Electric Field Formulation for Thin Film Magnetization Problems

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Introduction

Magnetization of type-II superconductors: an eddy current problem with a highly nonlinear $e(j)$ relation. Typically,

- **Power law**: $e = e_0 \left( \frac{j}{j_c} \right)^{p-1} \frac{j}{j_c}$,
- **Critical state**: the multivalued $p \to \infty$ limit.

A variety of numerical methods can be used to compute the magnetic field and the current density. Although the electric field is also needed, it is often the most difficult to compute.
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A variety of numerical methods can be used to compute the magnetic field and the current density. Although the electric field is also needed, it is often the most difficult to compute.

Long cylinder (strip) in a perpendicular field is an exception:

- **A-J formulation** to compute $j$;
- $e = -\partial_t A(j) + C$.

Here $C(t)$ is the Lagrange multiplier related to total current constraint; LP and Sokolovsky, 2011.
Introduction

Long cylinder in a parallel field:

- Magnetic field $h$ is expressed via the $\text{dist}(x, \partial \Omega)$ function or computed numerically;
- $j = \text{Curl } h$.

Thin film in a perpendicular field:

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Computing the electric field can be a problem:

The main variable ($h$ or $g$) can be found with high accuracy. Accuracy of $j$ is much lower due to differentiation in $\text{Curl}$. Substitution of $j$ into the power law, $e \propto j^p$, makes the error unacceptable if the power $p$ is large. The critical-state model situation is even worse: knowing $j$ we cannot compute $e$. 
Power law: a caveat

The situation is not hopeless for the power law model...

• Brandt and co-workers: pioneering works starting from 90-th. Numerical simulations based mostly on the power law model. For thin film geometry, the electric field was computed (with moderately big $p$ values) and also found analytically for the rectangular and related film shapes under a simplifying assumption: $\partial_t h = \dot{h}_e$.

• Vestgarden, Shantsev, Galperin, Johansen, 2007-2012: studied thin film problems extensively. These authors used a similar numerical approach and, recently, supplemented Brandt’s method by an efficient FFT-based algorithm.

As a result of these and some other studies, the peculiarities of thin film magnetization are now understood. However, the accuracy of the electric field simulation is questionable.
Critical state models

Longitudinal geometry: determining both \( \{e, h\} \)

1. Badía-Majós and Lopez 2004
   - solved a variational problem for \( h \);
   - evaluated \( e \) by integration along the paths of vortex penetration.

2. Barrett and LP 2006,2010
   - solved a dual variational problem for \( e \);
   - found \( h \) from the Faraday law,
     \[ h = h_0 - \frac{1}{\mu_0} \int_0^t \text{Curl } e \, dt. \]

Thin films:
LP 1998; Navau, Sanchez, Dell-Valle and Chen 2008

- solved a variational problem for \( g \);
- this gives \( j \) and \( h \);
- the electric field \( e \) was not found.
This work

• We derive a new variational formulation for the thin film problem written for two variables: the electric field $e$ and an auxiliary variable, the jump of the scalar magnetic potential across the film, $g$. The latter variable is analogous to the magnetization function.

• Our numerical scheme, based upon this formulation, allows us to determine simultaneously the electric field and current density in the superconducting film. Magnetic field can also be calculated. The numerical scheme is applicable for any value of the power $p$. It suffers no accuracy loss if the power is high. Hence, this algorithm can be used for both the power law and critical state models.
Magnetization problem

I. Thin film: \( \{ \Omega \times 0 \} \subset \mathbb{R}^3 \), where \( \Omega \) is a 2D domain.
We rewrite the power law,
\[
j = j_c(e/e_0)^{r-1}e/e_0,
\]
where \( r = 1/p \), \( j \) is the sheet critical current density, \( e \) the parallel to film electric field component. We may assume \( \Omega \) is simply connected: holes can be filled with the sheet critical current density \( j_c \) there set to zero or very small).

II. Outer space: \( \omega = \mathbb{R}^3 \setminus \{ \Omega \times 0 \} \).
\[
\mu_0 \partial_t H + \nabla \times E = 0, \quad \nabla \times H = 0,
\]
\[
H|_{t=0} = H_0, \quad H \to H_e(t) \text{ for } |x| \to \infty,
\]
where \( x = (x_1, x_2, x_3) \) and \( H_e = (0, 0, h_e(t)) \).

