

Solutions to Monge-Kantorovich equations as stationary points of a dynamical system

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Abstract

Solutions to Monge-Kantorovich equations, expressing optimality condition in mass transportation problem with cost equal to distance, are stationary points of a critical-slope model for sand surface evolution. Using a dual variational formulation of sand model, we compute both the optimal transport density and Kantorovich potential as $t \rightarrow \infty$ limit of evolving sand flux and sand surface, respectively.

1 Introduction

The Monge-Kantorovich equations,

$$f^+ - f^- = -\nabla \cdot (a\nabla u), \quad (1)$$

$$|\nabla u| \leq 1, \quad a \geq 0, \quad |\nabla u| < 1 \longrightarrow a = 0, \quad (2)$$

appear as a condition of optimality in the classical Monge-Kantorovich problem of optimal mass transportation with the cost function equal to the distance of transportation [1, 2, 3]. Here f^\pm are two given distributions of mass, a is the transport density, and u is the Kantorovich potential. Let $\mathcal{M}(R^n)$ be the set of bounded Radon measures on R^n , $\mathcal{M}_+(R^n)$ the subset of nonnegative measures, and $\text{Lip}_1(R^n)$ the set of real valued Lipschitz functions with Lipschitz constant not greater than 1. If $f^\pm \in \mathcal{M}_+(R^n)$ and satisfy $f^+(R^n) = f^-(R^n)$, there exist a transport density and a corresponding potential, $\{a, u\} \in \mathcal{M}_+(R^n) \times \text{Lip}_1(R^n)$, satisfying (1)-(2) in a weak sense, see [1]. If either f^+ or f^- are absolutely continuous with respect to Lebesgue measure in R^n the transport density a is unique and also absolutely continuous [1]. Obviously, the Kantorovich potential u is determined up to an additive constant. If the constant is fixed, the potential is still not unique outside $\text{spt}(a)$. Note that the system (1)-(2) arises also in a mass optimization problem [4].

A critical-slope model for sand surface evolution ([5], see also [6, 7])

$$f - \partial_t u = -\nabla \cdot (a \nabla u), \quad u|_{t=0} = u_0(x), \quad (3)$$

$$|\nabla u| \leq k, \quad a \geq 0, \quad |\nabla u| < k \longrightarrow a = 0, \quad (4)$$

can be regarded as a non-stationary version of the Monge-Kantorovich equations. Here $x \in R^2$, $u(x, t)$ is the evolving sand surface, $f(x, t)$ is the intensity of external source, k is the tangent of the sand angle of repose, $a(x, t)$ is the sand transport density (if $k = 1$) or proportional to it, and the initial surface u_0 satisfies the equilibrium constraint $|\nabla u_0| \leq k$.

Indeed, it can be proved that if the source intensity f is nonnegative, the term $\partial_t u$ is nonnegative too and, as was noted in [8], comparison of the two systems shows that at each time moment sand from the source is instantly incorporated into the bulk in a way that minimizes sand transport under the constraint $|\nabla u| \leq k$. The height function u plays here the role of Kantorovich potential.

In this note we explore an opposite approach to this similarity. Let the source intensity be a time-independent Radon measure, $f = f^+ - f^-$, such that $f^\pm \in \mathcal{M}_+(R^n)$ and satisfy $f^+(R^n) = f^-(R^n)$. Starting with an admissible initial condition u_0 , we seek a solution to Monge-Kantorovich equations as the $t \rightarrow \infty$ limit of solution to (3)-(4). To solve this latter evolutionary problem we employ the numerical algorithm [9] based on a dual variational formulation of the sand model written for flux of sand pouring down the evolving sand surface [10].

2 Variational formulations of sand model

To study the evolution of sand surface, it is convenient to exclude from equations (3)-(4) the transport density a , the Lagrange multiplier related to the constraint $|\nabla u| \leq k$, and rewrite the model as a variational inequality for u alone. For bounded domains and appropriate boundary conditions, the inequality was obtained in [5, 6]. Similar formulation of the initial value problem has been independently derived and studied in [7]:

$$u(\cdot, t) \in K : (\partial_t u - f, \varphi - u) \geq 0, \quad \forall \varphi \in K, \quad (5)$$

$$u|_{t=0} = u_0,$$

where $K = \{\varphi \in L^2(R^2) : |\nabla \varphi| \leq k \text{ a.e.}\}$. We assume both $u_0 \in K$ and f have compact supports. In this case $\text{spt}(u)$ remains compact for each $t > 0$, see [7].

