Universal Abelian Groups

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ABSTRACT

We examine the existence of universal elements in classes of infinite abelian groups. The method is defining some invariant of a group relative to a club guessing sequence, a combinatorial tool marketed here to algebraists. We prove, for example:

Theorem: For $n \geq 2$, there is a purely universal separable p-group in $\aleph_n$ if, and only if, $2^{\aleph_0} \leq \aleph_n$.

Theorem: If $2^{\aleph_0} \leq \lambda$ and there is μ such that $\mu^+ < \lambda < \mu^{\aleph_0}$, then there is no universal separable p-group of cardinality λ.

§0 Introduction

In this paper "group" will always mean "infinite abelian group", and "cardinal" and "cardinality" always refer to infinite cardinals and infinite cardinalities.

We call a group G **universal for a class of groups** K **in cardinality** λ if every $H \in K$ such that $|H| \leq \lambda$ is isomorphic to a subgroup of G. The objective of this paper is to examine the existence of universal groups in various well-known classes of infinite abelian groups. We also investigate the existence of **purely universal** groups, groups with the property that every other group of equal cardinality in the class is isomorphic to one of its pure subgroups.

The main set theoretic tool we use is a club guessing sequence. This is a prediction principle which has enough power to control properties of an infinite object which are defined by looking at all its possible enumerations. Unlike the diamond, club guessing sequences are proved to exist in ZFC. Therefore using them does not require any additional

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axioms beyond the usual axioms of ZFC. An important feature of this principle is its usefulness in proving theorems from **negations** of CH and GCH.

The paper is organized as follows: in Section 1 we define an invariant of a group relative to a club guessing sequence, and show that it is monotone in pure embeddings. In Section 2 we show how to construct various groups with prescribed demands on their invariant. In Section 3 all this is used to investigate the existence of universal groups in classes of torsion groups and classes of torsion free groups. It appears that cardinal arithmetic decides the question of existence of a universal group in quite many cardinals. For example: there is a purely universal separable p-group in $\aleph_n$ iff $\aleph_n \geq 2^{\aleph_0}$ for all $n \geq 2$ (for $n = 1$ only the "if" part holds).

This paper follows two other papers by the same authors, [KjSh 409] and [KjSh 447], in which the existence of universal linear orders, boolean algebras and models of unstable and stable unsuperstable first order theories were examined using the same method.

All the abelian group theory one needs here, and more, is found in [Fu], to whose system of notation we adhere. An acquaintance with ordinals and cardinals is necessary, and some familiarity with stationary sets helps. Knowledge of chapter II in [EM] is more than enough.

Before getting on, we first notice that in every infinite cardinality there are universal groups which are divisible:

0.1 Theorem: In every cardinality there is a universal group, universal p-group (for every prime p), universal torsion group and universal torsion-free group.

Proof: These are, respectively, the direct sum of λ copies of the rational group Q together with λ copies of $Z(p^\infty)$ for every prime p; the direct sum of λ copies of $Z(p^\infty)$; the direct sum of λ copies of $Z(p^\infty)$ for every prime p; and the direct sum of λ copis of Q. That these groups are universal, each for its class, follows from the structure theorem for divisible groups and the fact that every group (p-group, torsion group, torsion-free group) is embeddable in a divisible group (p-group, torsion group, torsion-free group) of the same cardinality ([Fu] I, 23 and 24]) 0.1

§1 The invariant of a group relative to the ideal id($\overline{C}$)

A fixed assumption in this Section is that λ is a regular uncountable cardinal. We assume the reader is familiar with the basic properties of closed and unbounded sets of λ, and with the definition and basic properties of stationary sets.

1.1 Definition: For a group G, $nG \stackrel{\text{def}}{=} \{ng : g \in G\}$. Two elements $g, h \in G$ are **n-congruent** if $g - h \in nG$. If g, h are n-congruent, we also say that h is an **n-congruent** of g.

1.2 Definition:

(1) ([Fuch, p.113]) Let G be group. A subgroup $H \subseteq G$ is a **pure** subgroup, denoted by $H \subseteq_{pr} G$, if for all natural n, $nH = nG \cap H$.

(2) An embedding of groups $h : H \to G$ is a **pure** embedding if its image $h(H)$ is a pure subgroup of G.

1.3 Definition: Suppose that λ is a regular uncountable cardinal and that G is a group of cardinality λ. A sequence $\overline{G} = \langle G_\alpha : \alpha < \lambda \rangle$ is called a **λ-filtration** of G iff for all α

(1) $G_\alpha \subseteq G_{\alpha+1}$

(2) G_α is of cardinality smaller than λ

(3) if α is limit, then $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$

(4) $G = \bigcup\limits_{\alpha<\lambda} G_\alpha$.

Suppose $\overline{G} = \langle G_\alpha : \alpha < \lambda \rangle$ is a given representation of a group G. Suppose $c \subseteq \lambda$ is a set of ordinals, and the increasing enumeration of c is $\langle \alpha_i : i < i(*) \rangle$. Let $g \in G$ be an element. We define a way in which g chooses a subset of c:

1.4 Definition: $\mathrm{Inv}_{\overline{G}}(g, c) = \{\alpha_i \in c : g \in \bigcup_n ((G_{\alpha_i+1} + nG) - (G_{\alpha_i} + nG))\}$

We call $\mathrm{Inv}_{\overline{G}}(g,c)$ **the invariant of the element g relative to the λ-filtration $\overline{G}$ and the set of indices** c.

Worded otherwise, the invariant of an element g relative to a given λ-filtration and set of indices, is the subset of those indices α_i such that by increasing the group G_{α_i} to the bigger group $G_{\alpha_{i+1}}$, for some n an n-congruent for g is introduced.

