On the role of infinite cardinals

Menachem Kojman

Ben-Gurion University of the Negev

Prologue

- What is the role played by infinite cardinal arithmetic in mathematics? Can it be compared to the role played by the operations on natural numbers — $+, \cdot, \exp, n!, \binom{n}{k}$, etc. — in combinatorics, analysis, algebra, etc?
- Cantor believed that through his work "... a lot of light will be shed on old and new problems in cosmology and arithmetic" (1884), and thought that infinite cardinals and their arithmetic would be effective in studying the "physical universe" (namely, Euclidean space) too.

Cantor's cardinals

- An ordinal is a transitive set α such that $(\alpha, \in)$ is a linear well-ordering. Ordinals form a well-ordered proper class $(ON, \in)$.
- An ordinal that has no bijection with a smaller ordinal is a cardinal. The cardinals form a proper sub-class of the ordinals.
  $CN = \{0, 1, \ldots, \omega = \aleph_0, \aleph_1, \ldots, \aleph_\omega, \aleph_{\omega+1} \ldots\}$.
- The cofinality $\mathrm{cf}(\kappa)$ of a cardinal κ is the smallest ordertype of an unbounded set of κ. A cardinal is regular if it is its own cofinality and singular otherwise.

The arithmetic of cardinals

- For infinite κ, λ, $\kappa + \lambda = \kappa \times \lambda = \max\{\kappa, \lambda\}$.
- The exponent λ^κ is defined as $|\{f | f : \kappa \to \lambda\}|$ (Cantor 1895).
- An example of a rule of exponentiation (Cantor 1895):

$$2^\kappa \leq \kappa^\kappa \leq (2^\kappa)^\kappa = 2^{\kappa \times \kappa} = 2^\kappa$$

- So, for $\kappa \leq \lambda$ it makes no difference whether one uses 2^λ or κ^λ.
- Let $\exp(x)$ denote 2^x.

λ^κ ***for*** $\kappa < \lambda$

Suppose now that $\kappa < \lambda$?

- Trivially, $\lambda^\kappa \geq \exp(\kappa)$. It can be thought of as: "$\lambda^\kappa = |[\lambda]^\kappa|$, the number of κ-subsets of λ; and even a single κ-subset of λ has 2^κ κ-subsets"
- The main point: λ^k is determined by another function, $\binom{\lambda}{\kappa}$, obtained from λ^κ by removing the factor $\exp(\kappa)$ in the equation:

$$\lambda^k = \exp(\kappa) \times \binom{\lambda}{\kappa}$$

The operation $\binom{\lambda}{\kappa}$

- To avoid counting $\exp(\kappa)$ k-subsets in a single κ-subset, count κ-subsets of λ up to inclusion: when counting a set $X \in [\lambda]^\kappa$, delete all its subsets. Now

$$\binom{\lambda}{\kappa} = \{\min |\mathcal{F}| : \mathcal{F} \subseteq [\lambda]^\kappa \;\&\; \bigcup_{X \in \mathcal{F}} [X]^\kappa = [\lambda]^\kappa\}$$

- Clearly now $\lambda^\kappa = \exp(\kappa) \times \binom{\lambda}{\kappa}$. The point in writing λ^κ in this way is that all three relations, $\exp(\kappa) < \binom{\lambda}{\kappa}$, $\exp(\kappa) = \binom{\lambda}{\kappa}$ and $\exp(\kappa) > \binom{\lambda}{\kappa}$ are in fact possible.

Relation to the finite binomial

Since "finite" = "Dedekind finite", for any k-set, X, where k is a natural number, it follows that $[X]^k = \{X\}$.
Now,

$$\binom{n}{k} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [n]^k \ \& \ \bigcup_{X \in \mathcal{F}} [X]^k = [n]^k\}$$

The infinite binomial is thus an extension of the finite one.

