FFT-based Solution of 3D Problems in Type-II Superconductivity

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- Vestgården et al. 2012,2013: an FFT-based method for thin film magnetization problems;
- This work: a new FFT-based method for 3D bulk magnetization problems.

Nonlinear 3D eddy current problem: formulation

Let Ω be the SC domain, Γ its boundary, $\Omega_{\text{out}} = R^3 \setminus \Omega$,
$\boldsymbol{h}$ - the magnetic field, $\boldsymbol{j}$ - the current density, $\boldsymbol{e}$ - the electric field.

For simplicity, here we assume:

1. the SC domain Ω is countourwise simply connected;
2. SC obeys $\boldsymbol{e} = \rho(\boldsymbol{j})\boldsymbol{j}$ relation with $\rho = \frac{e_0}{j_c}\left(\frac{|\boldsymbol{j}|}{j_c}\right)^{n-1}$ in Ω;
3. the external magnetic field $\boldsymbol{h}_e(t)$ is uniform.

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Our formulation is written for the magnetic field but differs from the popular h-formulation.

Let $\boldsymbol{h}$ be known at time t and $\boldsymbol{j} = \nabla \times \boldsymbol{h} = \mathbf{0}$ in Ω_{out}.

Knowing how to find $\partial_t \boldsymbol{h}$ for a given $\boldsymbol{h}$ we can integrate $\boldsymbol{h}$ in time.

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An outlook of the numerical method:

Since $\boldsymbol{j} = \nabla \times \boldsymbol{h}$ (Ampere's law) we can find $\boldsymbol{e} = \rho(\boldsymbol{j})\boldsymbol{j}$ in Ω.

Although $\boldsymbol{e}$ in Ω_{out} remains unknown, it should be such that

$$\partial_t \boldsymbol{h} = -\mu_0^{-1} \nabla \times \boldsymbol{e} \quad \text{(the Faraday law)}$$

yields

$$\partial_t \boldsymbol{j} = \nabla \times \partial_t \boldsymbol{h} = \boldsymbol{0} \quad \text{in } \Omega_{\text{out}}.$$

This condition is used to find the time derivative $\partial_t \boldsymbol{h}$ iteratively.

Iterative algorithm

Let $\boldsymbol{h}$ at time t be known; then $\boldsymbol{j} = \nabla \times \boldsymbol{h}$. We set

$$\boldsymbol{e}^0 = \begin{cases} \rho(\boldsymbol{j})\boldsymbol{j} & \text{in } \Omega, \\ \rho_{\text{out}}\boldsymbol{j} & \text{in } \Omega_{\text{out}}, \end{cases}$$

with a sufficiently high fictitious resistivity ρ_{out} and set, as an initial approximation,

$$\partial_t \boldsymbol{h}^0 = -\mu_0^{-1} \nabla \times \boldsymbol{e}^0.$$

A very high ρ_{out} can suppress $\boldsymbol{j}$ in Ω_{out} but makes the evolutionary problem for $\boldsymbol{h}$ stiff, which inhibits the integration.

Instead, we used a lower value of ρ_{out} and enforced the condition

$$\partial_t \boldsymbol{j}|_{\Omega_{\text{out}}} = \nabla \times \partial_t \boldsymbol{h}|_{\Omega_{\text{out}}} = \boldsymbol{0}$$

by iterative updating $\partial_t \boldsymbol{h}|_{\Omega_{\text{out}}}$.

The role of ρ_{out} - suppressing the stray current in a thin outside boundary layer around Ω.

Iterative algorithm: the details

The Biot-Savart law can be written as $\boldsymbol{h} = \boldsymbol{h}_e(t) + \boldsymbol{\Phi}[\boldsymbol{j}]$ with

$$\boldsymbol{\Phi}[\boldsymbol{j}] = \nabla \times \int_{R^3} G(\boldsymbol{r} - \boldsymbol{r}')\boldsymbol{j}(\boldsymbol{r}', t)\, d\boldsymbol{r}',$$

where $G(\boldsymbol{r}) = (4\pi|\boldsymbol{r}|)^{-1}$ is the Green function. Clearly,

$$\nabla \times \boldsymbol{h} = \nabla \times \boldsymbol{\Phi}[\boldsymbol{j}] = \boldsymbol{j}.$$

To find $\partial_t \boldsymbol{h}$, on the i-th iteration we set

$$\partial_t \boldsymbol{j}_{\text{in}}^{i-1} = \begin{cases} \nabla \times \partial_t \boldsymbol{h}^{i-1} & \text{in } \Omega, \\ \boldsymbol{0} & \text{in } \Omega_{\text{out}} \end{cases}$$

and use the time derivative of the Biot-Savart law to compute

$$\partial_t \boldsymbol{h}^i|_{\Omega_{\text{out}}} = d\boldsymbol{h}_e/dt + \boldsymbol{\Phi}[\partial_t \boldsymbol{j}_{\text{in}}^{i-1}]|_{\Omega_{\text{out}}}.$$

If these iterations converge, which is usually fast, we obtain $\nabla \times \partial_t \boldsymbol{h}|_{\Omega_{\text{out}}} = \partial_t \boldsymbol{j}_{\text{in}}|_{\Omega_{\text{out}}} = \boldsymbol{0}$ as desired.