The inequality (5) can be used for efficient computation of the evolving sand surface u . However, the dual variable, a , remains unknown and difficult to find. Such situation is typical also of other critical-state problems. Because of this reason the dual variational formulations [10], resembling mixed variational inequalities in elasto-plasticity [11] and allowing to find both variables simultaneously (see [9]), can be preferable. Having in mind the application to Monge-Kantorovich problem, we assume $x \in R^n$ and allow for non-constant coefficients $k = k(x) \geq 0$ to treat problems with, say, subregions through which mass transportation is forbidden or, on the contrary, where it is free of charge.

Suppose the pair $\{u, a\}$ satisfies the model relations (3)-(4). Then, for the horizontal projection of surface flux of sand $\mathbf{q} = -a \nabla u$ and arbitrary test field $\boldsymbol{\psi}$ we obtain

$$\nabla u \cdot (\boldsymbol{\psi} - \mathbf{q}) \geq -|\nabla u| |\boldsymbol{\psi}| - \nabla u \cdot \mathbf{q} = -|\nabla u| |\boldsymbol{\psi}| + k |\mathbf{q}| \geq -k |\boldsymbol{\psi}| + k |\mathbf{q}|.$$

Noting that support of u is bounded and integrating, we obtain $(\nabla u, \boldsymbol{\psi} - \mathbf{q}) \geq \phi(\mathbf{q}) - \phi(\boldsymbol{\psi})$, where $\phi(\mathbf{q}) = \int_{R^n} k|\mathbf{q}|$, and also $(\nabla u, \boldsymbol{\psi} - \mathbf{q}) = -(u, \nabla \cdot \{\boldsymbol{\psi} - \mathbf{q}\})$. Therefore,

$$\phi(\boldsymbol{\psi}) - \phi(\mathbf{q}) - (u, \nabla \cdot \{\boldsymbol{\psi} - \mathbf{q}\}) \geq 0.$$

Integrating now the balance equation (3) in time we get

$$u = u_0 + \int_0^t f dt - \nabla \cdot \int_0^t \mathbf{q} dt \quad (6)$$

and arrive at the mixed variational inequality written for the sand flux alone:

$$\mathbf{q}(\cdot, t) \in V : \left(\nabla \cdot \int_0^t \mathbf{q} dt - \mathcal{F}, \nabla \cdot \{\boldsymbol{\psi} - \mathbf{q}\} \right) + \phi(\boldsymbol{\psi}) - \phi(\mathbf{q}) \geq 0 \quad (7)$$

for any $\boldsymbol{\psi} \in V$. Here $\mathcal{F} = u_0 + \int_0^t f dt$ and we define

$$V = \{\boldsymbol{\psi} \in [\mathcal{M}(R^n)]^n : \nabla \cdot \boldsymbol{\psi} \in L^2(R^n)\}.$$

Several comments about the variational formulation (7) can be necessary.

The problem is not coercive in the usual Sobolev spaces. Nevertheless, under suitable conditions on the source function f , existence of a solution can be established [12].

Since the divergence of flux \mathbf{q} belongs to L^2 , we expect the continuity of normal component of sand flux; the tangential component of \mathbf{q} can be discontinuous (this corresponds to continuity of transport density along the transport rays). To solve such a problem numerically, one should use the divergence-conforming finite elements that do not enforce any additional smoothness on the solution.

Suppose the flux has been found as a solution to (7). Obviously, the transport density is easily determined for $k > 0$ as $a = |\mathbf{q}|/k$. The surface u (potential in the Monge-Kantorovich problem) can be calculated by means of the equation (6).

3 Solution of Monge-Kantorovich equations

To approximate the inequality (7) numerically, we smoothed the non-differentiable functional ϕ by introducing $|\mathbf{q}|_\varepsilon = (|\mathbf{q}|^2 + \varepsilon^2)^{1/2}$, discretized the regularized equality problem in time, employed the divergence-conforming Raviart-Thomas finite elements of lowest order (see, e.g., [13]) and used vertex sampling on the nonlinear term. Resulting nonlinear algebraic systems were solved at each time level iteratively using a form of successive over-relaxation. We refer to [12] for details and make only several remarks related to efficient implementation of this algorithm to Monge-Kantorovich equations.

i) The needed solutions are obtained as the $t \rightarrow \infty$ limit of solutions to (7) with the time-independent sources satisfying $f(R^n) = 0$. To find this limit, we need not solve the nonlinear equations at each time level with high accuracy: only a few iterations are needed before the transition to a new time level. If, however, the Kantorovich potential is also of interest by some reason, the time step and the number of iterations should ensure accurate calculation of the integral in (6).

ii) Suppose that transportation through an open domain Ω is forbidden. To model this situation, it is possible to solve the problem in $R^n \setminus \Omega$ with $\mathbf{q}_n|_{\partial\Omega} = 0$. Another possibility (used in this work) is to solve also in Ω but set $k|_{\Omega} = \infty$. Then, if \mathbf{q} solves (7), the flux $\mathbf{q}|_{\Omega}$ must be zero. Since the normal component of flux is continuous, only tangential to domain boundary flux of mass is permitted: the optimal mass transportation rounds the obstacle.