Cardinal Arithmetic divided into two

- Tarski 1925: the function $\lambda \mapsto \lambda^{\mathrm{cf}\lambda}$ determines the function $(\lambda, \kappa) \mapsto \lambda^\kappa$ for all λ, κ.
- For a regular κ, $\kappa^{\mathrm{cf}\kappa} = \kappa^\kappa = \exp(\kappa)$.
- For a singular μ, let $\mathrm{binom}(\mu)$ denote $\binom{\mu}{\mathrm{cf}\mu}$. Thus $\mu^{\mathrm{cf}\mu} = \mathrm{binom}(\mu) \times \exp(\mathrm{cf}\mu)$.

All of infinite cardinal exponentiation is thus determined by:

- $\exp$ on regular cardinals;
- binom on singular cardinals.
- The functions $\exp$ and binom behave totally differently in ZFC.

Properties of $\exp$

- Weak monotonicity: $\kappa < \lambda \Rightarrow \exp(\kappa) \leq \exp(\lambda)$;
- Cantor's $\exp(\kappa) > \kappa$ and, more generally, König's Lemma: $\operatorname{cf} \exp(\kappa) > \kappa$.
- Easton: These are the only rules for $\exp$ on regular cardinals: any function satisfying those rules is as consistent with ZFC as ZFC itself.
- The Continuum Hypothesis is the statement $\exp(\aleph_0) = \aleph_1$
- The Generalized Continuum Hypothesis is the statement: for every cardinal κ, $\exp(\kappa) = \kappa^+$.

The CH

- $\mathfrak{c} := \exp(\aleph_0) = |\mathbb{R}|$ is a particularly interesting value.
- Cantor: CH should be true! A "dogma" of Cantor.
- Godel 1947: "certain facts (not known or not existing in Cantor's time) . . . seem to indicate that CH will turn out to be wrong." Gödel quotes the existence of Lusin and Sierpinski sets as example of "non-verifiable" consequences of CH, namely consequences of CH which are not known to hold without it.
- Gödel also mentions that " Not even an upper bound, however high, can be assigned to the power of the continuum. Nor [is it known] . . . whether this number is regular or singular, accessible or inaccessible . . . and what its character of cofinality is."

CH and ZFC

- After Cohen's invention of forcing, it became clear that $\exp(\aleph_0)$ could indeed assume every value which is not countably cofinal.
- The "complete freedom" governing the value of $\mathfrak{c}$ extends into the vast space of cardinal invariants of $\mathfrak{c}$. Cardinal invariants are definitions of uncountable cardinals which quantify the properties of various topological, algebraic and combinatorial structure on the continuum.
- An example of an interesting property of cardinal invariants is exhibited in models that satisfy Martin's Axiom and $\mathfrak{c} > \aleph_1$. In such models no cardinal between $\aleph_0$ and $\mathfrak{c}$ is realized as a cardinal invariant of the continuum.

The Goldstern-Shelah chaos

- Goldstern and Shelah, in reply to Blass: There are uncountably many simple cardinal invariants of the continuum that can be assigned regular values arbitrarily.
- In particular, one can arrange — in contrast to the situation in MA models — that every regular cardinal between $\aleph_0$ and $\mathfrak{c}$ is the covering number of some simple meager ideal.
- The same result shows that there is no classification of even the simple cardinal invariants of the continuum. The situation is is in fact worse than that.

Properties of binom

- $\mathrm{binom}(\mu) > \mu$.
- $\mathrm{binom}(\aleph_\omega)$ cannot be increased by small forcing.
- The Binomial Hypothesis: $\mathrm{binom}(\aleph_\omega) = \aleph_{\omega+1}$.
- The GBH: for every singular μ, $\mathrm{binom}(\mu) = \mu^+$.