III. Relation between the two:
\[
E_\tau|_{\Omega^+} = E_\tau|_{\Omega^-} = e. \quad j = n^+ \times [H], \quad n^+ \cdot [H] = 0.
\]
Here \( [f] = f|_{\Omega^+} - f|_{\Omega^-} \) is the jump of \( f \) across the film and \( n^+ = (0, 0, 1) \).
Scalar magnetic potential

Since $\nabla \times \mathbf{H} = 0$ in $\omega$, there exists a potential $w(x,t)$ such that $\mathbf{H} - \mathbf{H}_e = -\nabla w$. Furthermore, $\nabla \cdot (\mathbf{H} - \mathbf{H}_e) = 0$ yields

$$\Delta w = 0 \quad \text{in} \ \omega. \quad (1)$$

Integrating the Faraday law in time, using the continuity on the film of $\mathbf{n}^+ \times \mathbf{E}$ and $\mathbf{n}^+ \cdot \mathbf{H}$, we obtain

$$\frac{\partial w}{\partial \mathbf{n}^+} = \frac{1}{\mu_0} \text{Curl} \left( \int_0^t \mathbf{e} \, dt \right) + h_e - h_0 \quad \text{on} \ \Omega^\pm, \quad (2)$$

where $h_0 = \mathbf{n}^+ \cdot \mathbf{H}_0$.

Finally, we note that $\mathbf{j} = \mathbf{n}^+ \times [\mathbf{H}] = \text{Curl}[w]$, so

$$j_c(e/e_0)^{r-1}e/e_0 = \text{Curl}[w], \quad (3)$$

and $[w]$ should be zero on the film boundary $\{\partial \Omega \times 0\}$. 
2D formulation

Denote $g = [w]$. Since $j = \text{Curl } g$ and $g|_{\partial \Omega} = 0$ this is the Brandt’s magnetization function. Expressing solution of the exterior problem via the double layer potential we arrive at a convenient two-dimensional variational formulation:

$$a \left( \frac{\partial g}{\partial t}, \psi \right) + \frac{1}{\mu_0} (\text{Curl } e, \psi) = - \left( \frac{dhe}{dt}, \psi \right)$$

$$\text{Curl } g = j_c (e/e_0)^{r^{-1}} e/e_0$$

for any test function $\psi$ such that $\psi|_{\partial \Omega} = 0$.

Here $(\phi, \psi) = \int_{\Omega} \phi \psi \, dx$ and

$$a(\phi, \psi) = \frac{1}{4\pi} \int_{\Omega} \int_{\Omega} \frac{\text{Grad } \phi(x) \cdot \text{Grad } \psi(y)}{|x - y|} \, dx \, dy.$$
Numerical approximation

Important: curl conforming finite elements for the electric field. We set \( e = Rv \), where \( R \) is the rotation matrix

\[
R = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

Then \( |v| = |e| \), \( \text{Curl } R = -\text{Div} \) and \( R^T \text{Curl} = \text{Grad} \).

The problem for \( v \), \( g \) is

\[
a \left( \frac{\partial g}{\partial t}, \psi \right) - \frac{1}{\mu_0} \left( \text{Div } v, \psi \right) = - \left( \frac{dh_e}{dt}, \psi \right),
\]

\[
\text{Grad } g = j_c(v/e_0)r^{-1}v/e_0.
\]

We use div conforming Raviart-Thomas edge elements for \( v \) and continuous piecewise linear elements for \( g \).
Numerical approximation

On each time level $t^n = t^{n-1} + \tau^n$ we solve the problem:

Find $v^n \in V^h$ and $g^n \in S^h_0$ such that

$$a(g^n, \psi^h) - \frac{\tau^n}{\mu_0} (\text{Div } v^n, \psi^h) = a(g^{n-1}, \psi^h) - (h^n_e - h^{n-1}_e, \psi^h),$$

$$e_0^{-r}(j_c|v^n|^{r-1}v^n, \eta^h) + (g^n, \text{Div } \eta^h) = 0$$

for all $\eta^h \in V^h$ and $\psi^h \in S^h_0$

Here $h$ is the maximal element size characterizing the regular partition $\mathcal{T}^h$ of $\Omega$ into triangles, $S^h_0$ is the set of continuous functions, linear on each triangle and zero in the boundary nodes; $V^h$ is the Raviart-Thomas space of vector functions, linear on each triangle and having continuous normal component on the mesh edges.
Numerical approximation

On each time level $t^n = t^{n-1} + \tau^n$ we solve the problem:

Find $v^n \in V^h$ and $g^n \in S_0^h$ such that

$$a\left(g^n, \psi^h\right) - \frac{\tau^n}{\mu_0} (\text{Div} v^n, \psi^h) = a\left(g^{n-1}, \psi^h\right) - \left(h^n - h^{n-1}, \psi^h\right),$$

$$e_0^{-r}(j_c |v^n|^{r-1} v^n, \eta^h) + (g^n, \text{Div} \eta^h) = 0$$

for all $\eta^h \in V^h$ and $\psi^h \in S_0^h$

The nonlinearity was resolved iteratively; at each iteration a linear algebraic system

$$AG_{n,j} - \tau^n BV_{n,j} = d^n$$

$$B^T G_{n,j} - M_{n,j-1} V_{n,j} = g_{n,j-1}$$

was solved. Here $A$ is a full matrix but is computed only once for each mesh, the other matrices are sparse.
Simulation results