iii) Numerical solution of (7) is possible only in a bounded domain and it is desirable to make the computational domain as small as possible. However, if the domain is not large enough, at some (depending on u_0) time moment $\text{spt}(\mathbf{q}(\cdot, t))$ can reach the domain boundary. To avoid any material loss through the boundary, we can surround the domain by an obstacle (see previous comment). On the other hand, we do not want the artificial obstacle to affect our solution. Since the transport set in Monge-Kantorovich problem (without obstacles) consists of straight transport rays that begin in $\text{spt}(f^+)$ and end in $\text{spt}(f^-)$, we can choose any computational domain containing the convex hull of $\text{spt}(f) = \text{spt}(f^+) \cup \text{spt}(f^-)$ and surround it by an obstacle. Our choice of domain (and u_0) can influence only the Kantorovich potential outside the establishing in $t \rightarrow \infty$ limit transport set, where this potential is not unique.

In all examples below, the computations were performed in a square surrounded by an obstacle (we set $k = 10^6$ in a thin border around the square). Matlab Partial Differential Equation Toolbox was used for generation of finite element meshes.

Example 1. Support of f^+ consists of two upper ellipses in which the density of f^+ is 1 (see Fig. 1). The mass from f^+ is to be uniformly distributed in the third ellipse below.

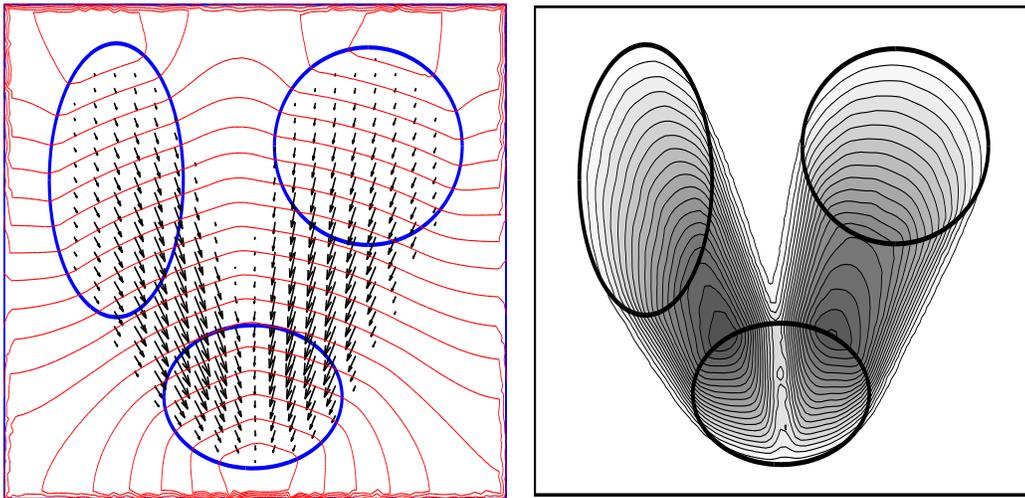


Figure 1: Solution to Monge-Kantorovich equations. Left: the optimal flux and levels of Kantorovich potential. Right: contour plot of $|\mathbf{q}|$.

Example 2. Let f^+ and f^- be uniformly distributed in upper and lower rectangles, respectively, the ellipse between them is an obstacle (see Fig. 2). In this case the transport density is not absolutely continuous: it becomes concentrated on part of the obstacle boundary (see also [14] for analysis of transport density in a sandpile growth problem

with an obstacle).

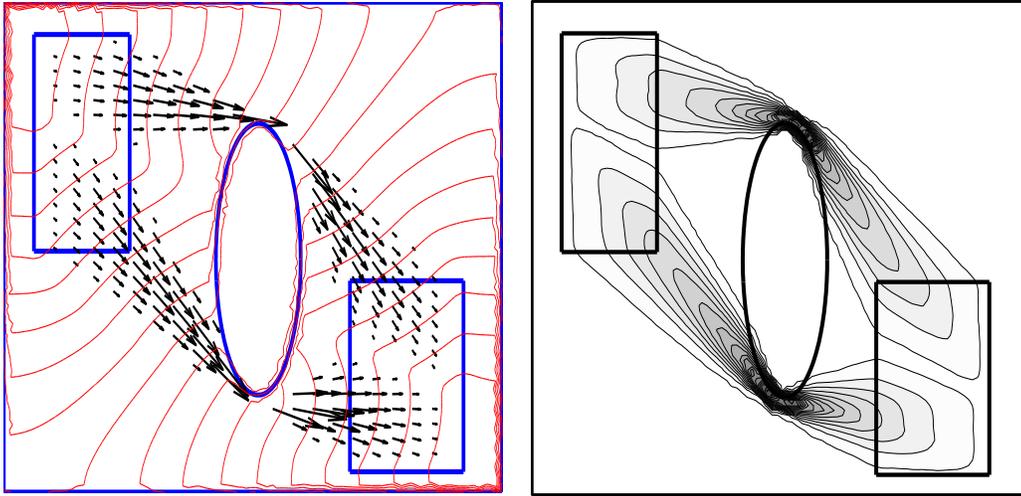


Figure 2: As in fig. 1; problem with an obstacle to mass transportation.