***BH* and ZFC**

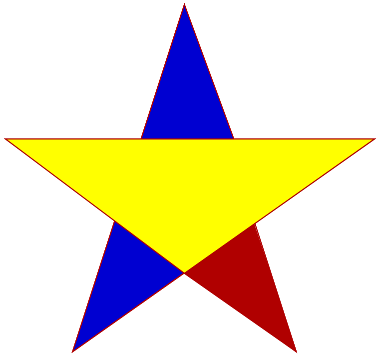
- Shelah 1990: There is an absolute bound:

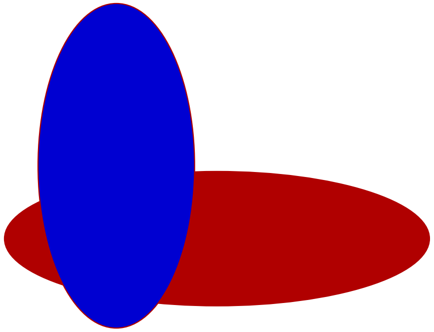
$$\text{ZFC} \vdash \mathrm{binom}(\aleph_\omega) < \aleph_{\omega_4}.$$

- BH can fail; however: the consistency of $\neg$BH is less credible than the consistency of ZFC;
- Many interesting consequences of the BH are in fact ZFC theorems!.
- The regular cardinals between $\aleph_\omega$ and $\mathrm{binom}(\aleph_\omega)$ are represented ias natural invariants of the combinatorial structure of $[\aleph_\omega]^\omega$.
- A variant of binom satisfies a form of GCH eventually in ZFC (Shelah).

Part II: $\mathbb{R}^d$

- Let $S \subseteq \mathbb{R}^d$ be a closed set. Let $\gamma(S)$ be the least number of convex subsets of S required to cover S.





- When $\gamma(S) > \aleph_0$, it is the covering number of some meager ideal.
  Let the convexity spectrum of $\mathbb{R}^d$ be the set of all uncountable convexity numbers of closed subsets $S \subseteq \mathbb{R}^d$.

Recent results

Geschke-K. 2002.

- For every $d \geq 3$ there is a closed set $S_d \subseteq \mathbb{R}^d$ so that $\gamma(S_d) \geq \gamma(S_{d+1})$ and so that for any $n > 3$ and a sequence $\kappa_3 > \kappa_4 \cdots > \kappa_n$ of regular cardinals there is a model of ZFC in which $\mathfrak{c} > \kappa_3$ and $\gamma(S_d) = \kappa_d$ for $3 \leq d \leq n$.
- The Dimension Conjecture: n, but no more than n, uncountable convexity numbers can be simultaneously realized in $\mathbb{R}^n$.
- Geschke 2003: The set $\gamma(S_{d+1})$ can consistently be smaller than $\gamma(S)$ for every closed $S \subseteq \mathbb{R}^d$,
- In $\mathbb{R}^2$ (and in $\mathbb{R}^1$) the dimension conjecture is true.

$\mathbb{R}^2$

(Geschke, K., Kubis, Schipperus 2001)

- Theorem: For every closed set $S \subseteq \mathbb{R}^2$ either there is a perfect $P \subseteq S$ with no 3 points from P in a single convex subset of S (and in this case $\gamma(S) = \mathfrak{c}$ in all models) or else there is a continuous pair coloring $c : [2^\omega]^2 \to 2$ so that $\gamma(S) = \mathfrak{hm}(c) := \operatorname{Cov} \mathcal{I}_c$, where $\mathcal{I}_c$ is the σ-ideal generated by c-monochromatic sets. In the latter case, in the Sacks model $\gamma(S) < \mathfrak{c}$.
- A closed $S \subseteq \mathbb{R}^2$ contains a perfect $P \subseteq S$ with no 3 points in a convex subset iff in the Sacks forcing extension $\gamma(S) = \mathfrak{c}$: This is a meta-mathematical characterization of a geometric fact.

Classification of continuus colorings

Geschke,Goldstern, K. 200?.

