x_2}) \quad \text{in } \Omega\ .\end{aligned} \tag{27}$$

Since $\psi = \psi' = 0$ for $|\nabla\widehat{h}| > k$, the function $|\nabla\widehat{h}|^2$ cannot have an interior maximum greater than k^2. Let us assume that this function reaches the maximum value greater than k^2 at a point $x = (x_1, x_2) \in \Gamma$. Without the loss of generality, we assume that $(x_1, x_2) = (0, 0)$, the vector $(0, 1)$ is the outward normal direction to the boundary Γ at this point, and the boundary Γ can be locally represented as $x_2 = g(x_1)$ with $g(0) = 0$, $g'(0) = 0$.

In the vicinity of $(0,0)$ the boundary condition (26) can be written as follows

$$\widehat{h}_{x_2}(x_1, g(x_1)) - g'(x_1)\,\widehat{h}_{x_1}(x_1, g(x_1)) \;=\; 0 \;. \tag{28}$$

Differentiating (28) in x_1 and substituting $x_1 = 0$ yields

$$\widehat{h}_{x_2 x_1}(0,0) - g''(0)\,\widehat{h}_{x_1}(0,0) \;=\; 0 \;. \tag{29}$$

Since $|\nabla\widehat{h}|^2 > k^2$ in the vicinity of $(0,0)$, we can apply Hopf's lemma to the equation (27) and obtain

$$\begin{aligned} \partial_n |\nabla\widehat{h}|^2\Big|_{(0,0)} &= \partial_{x_2} |\nabla\widehat{h}|^2\Big|_{(0,0)} \\ &= 2\left[\widehat{h}_{x_1 x_2}(0,0)\,\widehat{h}_{x_1}(0,0) + \widehat{h}_{x_2 x_2}(0,0)\,\widehat{h}_{x_2}(0,0)\right] \;>\; 0 \;. \end{aligned} \tag{30}$$

Since $\widehat{h}_{x_2}(0,0) = 0$, substitution of (29) into (30) yields $g''(0)\widehat{h}^2_{x_1}(0,0) > 0$ and so $g''(0) > 0$, which contradicts the convexity of Ω. ∎

Having proved that $|\nabla\widehat{h}| \leq k$ we can return to the original definition of ψ given by (4): we see that correction (19) makes no difference.

Using the maximum principle, we deduce

$$0 \;\leq\; \widehat{h} \;\leq\; \max\left(h^j + \tau\,\frac{m}{\nu}\right) \tag{31}$$

and, consequently,

$$|\Delta\widehat{h}| \;\leq\; \frac{2}{\varepsilon_h}\,\max\left(\frac{h^j}{\tau} + \frac{m}{\nu}\right) \;. \tag{32}$$

The estimates (32), (24) yield the boundedness of $\widehat{h}$ in $W^{2,p}(\Omega)$ for any $1 < p < \infty$ and, using the embedding theorems (see [18]), we obtain

$$\|\widehat{h}\|_{C^{1,\beta}(\overline{\Omega})} \leq M_1(\beta)\,, \qquad \forall\beta \in (0,1) \;. \tag{33}$$

Let us prove now the uniqueness and stability of the obtained solution. To do this we consider solutions $\widehat{h}_1$ and $\widehat{h}_2$ corresponding, respectively, to functions m_1 and m_2 in (25)–(26). Let us define $\overline{h}$ as $\widehat{h}_1 - \widehat{h}_2$ and $\overline{m}$ as $m_1 - m_2$. Then

$$\begin{aligned} \frac{\overline{h}}{\tau} &= \frac{\overline{m}}{\nu}\,\psi(|\nabla\widehat{h}_1|^2) - \frac{m_2}{\nu k^2}\,\nabla\overline{h}\cdot\nabla(\widehat{h}_1 + \widehat{h}_2) + \varepsilon_h\,\Delta\overline{h}\;, \\ \partial_n\overline{h}|_\Gamma &= 0\;. \end{aligned}$$

Applying the maximum principle (see [17]) to the function $\overline{h} - \max \frac{\tau}{\nu}|\overline{m}|$ we deduce the uniqueness of a solution to this problem as well as the following comparison result

$$\max|\overline{h}| \leq \frac{\tau}{\nu}\max|\overline{m}| \tag{34}$$

and, consequently,

$$|\Delta\overline{h}| \leq \frac{2}{\nu\,\varepsilon_h}\max\left(\frac{m_2}{k}|\nabla\overline{h}| + |\overline{m}|\right) .$$

The last inequality together with estimate (34) yield

$$|\Delta\overline{h}| \leq M_2 \max\overline{|m|} \tag{35}$$

for some positive constant M_2. Using the embedding theorems (see [18]) we establish the continuous dependence of the solution $\widehat{h}$ on the function m in the following sense:

$$\|\overline{h}\|_{C^{1,\alpha}(\overline{\Omega})} \leq M_3 \max|\overline{m}| \ . \tag{36}$$

Making use of (36), we deduce the continuity of the operator $Q(m) = \widehat{h}$ mapping $C^{2,\alpha}(\overline{\Omega})$ into $C^{1,\alpha}(\overline{\Omega})$.