I. We used dimensionless variables:

\[ x = \frac{x'}{L}, \quad t = \frac{t'}{t_0}, \quad e = \frac{e'}{e_0}, \quad j = \frac{j'}{j_{c0}}, \quad h = \frac{h'}{j_{c0}}, \quad g = \frac{g'}{j_{c0}L}, \]

where the time scale \( t_0 = \mu_0 j_{c0} L / e_0 \), \( 2L \) is the length of the projection of \( \Omega \) onto the \( x_1 \)-axis, and \( j_{c0} \) is a characteristic value of the sheet critical current density (\( j_c \) if it is constant).

II. Usually, we set \( p = 1000 \) or \( p = 100 \) but could also solve for small power values; the computation time did not depend on \( p \). The normal to film component of magnetic field, \( h \), was found numerically by means of the Biot-Savart law.

III. All simulations were done in Matlab, on a PC with the Intel Core i5-2400 3.10Hz processor and 4Gb RAM.
The analytical solution for the Bean model, Mikheenko and Kuzovlev 1993, Clem and Sanchez 1994, was used for the accuracy evaluation.

Numerically, we solved for $h_e(t) = t$ with $p = 1000$ on two different meshes: $h = 0.05$ (4200 el.) and $h = 0.03$ (12000 el.).

For $t = 0.5$ the relative errors of the numerical solution are:

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\delta(j)%$</th>
<th>$\delta(e)%$</th>
<th>$\delta(h)%$</th>
<th>CPU time (min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>1</td>
<td>4.9</td>
<td>2.5</td>
<td>5</td>
</tr>
<tr>
<td>0.03</td>
<td>0.6</td>
<td>3.1</td>
<td>1.4</td>
<td>57</td>
</tr>
</tbody>
</table>
Corners and boundary indentations

The electric field is strong near the boundary indentations and, especially, in vicinity of concave film corners (Schuster, Kuhn and Brandt 1996, Vestgarden et al. 2007). In our simulation we used a power law with $p = 1000$ and $h_e(t) = t$. The mesh contained about 9000 elements and was refined near the film boundary:
Corners and boundary indentations

Electric field $|e|$ (shown for $h_e = 0.4$)
Corners and boundary indentations

Current streamlines, $h_e = 0.4$
Corners and boundary indentations

Normal magnetic field level contours, $h_e = 0.4$

The penetration zone differs from that for a cylinder.
Inhomogeneous film

The electric field near the boundaries between regions of different critical current densities can be orders of magnitude higher than in the other film parts. Example: $p = 1000$ and $h_e(t) = t$. Two regions: the rectangular area with $j_c = 0.5$; in the rest of the film $j_c = 1$. Mesh (10.6 thousand el.):
Inhomogeneous film

Electric field $|e|$ (shown for $h_e = 0.5$)
Inhomogeneous film

Current streamlines, $h_e = 0.5$
Inhomogeneous film

Level contours of the normal to film magnetic field; $h_e = 0.5$

The penetration is deeper in the lower $j_c$ area.
Film with three holes

To solve this problem we set $j_c = 0.002$ in the holes. Flux penetration into the holes causes very strong electric field along the penetration paths. In this example: $p = 100$ and $h_e(t) = t$. Mesh (10.4 thousand el.):
Film with three holes

Electric field $|e|$ (shown for $h_e = 0.5$)
Film with three holes

Current streamlines, $h_e = 0.5$
Film with three holes

Level contours of the normal to film magnetic field; $h_e = 0.5$

Flux penetration from the film holes is different to that for long cylinders.
Conclusion

• The formulation of thin film magnetization problems in terms of the electric field AND scalar magnetization function is preferable to formulations written for the magnetization function alone. Its advantage is in simultaneous determination of the electric field and current density instead of computing the electric field by substitution of the current density approximation into the power law current-voltage relation. The latter approach is inaccurate if the power is high.

• Our numerical algorithm remains accurate for any power in the power law. For high powers we obtain a good approximation to the Bean model solution.

• The approach can be generalized to thin film problems with field-dependent and anisotropic sheet critical current densities.
Thank you!

J.W. Barrett and L.P.
Electric field formulation for thin film magnetization problems.
Preprint: see www.math.bgu.ac.il\~leonid