As the definition of the invariant depends on a λ-filtration, one may think that the invariant does not deserve its name. Indeed, given a group G equipped with two respective λ-filtrations $\overline{G}$ and $\overline{G}'$, it is not necessarily true that for $g \in G$

$$\mathrm{Inv}_{\overline{G}}(g,c) = \mathrm{Inv}_{\overline{G}'}(g,c) \tag{1}$$

The amendment to this problem is working with a club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S\rangle$. The idea is as follows: for any pair of λ-filtrations $\overline{G}$ and $\overline{G}'$ of the same group G there is a club of indices $E \subseteq \lambda$ such that for every $\alpha \in E$, $G_\alpha = G'_\alpha$. So if we chose our set c in the definition of invariant to consist only of such "good" α-s, namely if $c \subseteq E$, then it would not matter according to which λ-filtration we work. But we cannot choose a set c which will be a subset of every club E resulting from some pair of λ-filtrations. What we **can** do is find a sequence of c-s with the property that for every club $E \subseteq \lambda$, stationarily many of them are susets of E. Thus we will be able to define an invariant that will prove to be independent of a particular choice of a λ-filtration. Here is the precise formulation of this:

1.5 Definition: A sequence $\langle c_\delta : \delta \in S\rangle$, where $S \subseteq \lambda$ is a stationary set, $c_\delta \subseteq \delta$ and $\delta = \sup c_\delta$ for every δ, is called a **club guessing sequence** if for every club $E \subseteq \lambda$ the set $\{\delta \in S : c_\delta \subseteq E\}$ is a stationary subset of λ.

The theorems asserting the existence of club guessing sequences will be quoted later. A club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S\rangle$ gives rise to an ideal $\mathrm{id}(\overline{C})$ over λ, the **guessing ideal**.

1.6 Definition: Suppose that $\overline{C} = \langle c_\delta : \delta \in S \rangle$ is a club gussing sequence. We define a proper ideal (namely a notion of "small" sets) $\mathrm{id}(\overline{C})$ as follows:

$$A \in \mathrm{id}(\overline{C}) \Leftrightarrow A \subseteq \lambda \ \& \ \exists E \subseteq \lambda,\ E \text{ club},\ \& \ \forall \delta \in E \cap S, c_\delta \not\subseteq E.$$

So a set of δ-s is small if there is a club E which it fails to guess stationarily often, namely there is no $\delta \in A$ such that $c_\delta \subseteq E$.

1.7 Lemma: If $\overline{C}$ is a club guessing sequence as above, then $\mathrm{id}(\overline{C})$ is a proper, λ complete ideal.

Proof: That $\mathrm{id}(\overline{C})$ is proper means that it does not contains every subset of λ. Indeed, $S \notin \mathrm{id}(\overline{C})$, as it guesses every club. That $\mathrm{id}(\overline{C})$ is downward closed is immediate from the definition. Suppose, finally, that $A_i, i < (*)$, $i(*) < \lambda$ are less than λ sets in the ideal. We wish to show that their union $A \stackrel{\text{def}}{=} \bigcup\limits_{i<i(*)} A_i$ is in the ideal. Pick a club E_i for every $i < i(*)$ which demonstrates that $A_i \in \mathrm{id}(\overline{C})$, namely $\delta \in A_i \Rightarrow c_\delta \not\subseteq E_i$. The set $E = \bigcap\limits_{i<i(*)} E_i$ is a club. We show that E demonstrates that $A \in \mathrm{id}(\overline{C})$: Suppose that $\delta \in A$. Then there is some $i < i(*)$ such that $\delta \in A_i$. Therefore $c_\delta \not\subseteq E_i$. But $E \subseteq E_i$, so necessarily $c_\delta \not\subseteq E$. 1.7

We adopt the term "for almost every δ in S", by which we mean "for all $\delta \in S$ except for a set in $\mathrm{id}(\overline{C})$".

1.8 Lemma: Suppose that $\overline{C} = \langle c_\delta : \delta \in S \rangle$ is a club guessing sequence on $S \subseteq \lambda$. Suppose that $\overline{G}$ and $\overline{G}'$ are two λ-filtrations of a group G of cardinality λ. Then for almost every $\delta \in S$, (1) holds for every $g \in G$.

Proof: : The set of $\alpha < \lambda$ for which $G_\alpha = G'_a$ is a club. Let us denote it by E. If for some δ, $c_\delta \subseteq E$ holds, then for every $g \in G$ it is true that $\mathrm{Inv}_{\overline{G}}(g, c_\delta) = \mathrm{Inv}_{\overline{G}'}(g, c_\delta)$. But as $\overline{C}$ is a club guessing sequence, by definition, for almost every δ, $c_\delta \subseteq E$. 1.8

We define now an invariant of a group.

1.9 Definition: Suppose that $\overline{C}$ is a club guessing sequence and that $\overline{G}$ is a λ-filtration of a group G of cardinality λ. Let

(1) $P_\delta(\overline{G},\overline{C}) = \{\mathrm{Inv}_{\overline{G}}(g,c_\delta) : g \in G\}$

(2) $\mathrm{INV}(G,\overline{C}) = [\langle P_\delta(\overline{G},\overline{C}) : \delta \in S\rangle]_{\mathrm{id}(\overline{C})}$

The P_δ is the collection of all subsets of c_δ which serve as the invariant of some $g \in G$. The second item should read "the equivalence class of the sequence of P_δ modulo the ideal $\mathrm{id}(\overline{C})$", where two sequences are equivalent modulo an ideal if the set of coordinate in which the sequences differ is in the ideal.