Fourier transform

The operator

$$\mathbf{\Phi}[\boldsymbol{j}] = \nabla \times \int_{R^3} G(\boldsymbol{r} - \boldsymbol{r}')\boldsymbol{j}(\boldsymbol{r}', t)\, d\boldsymbol{r}'$$

can be expressed by means of convolutions in R^3 and computed in the Fourier space. Since

$$F(G) := \int_{R^3} G(\boldsymbol{r}) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{r} = \frac{1}{|\boldsymbol{k}^2|}$$

we obtain

$$\mathbf{\Phi}[\boldsymbol{j}] = \begin{pmatrix} j_z * \partial_y G - j_y * \partial_z G \\ j_x * \partial_z G - j_z * \partial_x G \\ j_y * \partial_x G - j_x * \partial_y G \end{pmatrix} = F^{-1}\left(\frac{\mathrm{i}}{|\boldsymbol{k}|^2} \begin{bmatrix} k_y F(j_z) - k_z F(j_y) \\ k_z F(j_x) - k_x F(j_z) \\ k_x F(j_y) - k_y F(j_x) \end{bmatrix} \right)$$

We use this representation of $\mathbf{\Phi}$ for our iterations

$$\partial_t \boldsymbol{h}^i|_{\Omega_{\text{out}}} = d\boldsymbol{h}_e/dt + \mathbf{\Phi}[\partial_t \boldsymbol{j}_{\text{in}}^{i-1}]|_{\Omega_{\text{out}}}.$$

All spatial derivatives can also be computed in the Fourier space.

Implementation (in Matlab)

To make this algorithm practical, we

- define a uniform $N_x \times N_y \times N_z$ grid in a rectangular domain containing Ω and some empty space around it;
- compute the operator $\boldsymbol{\Phi}$ and all spatial derivatives in the Fourier space (with Gaussian smoothing for the derivatives);
- replace Fourier transform by its discrete counterpart on this grid and use the standard FFT software;
- employ a standard ODE solver for integration in time (the method of lines).

Implementation of this method is not complicated: our Matlab program is short ($\approx$ 120 command lines + comments).

FFT vs FEM; benchmark problem, other examples

Finite element methods for 3D magnetization problems:
Pecher et al. 2003; Kashima and Elliot 2007,2008; Zhang and Coombs 2011; Grilli et al. 2013; Stenvall et al. 2014; Pardo and Kapolka 2017, 2018; Kapolka et al. 2017; Olm et al. 2017, ...

Benchmark problem 5 (H.T.S Modeling Workgroup).
Our solution (LP and Sokolovsky, SuST, 2018) is similar to those obtained using two different FEM methods by Kapolka, Pardo, Zermeno, Grilli, Morandi and Ribani (2016). The FFT method is simpler; our preliminary comparison suggests the efficiency may be comparable.

Two new examples: magnetic shield and magnetic lens. Scaling:

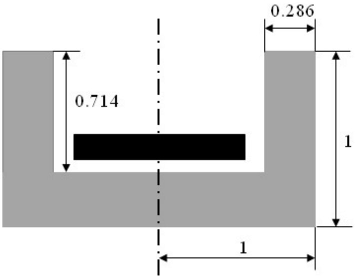
$$(x', y', z') = \frac{(x, y, z)}{l}, \quad t' = \frac{t}{t_0}, \quad \boldsymbol{e'} = \frac{\boldsymbol{e}}{e_0}, \quad \boldsymbol{j'} = \frac{\boldsymbol{j}}{j_c}, \quad \boldsymbol{h'} = \frac{\boldsymbol{h}}{j_c l},$$

where l is the characteristic size and $t_0 = \mu_0 j_c l^2 / e_0$; $n = 30$.

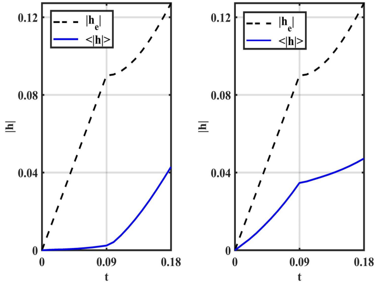
Example 1: Magnetic shield

Hollow sc cylinder closed from one end: Gozzelino et al. (2016).

1. $\boldsymbol{h}_\mathrm{e}$ first grows along the cylinder axis, then perpendicular to it;
2. $\boldsymbol{h}_\mathrm{e}$ first grows perpendicularly to the axis, then along the axis.