Choosing the constant $\widetilde{M}$ in (20) as $\widetilde{M} = M_1(\alpha)$, see (33), and using the inequalities (33), (36), we prove that $W \stackrel{\text{def}}{=} Q \circ P$ is a continuous operator mapping the set A into a relatively compact subset. Making use of Schauder's theorem we establish the existence of a fixed point h. Together, $h^{j+1} = h$ and $m^{j+1} = P(h)$ make the solution to (10)–(13) satisfying (14) and this completes the proof of the existence result for all $j = 1..N$.

Lemma 2. *If the conditions of Theorem 1 hold then for all $j = 1..N$ the estimates (15)–(18) hold.*

Proof: Integrating equations (10) and (11), $j = 1..N-1$, over Ω, using integration by parts and summing up, we obtain

$$\lambda\int_\Omega (m^{j+1} - m_0)\,dx + \int_\Omega (h^{j+1} - h_0)\,dx \ = \ \tau\sum_{l=0}^{j}\int_\Omega w^l\,dx \ . \tag{37}$$

The equality (37) and estimate (14) yield the uniform boundedness of the integral $\int_\Omega h^{j+1}dx$. Since $|\nabla h^{j+1}| \leq k$ we obtain that $h^{j+1} \leq \|w\|_{C(0,T;L^\infty(\Omega))}\,T + \|h_0\|_{C(\Omega)} + \lambda\,\|m_0\|_{C(\Omega)} + k\,\mathrm{diam}(\Omega)$ and the statement (15) is proved (we assume $\lambda < 1$). Integrating equations (10), $j = 0..N-1$, over Ω, summing them up, and making use of the equality (37) we obtain (16).

Integrating equations (11), $j = 0..N-1$, over Ω, summing them up, and using the estimate (14) we find

$$\frac{\tau}{\nu} \sum_{l=0}^{j} \int_\Omega m^{l+1}\, \psi(|\nabla h^{l+1}|^2)\, dx \;\leq\; \tau \sum_{l=0}^{j} \int_\Omega w^l\, dx + \lambda \int_\Omega m_0\, dx, \quad j = 0..N-1\,. \tag{38}$$

Multiplying equations (11) on $h^{j+1}(x)$, $j = 0..N-1$, and integrating we find

$$\int_\Omega m^{j+1} |\nabla h^{j+1}|^2\, dx \;\leq\; C_0 \Big(\int_\Omega w^j dx + \frac{\lambda}{\tau} \int_\Omega m^j\, dx \Big) + \varepsilon_m \int_\Omega m^{j+1} \Delta h^{j+1}\, dx\ . \tag{39}$$

To estimate the integral $\int_\Omega m^{j+1} \Delta h^{j+1}\, dx$ let us multiply the equation (10) by m^{j+1} and integrate it again. Integration by parts yields

$$\varepsilon_m \int_\Omega m^{j+1} \Delta h^{j+1}\, dx \;\leq\; \frac{\varepsilon_m\, C_0}{\varepsilon_h\, \tau} \int_\Omega m^{j+1}\, dx\ . \tag{40}$$

Substituting (40) into (39) and summing up the equations we find

$$\sum_{l=0}^{j} \int_\Omega m^{l+1}\, |\nabla h^{l+1}|^2\, dx \;\leq\; C_0 \sum_{l=0}^{j} \int_\Omega \Big[w^l + \frac{\lambda}{\tau}\, m^l + \frac{\varepsilon_m}{\varepsilon_h \tau}\, m^{l+1} \Big]\, dx\ . \tag{41}$$

Finally, combining the inequalities (38), (41) with definition (4) we obtain the estimate

$$\Big(1 - \frac{\lambda\, C_0}{k^2\, \tau} - \frac{\varepsilon_m\, C_0}{k^2\, \varepsilon_h\, \tau}\Big) \sum_{l=0}^{j} \int_\Omega m^{l+1} dx \;\leq\; \Big(\nu + \frac{C_0}{k^2}\Big) \Big[\sum_{l=0}^{j} \int_\Omega w^l dx + \frac{\lambda}{\tau} \int_\Omega m_0\, dx \Big]. \tag{42}$$

and complete the proof of Lemma 2. ∎

Using Lemma 2 completes the proof of Theorem 1 in the case of a smooth boundary Γ, $w \in C(0,T; C^\alpha(\overline{\Omega}))$, and $h_0 \in C^{1,\alpha}(\overline{\Omega})$. To prove Theorem 1 for the general data (Jordan curve Γ, $w \in C(0,T; L^\infty(\Omega))$ and Lipschitz-continuous h_0) we approximate them by smoother data, use the estimates obtained, and go to the limit. ∎