1.10 Lemma: The definition of $\mathrm{INV}(G,\overline{G})$ does not depend on the λ-filtration $\overline{G}$.

Proof: Suppose that $\overline{G}'$ is another λ-filtration. There is a club E such that for every $\alpha \in E$, $G_\alpha = G'_\alpha$. Therefore for every δ such that $c_\delta \subseteq E$ and every $g \in G$, $\mathrm{Inv}_{\overline{G}}(g,c_\delta) = \mathrm{Inv}_{\overline{G}'}(g,c_\delta)$. This means that for every δ such that $c_\delta \subseteq E$, $P_\delta(\overline{G},\overline{C}) = P_\delta(\overline{G}',\overline{C})$. But for almost all δ it is true that $c_\delta \subseteq E$, therefore the sequqnces $\langle P_\delta(\overline{G},\overline{C}) : \delta \in S\rangle$ and $\langle P_\delta(\overline{G}',\overline{C}) : \delta \in S\rangle$ are equivalent modulo $\mathrm{id}(\overline{C})$. 1.10

We have exhibited here a general method of defining well an invariant of a group with the help of representing it as an increasing union of smaller groups. This depends, though, on the existence of a club guessing sequence! Strangely enough, we can prove the existence of club guessing sequences for regular uncountable cardinals λ for all such cardinals **except** $\aleph_1$. We believe that this method of making definitions may have further algebraic applications beyond those made in this paper.

Let us now quote the relevant theorems which assert the existence of club guessing sequences:

1.11 Theorem: If μ and λ are cardinals, $\mu^+ < \lambda$ and λ is regular, then there is a club guessing sequence $\langle c_\delta : \delta \in S\rangle$ such that the order type of each c_δ is μ.

By [Sh-g] Ch. III: [Sh 365] or see the appendix to [KjSh 409] for proofs of this.

Preservation of INV under pure embeddings

We procced to show that INV is preserved, in a way, under pure embeddings.

1.12 Lemma: Suppose that H and G are groups of cardinality λ and that $\overline{H}$ and $\overline{G}$ are λ-filtrations. Suppose that $\overline{C}$ is a club guessing sequence on $S \subseteq \lambda$. If $h : H \to G$ is a pure embedding, then for almost every $\delta \in S$ $P_\delta(\overline{H}, \overline{C}) \subseteq P_\delta(\overline{G}, \overline{C})$.

Proof:

Suppose for simplicity that $H \subseteq_{pr} G$, namely that the embedding is the identity function. The set $E_1 \stackrel{\text{def}}{=} \{\alpha : H \cap G_\alpha = H_\alpha\}$ is a club. Define for every natural number n a function $f_n(y)$ on G as follows:

$$f_n(y) = \begin{cases} \text{some} \quad x \in \{x : x \in H \,\&\, (x+y) \in nG\} & \text{if } \{x : x \in H \,\&\, (x+y) \in nG\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

There is a club E_2 such that G_α is closed under f_n for all n for every $\alpha \in E_2$. $E = E_1 \cap E_2$ is a club.

1.13 Claim: Suppose that $h \in H$ and that $\alpha \in E$. Then h has an n-congruent in G_α (in the sense of G) iff h has an n-congruent in H_α (in the sense of H).

Proof: One direction is trivial. Suppose, then, that there is an n-congruent $g \in G_\alpha$. Let $h' = f_n(g)$. By the definition of f_n, $h' + g \in nG_\alpha$; also $h - g \in nG$. Therefore $h - h' \in nG$. As $H \subseteq_{pr} G$, $h - h' \in nH$, and therefore h' is an n-congruent of h in the sense of H. 1.13

The proof of the Lemma follows now readily: For almost every $\delta \in S$ it is true $c_\delta \subseteq E$. Therefore for every such δ, every $h \in H$ and every n, h has an n-congruent in H_α iff h has an n-congruent in G_α. Therefore $\mathrm{Inv}_{\overline{H}}(h, c_\delta) \subseteq \mathrm{Inv}_{\overline{G}}(h, c_\delta)$. 1.12

§2 Constructing groups with prescribed INV

We need to be able to construct a group G such that for a prescribed list of sets $\langle A_\delta : \delta \in S\rangle$, $A_\delta \in P_\delta(\overline{G}, \overline{C})$. In this Section we construct several groups with such prescribed demands on their INV. These will be used in the next Section for showing that in certain cardinals universal groups do not exist. The technique in all the constructions will be attaching to a simply defined group points from a topological completion of the group.

a. Constructions of p-groups

2.1 Theorem: If λ is a regular uncountable cardinal, $\overline{C} = \langle c_\delta : \delta \in S\rangle$ is a club guessing sequence and $A_\delta \subseteq c_\delta$ is a given set of order type ω, then there is a separable p-group G of cardinality λ and λ-filtration $\overline{G}$ such that $A_\delta \in P_\delta(\overline{G}, \overline{C})$ for every $\delta \in S$.

2.2 Remark: This implies by Theorem 1.12 that for every separable p-group G' of cardinality λ into which G can be purely embedded and a λ-filtration $\overline{G}'$, for almost every δ, $A_\delta \in P_\delta(\overline{G}', \overline{C})$

Proof: For every n, let $B_n = \bigoplus_{\eta \in {}^n\lambda} A_\eta$ where A_η is a copy of Z_{p^n} with generator a_η. Let $G^0 = \bigoplus_n B_n$, and let G^1 be the torsion completion of G^0. G^1 may be identified with all sequences $(x_1, x_2, \cdots)$ where $x_n \in B_n$ and such that there is a (finite) bound to $\{o(x_n)\}_n$. For details see [Fuchs II,14–21]. The group we seek lies between G^0 and G^1, and is a pure subgroup of G^1.

Let us make a simple observation:

(1) If $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ belong to G^1 and $x - y \in p^n G^1$, then $x_i = y_i$ for all $i \le n$.

Proof: Let $z_i = x_i - y_i$. $z_i \in B_i$. As $B_i \cap p^n G^1 = 0$ for $i \le n$, we are done.