- There are two continuous pair colorings $c_{\min}$ and $c_{\max}$ so that for every Polish space X and a nontrivial continuous $c : [X]^2 \to 2$,

$$\mathfrak{hm}(c) = \mathfrak{hm}(c_{\min}) \quad \text{or} \quad \mathfrak{hm}(c) = \mathfrak{hm}(c_{\max})$$

- $\mathfrak{hm}(c_{\min}) = \mathrm{Cov}\,\mathrm{Lip}(2^\omega) \geq \mathfrak{d}$.
- It is consistent that $\mathfrak{hm}(c_{\min}) < \mathfrak{hm}(c_{\max})$
- (Zapletal) there is an optimal forcing notion for isolating $\mathfrak{hm}(c_{\min})$.

Covering numbers

- For a given set X, a set $\mathcal{F} \subseteq X^X$ of self-maps on X covers X^2 if for all $x, y \in X$ there is $f \in \mathcal{F}$ so that $f(x) = y \vee x = f(y)$. Equivalently, the graphs and inverses of graphs of all $f \in \mathcal{F}$ cover X^2.
- For a metric space X, let $\mathrm{Cov}(\mathrm{Cont}(X))$, respectively $\mathrm{Cov}(\mathrm{Lip}(X))$, denote the required numbers of continuous, respectively Lipschitz, self-maps required to cover X^2. $(\mathrm{Cov}(\mathrm{Fnc}(X))^+ \geq |X|$ for all infinite X.
- For a Hausdorff space X let $\mathfrak{d}(X)$ be number of compact subspaces of X required to cover X. In the Baire space $N^{\mathbb{N}}$ the number $\mathfrak{d}$ is the number of closed subsets of $\mathbb{R}$ not containing any rational number required to cover $\mathbb{R} \setminus \mathbb{Q}$. Also, $\mathfrak{d} = \mathrm{cf}(\mathbb{N}^{\mathbb{N}}, <^*)$.

The results

$$
\begin{array}{ccccc}
 & & & & 2^{\aleph_0} \\
 & & & & \uparrow \\
 & & & & \mathfrak{hm}(c_{\max}) \\
 & & & & \uparrow \\
\mathrm{Cov}(\mathrm{Lip}(\mathbb{R})) \geq & \mathrm{Cov}(\mathrm{Lip}(\omega^\omega)) & = \mathrm{Cov}(\mathrm{Lip}(2^\omega)) & = & \mathfrak{hm}(c_{\min}) \\
 & \uparrow & & & \\
\mathrm{Cov}(\mathrm{Cnt}(\mathbb{R})) = & \mathrm{Cov}(\mathrm{Cnt}(\omega^\omega)) & = \mathrm{Cov}(\mathrm{Cnt}(2^\omega)) & & \\
 & \uparrow & & & \\
 & \mathfrak{d} & & &
\end{array}
$$

- An arrow in the diagram is an irreversible ineqaulity. For any two rows in the diagram there is a model which separates them. Since at most two cardinals can appear in the diagram above $\mathfrak{d}$, four different models are required.

- The model in which $\mathrm{Cov}(\mathrm{Cnt}(\mathbb{R})) < \mathrm{Cov}(\mathrm{Lip}(\mathbb{R}))$ is a new optimal model. In it, covering $\mathbb{R}^2$ by continuous functions is easy, but covering it by Lipschitz continuous functions is hard.

- The model in which $\mathfrak{hm}(c_{\min}) < \mathfrak{hm}(c_{\max})$ is obtained by an iteration of a new tree-forcing, and required certain new finite combinatorial facts relating random graphs to perfect graphs, that were proved by Noga Alon for this purpose.

Is there a largest covering number?

