1. Let $X$ a set, and $\tau_1, \tau_2$ topologies on $X$. We say that $\tau_1 \prec \tau_2$ ($\tau_1$ refines $\tau_2$) if $U \in \tau_2 \Rightarrow U \in \tau_1$:

(a) Note that the refinement relation is an inverse inclusion relation ($\tau_1 \prec \tau_2 \Rightarrow \tau_2 \subseteq \tau_1$), hence, all axioms of partial order relation are trivial. Also note that for every topology $\tau$, and for every set $U \in \tau$, $U$ is a subset of $X$, and hence $U \in \mathcal{P}(X)$. Thus, $U \in \tau \Rightarrow U \in \mathcal{P}(X)$ and so $\mathcal{P}(X) \prec \tau$. So, $\mathcal{P}(X)$, the discrete topology is a minimal element. In the same manner, we see that $\tau \prec \{\phi, X\}$, for every topology $\tau$, and hence the trivial topology $\{\phi, X\}$ is a maximal element.

(b) There are 29 topologies on a set of three elements $X = \{a, b, c\}$: the trivial topology :\(\{\phi, X\}\) topologies with one non-trivial open set:

\begin{itemize}
  \item (2) $\{\phi, \{a\}, X\}$, (3) $\{\phi, \{b\}, X\}$, (4) $\{\phi, \{c\}, X\}$, (5) $\{\phi, \{a, b\}, X\}$, (6) $\{\phi, \{a, c\}, X\}$, (7) $\{\phi, \{b, c\}, X\}$
\end{itemize}

topologies with two non-trivial open sets(9):

\begin{itemize}
  \item (8) $\{\phi, \{a\}, \{b\}, X\}$, (9) $\{\phi, \{a\}, \{c\}, X\}$, (10) $\{\phi, \{b\}, \{c\}, X\}$, (11) $\{\phi, \{a, b\}, \{c\}, X\}$, (12) $\{\phi, \{a, c\}, \{b\}, X\}$
\end{itemize}

topologies with three non-trivial open sets:

\begin{itemize}
  \item (13) $\{\phi, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, (14) $\{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$, (15) $\{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, (16) $\{\phi, \{c\}, \{b, c\}, \{a, b\}, X\}$
\end{itemize}

and, finally, the discrete topology (29)$\mathcal{P}(X)$.

(c) Let $\tau_1 := \phi, X$ and $\tau_2 := \{\phi, \{a\}, X\}$. It is clear that for all $U \in \tau_2$, $\phi$ satisfies the claim. It is also clear that $\{a\}$ is in $\tau_2$ and not in $\tau_1$, and thus $\tau_1$ does not refine $\tau_2$.

3. Let $\mathcal{N} := \mathbb{N}^\mathbb{N}$. for $\eta_1, \eta_2 \in \mathcal{N}$ define:

$$d(\eta_1, \eta_2) := 2^{-\min_{n \geq 2}(\eta_1(n) \neq \eta_2(n))} \text{ (or 0 if } \eta_1 = \eta_2)$$

(a) The only property of a metric which is not trivial is the triangle inequality:

Let $\eta_1, \eta_2, \eta_3 \in \mathcal{N}$: denote $n_{i,j} := \min_n \{\eta_i(n) \neq \eta_j(n)\}$

If $n_{1,3} \leq n_{2,3}$ then for all $n < n_{1,3}$ : $\eta_1(n) = \eta_2(n) = \eta_3(n)$ and for $n_{1,3} \leq n$ : $\eta_1(n) \neq \eta_3(n)$, while $\eta_2(n)$ may or may not be equal to $\eta_1(n)$. This gives $n_{1,2} \geq n_{1,3}$.

In the same manner we see that $n_{2,3} \leq n_{1,3}$ $\Rightarrow$ $n_{1,2} \geq n_{2,3}$.

It follows that $n_{1,2} \geq \min\{n_{1,3}, n_{1,2}\}$, and so $2^{-n_{1,2}} \leq 2^{-\min\{n_{1,3}, n_{1,2}\}} = \max\{2^{-n_{2,3}}, 2^{-n_{1,3}}\}$, and the claim follows from this.
(b) Let \( \{\eta_i\}_{i=1}^{\infty} : \forall \epsilon > 0 \exists N \text{ s.t. } n, k > N \Rightarrow d(\eta_n, \eta_k) < \epsilon. \) so by simple arithmetics we get that \(\frac{1}{2} < 2^{\min_i \{\eta_n(n) \neq \eta_k(n)\}}\) and thus \(\log_2(\frac{1}{2}) < \min_i \{\eta_n(n) \neq \eta_k(n)\}.\) So a Cauchy sequence in \( \mathcal{N} \) is one in which for every \( \epsilon \) there exists a natural number \( N \), such that for every \( n, k > N \), the series \( \eta_n, \eta_k \) identify for every \( i \leq \lfloor \log_2(\frac{1}{2}) \rfloor \).