Theorem 2. *Let functions $\lambda(\nu)$, $\varepsilon_h(\nu)$, $\varepsilon_m(\nu)$ tend to zero as $\nu \to 0$ so that also $\varepsilon_m(\nu)/\varepsilon_h(\nu) \to 0$ and the condition (17) holds uniformly in ν. Then the solutions h^{j+1} to problem (10)–(13) converge in $L^2(\Omega)$ as $\nu \to 0$ to the solutions $\widetilde{h}^{j+1}$ of the following variational inequalities for all $j = 0..N-1$*

$$\widetilde{h}^{j+1} \in K\colon \quad \Big(\frac{\widetilde{h}^{j+1} - \widetilde{h}^j}{\tau} - w^j,\, \varphi - \widetilde{h}^{j+1} \Big) \geq 0, \quad \forall \varphi \in K\ , \tag{43}$$

$$\widetilde{h}^0 = h_0(x)\ . \tag{44}$$

Remark. Existence and uniqueness of solutions to these inequalities follow from the well-known results on variational inequalities (see, e.g. [19], Th. 3.1). □

Proof: Let us consider first smoother test functions $\varphi \in K \cap C^2(\overline{\Omega})$ such that $\partial_n \varphi|_\Gamma = 0$. Multiplying the equation (11) by $\varphi - h^{j+1}$, integrating it over Ω, and using the equation (10) we find

$$
\begin{aligned}
&\lambda\Big(\frac{m^{j+1} - m^j}{\tau}, \varphi - h^{j+1}\Big) + \Big(\frac{h^{j+1} - h^j}{\tau} - w^j, \varphi - h^{j+1}\Big) + \\
&\quad + \varepsilon_h(\nabla h^{j+1}, \nabla\varphi - \nabla h^{j+1}) + (m^{j+1}\nabla h^{j+1}, \nabla\varphi - \nabla h^{j+1}) = \\
&= \varepsilon_m(m^{j+1}, \Delta\varphi) - \varepsilon_m(m^{j+1}, \Delta h^{j+1}) \\
&= \varepsilon_m(m^{j+1}, \Delta\varphi) - \frac{\varepsilon_m}{\varepsilon_h}\Big(\frac{h^{j+1} - h^j}{\tau}, m^{j+1}\Big) \\
&\quad + \frac{\varepsilon_m}{\nu\varepsilon_h}\Big(m^{j+1}\psi(|\nabla h^{j+1}|^2), m^{j+1}\Big) \\
&\geq \varepsilon_m(m^{j+1}, \Delta\varphi) - \frac{\varepsilon_m}{\varepsilon_h}\Big(\frac{h^{j+1} - h^j}{\tau}, m^{j+1}\Big) .
\end{aligned}
\tag{45}
$$

Since $\nabla h^{j+1} \cdot \nabla\varphi \leq k^2$,

$$
\begin{aligned}
\Big(m^{j+1}\nabla h^{j+1}, \nabla\varphi - \nabla h^{j+1}\Big) &= \int_\Omega m^{j+1}(\nabla h^{j+1} \cdot \nabla\varphi) - k^2 + k^2\psi(|\nabla h^{j+1}|^2)\,dx \\
&\leq k^2 \int_\Omega m^{j+1}\psi(|\nabla h^{j+1}|^2)\,dx \leq k^2\frac{C_2}{\tau}\nu ,
\end{aligned}
$$

due to the estimate (16). We use now the estimates (14)–(16) and (18) and obtain from the inequality (45) the estimate

$$
\Big(\frac{h^{j+1} - h^j}{\tau} - w, \varphi - h^{j+1}\Big) + k^2\frac{C_2}{\tau}\nu + \frac{C_3}{\tau^2}\Big(\lambda + \frac{\varepsilon_m}{\varepsilon_h} + \varepsilon_m\tau\Big) + C_4\varepsilon_h \geq 0
\tag{46}
$$

with some constants C_2, C_3, C_4. Making use of the uniform in ν boundedness of h^j in $H^1(\Omega)$, $j = 1..N$, we choose a subsequence $\nu_l \to 0$ as $l \to \infty$, such that $h^j_l \to \widetilde{h}^j$ in $L^2(\Omega)$, $\widetilde{h}^j \in K$, $j = 1..N$, as $l \to \infty$. Taking the limit $l \to \infty$ in (46) we see that the variational inequality (43) holds for any $\varphi \in K \cap C^2(\overline{\Omega})$, $\partial_n\varphi|_\Gamma = 0$. Noting the density of such test functions in K completes the proof of Theorem 2. ∎

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