- The invariant $\mathrm{Cov}(\mathrm{Cont}(2^\omega))$ satisfies that its successor cardinal is at least $\mathfrak{c}$; is there a classification of all simple invariants which satisfy this or even a more general condition?
- In particular, is $\mathfrak{hm}(c_{\max})$ the largest covering number of a nontrivial simple meager ideal?
- in contrast to the Goldstern-Shelah chaos that governs simple cardinal invariants of the continuum, the subclass of simple invariants that come from convexity in finite dimensional Euclidean spaces behave well and may be classifiable.

Forcing-free handling of $\mathfrak{c}$

- The results about convexity and continuous Ramsey Theory can be placed in a broader context that recently emerged from a few exciting "third generation" developments in the theory of cardinal invariants of the continuum:
- Ciecielski-Pawlikowski axiomatized the Sacks model by their Covering Property Axiom.
- Dzamonja-Hrusak-Moore introduced forms of the diamond that accompany well-known cardinal invariants and show that their diamonds have to be present in canonical models related to the invariants.
- Roslanowski-Shelah investigate new forcing notions related to the continuum and classify them.

- Zapletal proves the existence of optimal models for a large collection of invariants, axiomatizes them by parametrized versions of the CPA and shows that natural questions about the possible relations between invariants are in fact descriptive set theoretic problems.
- It looks like a more organized picture of the universe of cardinal invariants is emerging: formulating a naturally-occuring problem about Euclidean space properly, one may hope to be able to sort out the possible behaviour of cardinal arithmetic for deciding the problem without using forcing, by consulting one of the works mentioned above.

Part III: binomial arithmetic

- Let X be a topological space. For every infinite regular cardinal κ the following are equivalent:
  1. Every open cover of X of cardinality κ has a smaller subcover;
  2. Every set $A \in [X]^\kappa$ has a point $x \in X$ so that for every neighborhood u of x, $|u \cap A| = \kappa$: a point of complete accumulation.
- Thus, X is compact if and only if for every regular cardinal k, X has the k-CAP property.
- Alexandrov-Uryson 1929: What if one drops $\aleph_0$ in compactness? Does one get the property of Lindelöf? At least every increasing open cover has a countable subcover: Lindelöfness.

LLnL spaces below $\mathrm{binom}(\aleph_\omega)$

- A Linearly Lindelöf not Lindelöf space is a space that satisfies CAP for all regular $\kappa > \aleph_0$ but is not Lindelöf.
- LLnL spaces were constructed by Miscenko 1962, Buzyakova-Grunhage 1995, Kunen 2001 (a locally compact space), Arkhangelskii-Buzyakova (a realcompact space from $2^{\aleph_0} = 2^{\aleph_0}$). All these spaces have exponential powers.
- However, the properties of LLnL spaces are not related to exponential arithmetic; the construction of such spaces is binomial in nature.

SLLnL space

- Consider a property stronger than CAP: For a regular cardinal κ and a topological space X, the SCAP property holds if for every $A \in [X]^\kappa$ there is $B \subseteq A$ in $[X]^\kappa$ that converges to a point, namely, has a unique CAP point.
- A space X satisfies SCAP for all regular k if and only if it is compact scattered (Mrowka, Rajagopalan and Soundararajan 1974).
- Call a space X Sequentially Linearly Lindelöf if it satisfies SCAP for all regular $\kappa > \aleph_0$

Results

- K.-Lubitch 2002

$$\aleph_\omega, \overbrace{\aleph_{\omega+1}, \aleph_{\omega+2}, \ldots, \aleph_{\alpha+1}}^{\text{SLLnL spaces}} = \mathrm{binom}\,\aleph_\omega, \ldots \ldots \aleph_{\omega_1} < \exp(\aleph_0)$$

One has simultaneously infinitely many different SLLnL spaces on all successor cardinals in an infinite interval of cardinals.

- All of this can happen below the continuum. The constructions rest on the solid combinatorial structure below $\mathrm{binom}(\aleph_\omega)$ and are not affected by adding reals.
- One can also have realcompact LLnL spaces below $2^{\aleph_\omega}$ in this way.