(c) By the same considerations as the prior segment:
\[ B(1, \epsilon) = \{\eta \in \mathcal{N} : \forall i \leq \lfloor \log_2(\epsilon^{-1}) \rfloor \eta(i) = 1\} \] (where \( \bar{1} = (1, 1, 1, \ldots) \))

5. (a) Show that \( U_a := \{U \subseteq X : a \in U\} \) is an ultra-filter over \( X \):

- \( a \in X, a \notin \phi \Rightarrow X \in U_a, \) and \( \phi \notin U_a \)
- \( U \in U_a \Rightarrow a \in U \Rightarrow \) if \( U \subseteq V \) then \( a \in V \Rightarrow V \in U_a \)
- \( U, V \in U_a \Rightarrow a \in U \) and \( a \in V \) \( \Rightarrow a \in U \cap V \Rightarrow U \cap V \in U_a \)

To show that \( U_a \) is an ultra-filter, we assume towards contradiction that there exists a filter \( \mathcal{U} \), such that \( U_a \subseteq \mathcal{U} \) and there exist \( U \in \mathcal{U} \setminus U_a. \)

\( U \notin U_a, \) so \( a \notin U. \) Hence, by filter properties \( \{a\} \cap U = \phi \in \mathcal{U} \) in contradiction.

(b) Let \( X \) a finite set, and \( \mathcal{U} \) an ultra-filter. Since \( X \) if finite, \( \mathbb{P}(X) \) is also finite, and so \( \mathcal{U} \subseteq \mathbb{P}(X) \) is also finite. Let \( \{U_1, U_2, \ldots, U_n\} \) be a numeration of \( \mathcal{U} \)'s elements. From closure under finite intersection, we get that \( \bigcap_{i=1}^{n} U_i = V \subseteq \mathcal{U} \) \( \forall V \neq \phi, \) and we assume that \( \mathcal{U} \) is not principal so, \( |V| \geq 2. \)

For some \( a \in V \) we shall observe the two subsets of \( V : \{a\} \) and \( V - \{a\}. \)

By an ultra filter property (shown in class) \( \{a\} \in \mathcal{U} \) or \( X - \{a\} \in \mathcal{U}. \) Since we assumed \( \mathcal{U} \) is not principal \( X - \{a\} \in \mathcal{U}, \) and thus \( (X - \{a\}) \cap V = V - \{a\} \in \mathcal{U}. \)

But by \( V \)'s definition, it is included in any set in \( \mathcal{U}, \) thus \( V \subseteq (V - \{a\}), \) so \( a \notin V \) in contradiction.

(c) Show that for an infinite set \( X, \mathcal{F} := \{U \subseteq X : |X - U| < \aleph_0\} \) is a filter, and not an ultra filter.

- \( |X - X| = 0, |X - \phi| = |X| \geq \aleph_0 \Rightarrow X \in \mathcal{F}, \) and \( \phi \notin \mathcal{F} \)
- \( U \in \mathcal{F} \) and \( U \subseteq V \) then \( |X - V| \leq |X - U| < \aleph_0 \) thus \( V \in \mathcal{F}. \)
- \( U, V \in \mathcal{F} \Rightarrow |X - U|, |X - V| < \aleph_0 \Rightarrow X = (X - (U \cap V)) \notin \mathcal{F} \)

To show that \( \mathcal{F} \) is not an ultra-filter, we define \( \iota : \mathbb{N} \rightarrow X, \) a 1:1 function from the natural numbers to \( X \) (existence is proved out of cardinality considerations).

Let \( A := \iota(2\mathbb{N}) \) (the image of the even numbers under \( \iota \).) If \( \mathcal{F} \) is an ultra-filter then \( A \in \mathcal{F} \) or \( X - A \in \mathcal{F}, \) but \( |X - (X - A)| = |A| = \aleph_0 \)
so $X - A \notin \mathcal{F}$ and $\iota(\{\text{The odd numbers}\}) \subseteq X - A$, and hence $|X - A| \geq \aleph_0$ so $A \notin \mathcal{F}$.

Hence $\mathcal{F}$ is not an ultra filter.

Q. Show that a finite metric space is discrete:
Let $(X, d)$ a finite metric space, denote $X = \{x_1, x_2, \ldots, x_n\}$ and let $x \in X$ be an element of $X$.
Define $\delta := \min_{x_i \neq x} \{d(x, x_i)\}$. $\delta > 0$ as a minimum of a finite number of strictly positive numbers. Now we shall notice that the open set $B(x, \frac{\delta}{2})$ does not contain any $x \neq x_i \in X$, and thus $\{x\}$ is an open set.
By that manner for every $y \in X : \{y\}$ is an open set, and so $X$ is discrete.