For every $\delta \in S$ let $\langle \alpha_n^\delta : n \in \omega\rangle$ be the increasing enumeration of A_δ. Denote by η_n^δ the sequence $\langle \alpha_1^\delta, \cdots, \alpha_n^\delta\rangle$. Let $b_\delta^0 \in G^1$ be $(x_1^\delta, x_2^\delta, \cdots)$ where $x_n^\delta = p^{n-1} a_{\eta_n^\delta}$. So x_n is of order p and height $n-1$. Consequently, b_δ^0 is of order p. Let us denote

$$\frac{x_k^\delta}{p^n} \stackrel{\text{def}}{=} \begin{cases} p^{k-n-1} a_{\eta_k^\delta} & \text{if } n < k \\ 0 & \text{otherwise} \end{cases}$$

and also let $\frac{0}{p^n} \stackrel{\text{def}}{=} 0$. Let $b_\delta^n = (\cdots, \frac{x_k^\delta}{p^n} \cdots)$. Let G be the subgroup of G^1 generated by G^0

together with $\{b^n_\delta : \delta \in S,\ n < \omega\}$. Having defined G, let us specify a λ-filtration $\overline{G}$. For every $i < \lambda$ let G_i be $\langle \{a_\eta : \eta \in {}^{<\omega}i\} \cup \{b^n_\delta : \delta \in S,\ \delta < i,\ n < \omega\}\rangle$.

2.3 Claim: $\mathrm{Inv}_{\overline{G}}(b^0_\delta, c_\delta) = A_\delta$.

Proof: We should show that the set of indices i with the property that in G_{i+1} some congruent of b^0_δ appears coincides with A_δ. Suppose first, then, that $i = \alpha_n$ for some n. $(x^\delta_1, \cdots, x^\delta_n, 0, 0, \cdots)$ is clearly a p^n-congruent of b^δ_δ, as $b^0_\delta - (x^\delta_1, \cdots, x^\delta_n) = p^n b^n_\delta$. Conversely, suppose that $i < \alpha_n$ and suppose to the contrary that there is some $y = (y_1, y_2, \cdots) \in G_i$ such that $b^0_\delta - y \in P^n G$. Then $y_i = x_i$ for all $i \leq n$ by (1). But $y \in G_i$ implies that $y_n \in G_i$ — a contradiction to $i < \alpha_n$. 2.3,2.1

We shall prove now a strengthening of the previous theorem: We shall construct a separable p-group such that the demands on its INV will hold in any separable p-groups of the same cardinality extending it — not necessarily as a pure extension. The price we pay for this is that the cardinality of the group we construct must be at least $2^{\aleph_0}$.

2.4 Theorem: Suppose that $\lambda \geq 2^{\aleph_0}$ is a regular cardinal, and that $\overline{C} = \langle c_\delta : \delta \in S\rangle$ is a club guessing sequence . Suppose also that for every δ a subset $A_\delta \subseteq c_\delta$ of order type ω is given. Then there exists a separable p-group G of cardinality λ and a λ-filtration $\overline{G}$ of G with the property that for every separable p-group H of cardinality λ (with λ-filtration $\overline{H}$) and embedding $\varphi : G \to H$, for almost every δ, $A_\delta \in P_\delta(\overline{H}, \overline{C})$.

Proof: We define $G^0 = \bigoplus\limits_n B_n$, where $B_n = \bigoplus\limits_{\eta \in {}^{<\omega}\lambda} A^n_\eta$ is a direct sum of isomorphic copies of Z_{p^n}. We fix a^n_η as a generator of A^n_η. G^1 is the torsion completion of G^0. The group G we seek lies between G^0 and G^1.

Enumerate in the sequence $\langle \Gamma_i : i < 2^{\aleph_0}\rangle$ all infinite subsets of ω. For every $i < 2^{\aleph_0}$ we throw into G an element $b^i_{\delta,0}$, (which will have A_δ as its invariant). Let $\langle \eta(n) : n < \omega\rangle$ be the increasing enumeration of A_δ. We define $b^i_{\delta,0} = \langle x^\delta_1, x^\delta_2, \cdots\rangle$, an element in G^1, as follows:

$$x_n^\delta = \begin{cases} 0 & \text{if } n \notin \Gamma_i \\ p^{n-1} a^n_{\eta\ k} & \text{if } n \text{ is the } k\text{-th member of } \Gamma_i \end{cases}$$

Let

$$\frac{x_l^\delta}{p^n} \stackrel{\text{def}}{=} \begin{cases} p^{l-n-1} a^n_{\eta\ k} & \text{if } x_l^\delta = p^{l-1} a^l_{\eta\ k} \text{ and } l > n \\ 0 & \text{otherwise} \end{cases}$$

Let G be the subgroup of G^1 generated by $G^0 \bigcup \{b^i_{\delta,n} : \delta \in S,\ i < 2^{\aleph_0},\ n < \omega\}$, where $b^i_{\delta,n}$, for $n > 0$, is $(\cdots, \frac{x^\delta_k}{p^n}, \cdots)$. Let $G_\alpha = \langle \{a^n_\nu : n < \omega,\ \nu \in {}^n\alpha\} \cup \{b^i_{\delta,n} : \delta \in S,\ \delta < \alpha,\ n < \omega\}\rangle$.

Suppose now that $\varphi : G \to H$ is an embedding, H any separable p-group of cardinality λ. By picking a basic subgroup $H_0 = \bigoplus_n C_n$ of H, we may identify elements of H with the bounded sequences $\langle y_1, y_2, \cdots\rangle$ such that $y_n \in C_n$. Fix any λ-filtration $\overline{H}$ of H. Clearly, the set $E = \{\alpha < \lambda : \varphi[G_\alpha] \subseteq H_\alpha \ \&\ \varphi^{-1}[H_\alpha] = G_\alpha\}$ is a club of λ. Therefore it is enough to show that for every $\delta \in S$ such that $c_\delta \subseteq E$, $A_\delta \in P_\delta(\overline{H}, \overline{C})$.