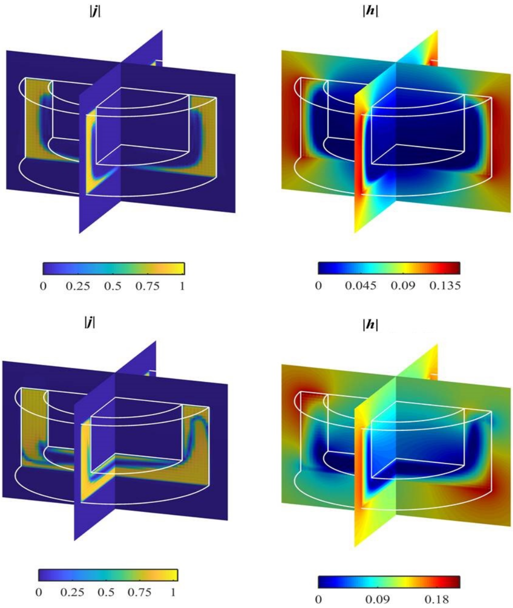


Gray: SC cylinder, black: shielded area. Scaled dimensionless units.



Left: case **1**, right: case **2**. Dashed line - $|\boldsymbol{h}_\mathrm{e}|$, solid line - average $|\boldsymbol{h}|$ in the shielded zone.

Example 1: Magnetic shield, case 1



Domain: $-1.6 \leq \frac{\{x,y,z\}}{R} \leq 1.6$
Mesh:
128^3: CPUtime 2h;
192^3: CPUtime 12h

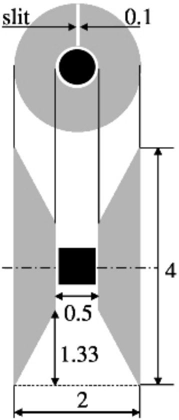
$\frac{|\boldsymbol{j}|}{j_c}$ (left) and $\frac{|\boldsymbol{h}|}{j_c R}$ (right).

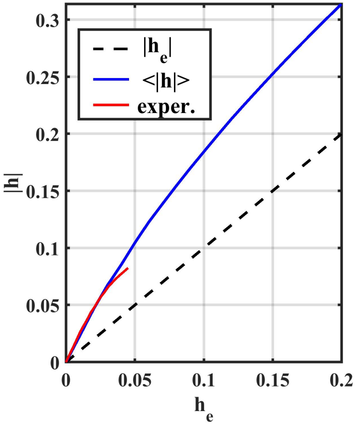
Top: $\frac{\boldsymbol{h}_e}{j_c R} = (0, 0, 0.09)$;

Bottom:
$\frac{\boldsymbol{h}_e}{j_c R} = (0.09, 0, 0.09)$.

Example 2: Magnetic lens

SC cylinder with a coaxial biconical hole and a narrow slit to prevent shielding circumference current: ZY Zhang et al. (2012).





Gray: superconductor, black: lens central area. Dimensionless units.

Dashed line - $|\boldsymbol{h}_e|$, blue - $< |\boldsymbol{h}| >$ in the central area, red - measured.

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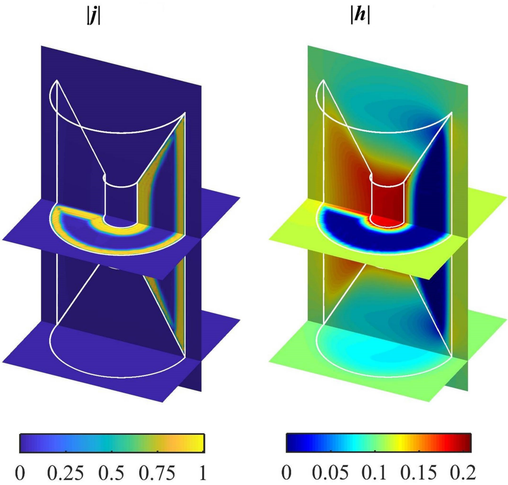

$\frac{|\boldsymbol{j}|}{j_c}$ (left) and $\frac{|\boldsymbol{h}|}{j_c R}$ (right) for $\frac{h_{e,z}}{j_c R} = 0.1$;
Mesh: $128 \times 128 \times 256$ in the domain $\frac{\{|x|,|y|\}}{R} \leq 1.8, \quad \frac{|z|}{R} \leq 3.6$
CPU time 6h (for $0 \leq t \leq 0.2$).

Conclusion

- A new FFT-based method for 3D magnetization problems has been derived, tested, and applied to two realistic problems;
- The main advantages are:
  - simplicity: much easier to implement than the FE methods, although further comparison is desirable;
  - generality: applicable also to multiply connected SC domains, the power law can be replaced by another $\boldsymbol{e}(\boldsymbol{j}, \boldsymbol{h})$ relation;
- The method can be efficient; its (further) parallelization is one of the improvements we are going to explore.
- Extending the FFT method to 3D modeling of SC/FM hybrid systems is a direction of our future research.

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Thank you!

1. LP and V. Sokolovsky, SuST **31** (2018) 055018;
2. LP and V. Sokolovsky, ArXiv 1803.01346