Two problems by Erdős and Hechler

- A Maximal Almost Disjoint family over a cardinal λ is a family $\mathcal{F} \subseteq [\lambda]^\lambda$ so that $|A \cap B| < \lambda$ for any two distinct members of $\mathcal{F}$ and $\mathcal{F}$ is maximal with respect to this property. Let $\mathfrak{a}_\lambda$ denote the least cardinality of a MAD family over λ (and $\mathfrak{a} = \mathfrak{a}_\omega$).
- Erdoős and Hechler proved in 1973 that the GCH implies tha existence of an $\aleph_\omega$-MAD family (that is, a MAD family over $\aleph_\omega$) of cardinality $\aleph_\omega$ and asked:
- (1) is it ever possible that an $\aleph_\omega$-MAD family of size $\aleph_\omega$ does not exist? In particular, does MA with $2^{\aleph_0} > \aleph_\omega$ imply that such a family does not exists?
- (2) does $2^{\aleph_0} < \aleph_0$ imply the existence of an $\aleph_\omega$-MAD family of size $\aleph_\omega$?

Results

- This is a great test question for comparing the use of exponential arithemtic with the use of binomial arithmetic.
- Answers: Yes to both questions (K.-Kubis-Shelah 2003); and much more.
- MAD families over $\aleph_\omega$ can be constructed using binomial properties of $\mathrm{binom}(\aleph_\omega)$ and can be destroyed using cardinal invariants of $\mathfrak{c}$. This gives an almost complete freedom in determining the cardinals in which $\aleph_\omega$-MAD families exist over $\aleph_\omega$.
- For example, for every $\alpha \leq \beta < \omega_1$ it is possible to have a universe of ZFC in which MAD families exist exactly in the interval $[\aleph_{\omega+\alpha+1}, \aleph_{\omega+\beta+2}]$.

A recipe for a paper

- Recently, Brendle constructed a model in which $\mathfrak{a} = \aleph_\omega$.
- Compute $\mathfrak{a}_{\aleph_\omega}$ in Brendle's model.
- If $\mathfrak{a}_{\aleph_\omega} < \aleph_\omega$ then you have solved
  Open Problem 1: "Is it consistent that $\mathfrak{a}_{\aleph_\omega} < \mathfrak{a}$?"
- Otherwise, you have solved
  Open Problem 2: "Is it consistent that $\mathfrak{a}_{\aleph_\omega} = \aleph_\omega$?"
- Please send the paper to Brendle and to me.

Universal objects

Let's finish with a model theoretic problem.

- If T is a first order theory with the strict order property (e.g. linear order, Boolean algebra, etc.) then it has a universal model, in fact a saturated model, at every strong limit cardinal μ (For linear ordering this was proved by Hausdorff in his **Mengenlehre** book, 1914).
- If μ is a singular below the first fixed point of second order then also the converse holds: if μ is not a strong limit then T does not have a universal model in μ (K.-Shelah 1991).
- Problem: Is the statement "there is a universal linear order in cardinality μ iff μ is a strong limit" true for all singular cardinals μ?

Concluding remarks

- Cardinal arithmetic is divided into two disjoint functions: exp on regulars and binom on singulars, each with its distinct properties.
- Only two rules govern the behaviour of exp; many rules have been discovered about binom. Accordingly, exp plays a role once its possible behaviours is known, whereas the rules governing binom can be used directly.
- There is an impressive and effective body of knowledge that allows one to find out what is possible and what is not possible in the realm of cardinal invariants of the continuum, and this has consequences about the geometry of $\mathbb{R}^d$.

- The rigid structure associated with binom enables many constructions in topology, algebra and combinatorics which are robust in the sense that they survive the easy manipulations of binom.
- To conclude: cardinal arithmetic has a good potential for applications. However, one must first verify whether the problem at hand is related to exponential arithmetic, binomial arithmetic (or both!), and then pull the right tool out of the toolbox.