Let us look, then, at one such δ. Observe that if $a \in G$ is of height k, then the height of $\varphi(a)$ is at least k. Define by induction x_n as follows: $x_1 = a^1_{\eta\ 1}$, and $x_{n+1} = p^{h(\varphi(x_n))} a^{h(\varphi(x_n))+1}_{\eta\ (n+1)}$.

There exists some index $i < 2^{\aleph_0}$ such that $b^i_{\delta,0} = \langle x_1^\delta, 0, 0, \cdots, x_2, 0 \cdots\rangle$ is such that x_k^δ is the k-th non-zero element of $b_\delta^{i,0}$. Let us see that $\mathrm{Inv}_{\overline{H}}(\varphi(b^i_{\delta,0}), c_\delta) = A_\delta$. Let α_k be the k-th member of A_δ. Let $n = h(\varphi(x_k)) + 1$. We shall see that in $H_{\alpha_{k+1}}$ a p^n-congruent to $\varphi(b^i_{\delta,0})$ is introduced. First, $\varphi(x_1, 0, \cdots 0, x_k)$ is clearly a p^n-congruent, for $\varphi(b^i_{\delta,0} - (x_1, 0 \cdots 0, x_k)) = p^n \varphi(b^i_{\delta,n})$. As $\alpha_k, \alpha_{k+1} \in E$, $\varphi((x_1, 0 \cdots, x_k)) \in G_{\alpha_{k+1}} \setminus G_{\alpha_k}$. On the other hand, $h(\varphi((x_1, 0, \cdots, 0, x_k))) < n$, so as a p^n-congruent is unique modulo $p^n H$, there is no p^n-congruent of $\varphi(b^i_{\delta,0})$ in H_{α_k}. 2.4

b. Constructions of torsion-free groups

We start by constructing a torsion-free homogeneous group of a given type $\mathbf{t} = (\infty, \cdots, \infty, 0, \infty \cdots)$. We recall that a characteristic $\chi(g)$ of an element $g \in G$ is the sequence $(k_1, k_2, \cdots)$ where k_l is the p-height of g for the l-th prime. The height can be ∞. A type $\mathbf{t}$ is an equivalence class of characteristics modulo the equivalence relation of having only a finite difference in a finite number of coordinates. A homogeneous group is a group in which all elements have the same type. We call a type $\mathbf{t}$ a p-type if $\mathbf{t} = (\infty, \cdots, \infty, 0, \infty \cdots)$ where the only coordinate in which there is 0 is the number of p in the list of primes.

2.5 Theorem: For every uncountable and regular cardinal λ, a club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S \rangle$ and given sets $A_\delta \in c_\delta$, each A_δ of order type ω, there is a homogeneous group G of cardinality λ with p-type $\mathbf{t}$ and a λ-filtration $\overline{G}$ such that for every $\delta \in S$, $A_\delta \in P_\delta(\overline{G}, \overline{C})$.

2.6 Remark: This means that for every pure extension G' of G, for almost every $\delta \in S$, $A_\delta \in P_\delta(\overline{G}', \overline{C})$.

Proof: This proof resembles the proof of Theorem 2.1. Let $G^0 = \bigoplus_\lambda Q_p$ (where Q_p is the group or rationals with denominators prime to p). We index the isomorphic copies of Q_p by $\eta \in {}^n\lambda$ and fix a_η, an element a_η of characteristic $(\infty, \cdots, \infty, 0, \infty \cdots)$ in the η-th copy of Q_p. Let G^1 be the completion of G^0 in the p-adic topology. Let $\langle \eta^\delta(n) : n < \omega \rangle$ be the increasing enumeration of A_δ, and let $\eta^\delta_n = \langle \eta^\delta(0), \cdots, \eta^\delta(n-1) \rangle$. Let $b_{\delta,n} = \sum_k p^{k-n} a_{\eta_k}$. The rest is as in the proof of Theorem 2.1. 2.5

Working above the continuum, we choose to construct two types of groups: homogeneous groups and slender groups.

2.7 Theorem: Suppose that $\lambda > 2^{\aleph_0}$ is a regular cardinal, and that $\overline{C} = \langle c_\delta : \delta \in S \rangle$ is a club guessing sequence. Suppose also that for every $\delta \in S$ a subset $A_\delta \subseteq c_\delta$ of order type ω is given. Then there exists a homogeneous group G of p-type $\mathbf{t}$ of cardinality λ and a λ-filtration $\overline{G}$ of G with the property that for every reduced torsion-free group H of cardinality λ (with λ-filtration $\overline{H}$) and embedding $\varphi : G \to H$, for almost every δ,

$A_\delta \in P_\delta(\overline{H}, \overline{C})$.

Proof: With the same adaptations as in the previous proof, this resembles the proof of Theorem 2.4. 2.7

We recall that a a slender group is a group into which there is no homomorphism of $P = \prod_{\aleph_0} \mathbb{Z}$ that sends an infinite number of e_n to non-zero images. Slender groups are reduced, and a group of cardinality smaller than $2^{\aleph_0}$ is slender iff it is reduced (see [Fu], II, 94). The demands on INV will be preserved under any embedding into a reduced group.

2.8 Theorem: Suppose that $\lambda > 2^{\aleph_0}$ is a regular cardinal, and that $\overline{C} = \langle c_\delta : \delta \in S\rangle$ is a club guessing sequence. Suppose also that for every δ a subset $A_\delta \subseteq c_\delta$ of order type ω is given. Then there exists a slender group G of cardinality λ and a λ-filtration $\overline{G}$ of G with the property that for every reduced torsion-free group H of cardinality λ (with λ-filtration $\overline{H}$) and embedding $\varphi : G \to H$, for almost every δ, $A_\delta \in P_\delta(\overline{H}, \overline{C})$.