Example 3. Let f^+ and f^- be uniformly distributed in upper and lower rectangles, respectively, and the third polygon in Fig. 3 be a “highway” where the transportation cost is significantly reduced. To model this situation, we set $k = 0.01$ in the polygon (which makes transportation through this area hundred times cheaper). Not surprisingly, the optimal transportation in this case is concentrated mostly upon the highway.

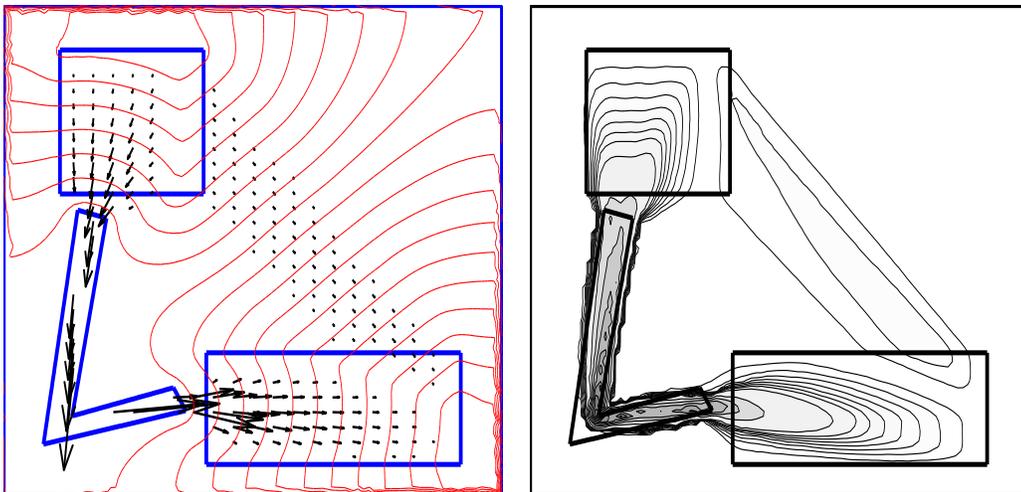


Figure 3: As in fig. 1; problem with a region of cheap transportation.

4 Conclusion

We used formally the similarity of the Monge-Kantorovich equations and a model for sand surface evolution to present solutions to Monge-Kantorovich equations as stationary points of a dynamical system and find them numerically. We note that, although it remains to prove rigorously the solutions to non-stationary problem do have the $t \rightarrow \infty$ limit, numerical examples confirm this statement and demonstrate the efficiency of our approach. Our hypothesis is the convergence, in fact, occurs in a finite time.

It seems interesting to give also an “optimal mass transportation” interpretation of sand model equations. Let $f^+(x, t)$ and $f^-(x, t)$ be, respectively, the production and consumptions rates of some homogeneous commodity and $u_0(x)$ its initial distribution. Let, at any time moment, the local price of this commodity be defined as $-u(x, t)$, i.e., the price is positive if there is a deficit, $u(x, t) < 0$, and is negative if there is a surplus $u(x, t) > 0$ which needs storing.

Whenever the price difference between two arbitrary points, x and y , exceeds the cost of transportation of one commodity unit from one of these points to another, $c(x, y) = |x - y|$, the prices adjust themselves instantaneously, since a cheaper price is available by buying elsewhere and transporting. Condition of price equilibrium can thus be written in the form $|u(x) - u(y)| \leq |x - y|$ or, locally, $|\nabla u| \leq 1$. Assuming the initial distribution of goods u_0 satisfies this condition, we describe its further evolution driven by production and consumption as

$$\partial_t u + \nabla \cdot \mathbf{q} = f^+ - f^-, \quad u|_{t=0} = u_0(x)$$

and assume that the mass flux \mathbf{q} is always directed towards the price gradient: $\mathbf{q} = -a\nabla u$, where $a(x, t) \geq 0$ is an unknown scalar function. Finally, if $|u(x) - u(y)| < |x - y|$, the transportation from one of these points to another can only increase the cost and is unprofitable. Locally, this can be reformulated as the condition $|\nabla u| < 1 \rightarrow a(x, t) = 0$, which brings us to the critical-state model (3)-(4) in which all transport occurs at the border of equilibrium.

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