Proof: For every pair $\langle \eta, \rho\rangle$ such that $\eta \in {}^{<\omega}\lambda$, $\rho \in {}^{<\omega}\omega$ and $\lg \eta = \lg \rho$, let $x_{\langle \eta, \rho\rangle}$ be a free generator. Let $G^0 \stackrel{\text{def}}{=} \bigoplus \langle x_{\langle \eta, \rho\rangle}\rangle$. Let $G^1 \stackrel{\text{def}}{=} \widehat{G}^0$, the Z-adic completion of G^0. The group we seek lies between G^0 and G^1. Let Δ be the set of increasing ω-sequences of integers $\rho = \langle \rho(0), \rho(1), \cdots\rangle$ such that $n!\rho(n) | \rho(n+1)$. For every $\delta \in S$ let η_δ enumerate the members of A_δ.

For every $\delta \in S$ we throw into G the elements $b_{\eta_\delta, \rho, k}$ for all $\rho \in \Delta$ and natural k, where $b_{\eta_\delta, \rho, 0} = \sum_{l<\omega} \rho(l) x_{\eta_\delta \restriction l, \rho \restriction l}$ and $b_{\eta_\delta, \rho, k+1} = \sum_{k<l<\omega} \frac{\rho(l)}{\rho(k)} x_{\eta \restriction l, \rho \restriction l}$.

Let $G_\alpha = \langle \{x_{\langle \eta, \rho\rangle} : \eta \in {}^{<\omega}\alpha,\ \rho \in {}^{<\omega}\omega,\ \lg \eta = \lg \rho\} \cup \{b_{\eta_\delta, \rho, k} : \delta \in S,\ \delta < \alpha,\ \rho \in \Delta,\ k < \omega, \}\rangle$. Let $G = \bigcup_{\alpha<\lambda} G_\alpha$.

For an element $g \in G$ we define the support $\operatorname{supp} g$ in the following manner: The quotient $G/n!G$ is naturally isomorphic to $\bigoplus \mathbb{Z}_{n!} x_{\langle \eta, \rho\rangle}$. Therefore we can define $\operatorname{supp}_n g$ as the support in this direct sum of $[g]_{n!G}$, (where by "support" of a sum we mean the finite set of $x_{\langle \eta, \rho, \rangle}$ with non zero coefficients in this sum). We let $\operatorname{supp} g \stackrel{\text{def}}{=} \bigcup_n \operatorname{supp}_n G$. Trivially, an element $g \in G$ is in the closure of $\bigoplus_{x_{\langle \eta,\rho,\rangle} \in \text{supp}\, g} x_{\langle \eta,\rho,\rangle}$. Also, $\text{supp}\,(g+h) \subseteq \text{supp}\, g \cup \text{supp}\, h$. We notice also the following obvious fact:

2.9 Fact: If $g \in G$ then there is a finite set $\{b^1_{\eta_{\delta^1},\rho^1,k^1}, \cdots, b^l_{\eta_{\delta^l},\rho^l,k^l}\}$ such that $\text{supp}\, g$ is almost included in $\bigcup_{0<i<l} \text{supp}\, b^i_{\eta_{\delta^i},\rho^i,k^i}$, where "almost" means "except for finitely many elements".

We should show that G is slender. We prove the following

2.10 Claim: For every set of non-zero elements $\{y_n : n < \omega\}$ such that $n!|y_n$, there is a sequence of integer coefficients a_n such that the series $\sum a_n y_n$ does not converge in G.

First, let us point out that this claim implies the slenderness of G. Suppose that $\varphi : P \to G$ is a homomorphism satisfying that (without loss of generality) for all n, $\varphi(e_n) \neq 0$. Then let $y_n = n!\varphi(e_n)$. Apply the claim to obtain a member $x \in P$ for which $\varphi(x)$ is not defined.

Proof: (of Claim) For a finite set $B \subseteq {}^\omega\lambda \times {}^\omega\omega$, we let $A_B = \{x_{\langle \eta \restriction n, \rho \restriction n\rangle} : \langle \eta, \rho\rangle \in B,\ n < \omega\}$. For $A = A_B$ for some finite B as above, we define an endomorphism $T_A : G \to G$ as follows:

$$T_A(x_{\langle \eta,\rho\rangle}) = \begin{cases} x_{\langle \eta,\rho\rangle} & \text{if } \langle \eta,\rho\rangle \in A \\ 0 & \text{otherwise} \end{cases}$$

The image of T_A is a closed and countable subgroup of G.

Let $y_n \in n!G$ be given for all $n < \omega$. $y_n = a_0^n + \sum_{l<l(n)} a_l^n b_{\eta_{\delta_l^n},\rho_l^n,k_l^n}$, where $a_0^n \in G_0$. We distinguish two cases:

First case: There is a finite set B as above such that for infinitely many n, $\text{supp}\, y_n \cap A_B$ is not empty. Let A denote A_B.

We assume for simplicity that for every n, $T_A(y_n) \neq 0$. If every infinite sum $\sum a_n y_n$ existed in G, then the image under T_A of every sum would exist in $\text{Im}T_A$. But this is a contradiction to The countability of $\text{Im}T_A$.

Second case: For every finite set $B \subseteq {}^{\omega}\lambda \times {}^{\omega}\omega$, there are only finitely many n for which $\operatorname{supp} y_n \cap A_B$ is not empty. We may assume by passing to a subsequence that the $\{\operatorname{supp} y_n : n < \omega\}$ is a collection of pairwise disjoint sets.

We choose coefficients c_n by induction: let c_{n+1} be so big, that for all $\sum_{m \leq n} c_m y_m \notin c_{n+1}G$. Call the resulting sum g. The choice of the coefficients guarantees that that $\operatorname{supp} g$ is not almost included in any finite union of supports of $b_{\eta_\delta,\rho,k}$, and therefore by Fact 2.9 is not in G. 2.10

Suppose now that $\overline{H}$ is a λ-filtration of a torsion-free group of cardinality λ and that $\varphi : G \to H$ is a monomorphism. It is enough to show that for every δ wich satisfies that $\alpha \in c_\delta \Rightarrow \varphi[G_\alpha] \subseteq H_\alpha \wedge \varphi^{-1}[H_\alpha] = G_\alpha$, $A_\delta \in P_\delta(\overline{H}, \overline{C})$. We define by induction an ω-sequence $\rho \in \Delta$. Let $\rho(0) = 1$. Suppose that $(\rho(0), \cdots, \rho(n-1))$ are defined, and for convenience denote by $\rho \; n$ this finite sequence. Let $\rho(n)$ be so big that $\sum\limits_{m<n} \rho(m)\varphi(x_{\eta_d \; m,\rho \; m} \notin \rho(n)G$ and $n!\rho(n-1)|\rho(n)$. The sequence ρ just defined is clearly in Δ. We let the reader verify that $\operatorname{Inv}_{\overline{H}}(\varphi(b_{\eta_\delta,\rho,o}), c_d) = A_\delta$. 2.8

§3 The Theorems

a. The Universality Spectrum of Torsion groups

There is universal torsion group in λ iff there is a universal p-group in λ for every prime p. We therefore may focus on p-groups alone. There is a universal divisible p-group in λ, the group $\bigoplus\limits_{\lambda} Z(p^\infty)$, therefore the first interesting question to ask in torsion groups is whether there is a universal **reduced** p-group. Here the answer is "no":

3.1 Theorem: If λ is an infinite cardinal (not necessarily regular, not necessarily uncountable) then ther is no universal reduced p-group in λ.

Proof: There are p-groups of cardinality λ of Ulm length σ for every ordinal $\sigma < \lambda^+$. As $u(A) \leq u(B)$ whenever $A \subseteq B$, and $u(A) < \lambda^+$, for every group of cardinality λ, no p-group of cardinality λ can be universal. 3.1

We put a further restriction on the class of p-groups, by demanding that the Ulm length of a group be at most ω.[1] We restrict ourselves then to the class of separable p-groups. On this class see [Fu] vol II, chapter XI.

b. Universal separable p-groups

In this Section we investigate the universality spectrum of the class or separable p-groups.

3.2 Theorem: If $\lambda = \lambda^{\aleph_0}$ then there is a purely universal separable p-group in λ.

Proof: Let $B = \bigoplus B_n$ where $B_n = \bigoplus_{\lambda} Z_{p^n}$. The torsion completion of B, denoted by G, is of cardinality $|B|^{\aleph_0} = \lambda^{\aleph_0} = \lambda$ and is puely universal in λ. To see this let A be any separable p-group of cardinality λ, and let B_A be its basic subgroup. B_A is purely embeddable in B, and this gives rise to a pure embedding of A into G. 3.2

We see then, that for every n such that $\aleph_n \geq 2^{\aleph_0}$ there is a purely universal separable p-group in $\aleph_n$. Therefore CH implies that in every $\aleph_n$ there is a purely universal separable p-group. It is not uncommon that CH adds enough information to decide questions in algebra. It is much less common, though, that a negation of CH adds information. (This, we believe, is the reason that although CH and $\neg$CH are equally consistent with ZFC, it is CH that has become popular among mathematicians). We shall see now that also a negation of CH may be useful.

3.3 Theorem: If λ is regular and uncountable, and $\aleph_1 < \lambda < 2^{\aleph_0}$, then there is no purely universal separable p-group.

Proof: Let G be any separable p-group of cardinality λ, and we shall show that it is not purely universal. We fix some presentation $\overline{G}$, and by Theorem 1.11 choose some club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S \rangle$ where S is a stationary subset of λ. We may assume

[1] One can make finer distictions here by considering the class of all p-groups of Ulm length which is bounded by an ordinal σ. But we do not do this here.

that the order type of each c_δ is ω. As $|G| = \lambda$, for every $\delta \in S$, $|P_\delta(\overline{G}, \overline{C})| = |\{\mathrm{Inv}_{\overline{G}}(g, c_\delta) : g \in G\}| \leq \lambda$. As $|\mathcal{P}(c_\delta)| = 2^{\aleph_0} > \lambda$, there is some $A_\delta \subseteq c_\delta$, $A_\delta \notin P_\delta(\overline{g}, \overline{c})$. Having thus chosen a_δ for every $\delta \in s$, we invoke Theorem 2.1 to obtain a group H and a λ-filtration $\overline{H}$ such that for every $\delta \in S$, $A_\delta \in P_\delta(\overline{H}, \overline{C})$. If H were purely embedded in G, then by theoren 1.12, for almost every δ in S, A_δ would belong to $P_\delta(\overline{G}, \overline{C})$. This clearly contradicts the choice of A_δ. Therefore H is not purely embeddable in G, and G is therefore not purely universal. 3.3

3.4 Corollary: For $n \geq 2$, there is a purely universal separable p-group in $\aleph_n$ if, and only if, $2^{\aleph_0} \leq \aleph_n$.

Next we handle cardinals which are bigger than $2^{\aleph_0}$.

3.5 Theorem: $\lambda > 2^{\aleph_0}$ is regular and there is some μ such that $\mu^+ < \lambda < \mu^{\aleph_0}$ then there is no universal separable p-group in λ.

Proof: By $\mu^+ < \lambda$ and Theorem 1.11, we may pick some club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S\rangle$ where S is a stationary set of λ and $\operatorname{otp} c_\delta = \mu$. Suppose G is a given separable p-group. We will show that G is not universal by presenting a separable p-group H of cardinality λ which is not embeddable in G. We choose a λ-filtration $\overline{G}$ of G and observe that $|P_\delta(\overline{G}, \overline{C})| \leq \lambda$ for every δ. As $\mu^{\aleph_0} > \lambda$, there is some $A_\delta \subseteq c_\delta$, of order type ω, which dose not belong to $P_\delta(\overline{G}, \overline{C})$. By Theorem 2.4, there is a group H of cardinality λ such that for every embedding $\varphi : H \to G$, for almost every δ, $A_\delta \in P_\delta(\overline{G}, \overline{C})$. This can hold only emptily, that is, if there are no such embeddings, because A_δ was chosen such that $A_\delta \notin P_\delta(\overline{G}, \overline{C})$ 3.5

Examining the conditions on λ in the last three theorems, one discovers that for purely universal separable p-groups there are cases for regular λ which are not covered by them: If $\operatorname{cf}\lambda = \lambda < \lambda^{\aleph_0}$, there is some first $\mu < \lambda$ for which $\mu^{\aleph_0} > \lambda$. This μ must be of cofinality $\aleph_0$, or else $\mu^{\aleph_0} = \bigcup_{\alpha<\mu} \alpha^{\aleph_0} \leq \lambda$, as for every $\alpha < \mu$, $\alpha^{\aleph_0} \leq \lambda$. So there are two cases: $\mu^+ < \lambda$

and $\mu^+ = \lambda$. The first case is covered by 3.2 (if $\mu = \aleph_0$) or by 3.5, (if $\mu > 2^{\aleph_0}$). The second case is not covered here. The authors plan to write a paper with consistency results, in which it will be shown that in cardinals which fall in the second case the existence of a universal separable p-group is consistent (namely, one cannot prove that there is no such group).

Also, Theorem 3.5 eliminates the existence of universal separable p-groups in λ, $\mu^+ < \lambda < \mu^{\aleph_0}$, while Theorem 3.2 eliminates only the existence of **purely**-universal separable p-groups in λ, $\aleph_1 < \lambda < 2^{\aleph_0}$. A consistency result showing that in a regular cardinal $\lambda < 2^{\aleph_0}$ there may exist a universal separable p-group will also be presented. We shall add here a theorem that shows that the existence of a universal separable p-group in cardinality λ, when $\lambda < 2^{\aleph_0}$, is equivalent to the existence of a universal separable p-group with countable dimensions.

3.6 Definition: For a cardinal λ let $G_\lambda = \bigoplus_n B_n^\lambda$ where $B_n^\lambda = \bigoplus_{i<\lambda} \mathbb{Z}(p^n)x_i^n$. Let $\widehat{G}_\lambda$ be the torsion completion of G_λ. The form of an element in $\widehat{G}_\lambda$ is $\sum_{n<\omega, i<\lambda} c_{n,i}x_i^n$ where for each n only a finite number of $c_{n,i}$ are not zero, and such that there is a common bound p^k to the orders of all the terms in the sum.

It is clear that every separable p-group of cardinality λ is realized as a pure subgroup of $\widehat{G}_\lambda$.

When $\lambda \leq 2^{\aleph_0}$ we observe the following fact:

3.7 Fact: If $\lambda \leq 2^{\aleph_0}$ than $\widehat{G}_\lambda$ is embeddable into $\widehat{G}_{\aleph_0}$

Proof: For brevity let us denote $G_{\aleph_0}$ as G_0. $G_0 = \bigoplus B_n^0$, where B_n^0 is a direct sum of $\aleph_0$ many copies of $\mathbb{Z}(p^n)$. As it makes no difference which countable index set we use to enumerate the summands of B_n^0, let $B_n^0 = \bigoplus_{\eta \in {}^n2, m<\omega} \mathbb{Z}(p^n)y_\eta^{n,m}$. As $\lambda < 2^{\aleph_0}$ we can find λ many sequences $\eta_i^n \in {}^\omega 2$ $(n < \omega, i < \lambda)$. We define a homomorphism $\varphi : \widehat{G}_\lambda \to G_0$ by specifying the images of each x_i^n. So $\varphi(x_i^n) = \sum_{n \leq m} p^{m-n} y^{m,n}_{\eta_i^m \restriction n}$. We extend φ to all of $\widehat{G}_\lambda$

by continuity, namely $\varphi(\sum_{n,i} c_{i,n} x_i^n) = \sum_{n,i} c_{i,n}\varphi(x_i^n)$.

We have yet to show that φ is 1-1. Suppose, then, that $\sum_{n,i} c_{i,n} x_i^n$ is a non zero element of $\widehat{G}_\lambda$. We shall see that its image under φ is not zero. Suppose that n_0, i_0 are such that $c_{n_0,i_0} x_{i_0}^{n_0} \neq 0$. Let all i for which $c_i^{n_0} \neq 0$ be in the set $\{i_0, \cdots, i_k\}$. There is some $l(*) > n_0$ such that for every $0 < j \leq k$, $\eta_{i_0} \ l(*) \neq \eta_{i_j} \ l(*)$. Now $\varphi(\sum_{n,i} c_{i,n} x_i^n) = \sum_{n,i} c_{i,n} \sum_{n \leq m} p^{m-n} y^{m,n}_{\eta_i \ n} = \sum_{i,n \leq m} c_{i,n} p^{m-n} y^{m,n}_{\eta_i \ n}$. As $c_{n_0,i_0} x_{i_0}^{n_0} \neq 0$, $p^{n_0} \not| c_{i_0}^{n_0}$, and it follows that $c_{n_0,i_0} p^{l(*)-n_0} y^{l(*),n_0}_{\eta_{i_0}} \ l(*) \neq 0$. But this term appears only once in the infinite sum above, and cannot be cacelled, as all other therms are in a direct summand of G_0 not containing it. We conclude that the image is not zero. 3.7

In light of the last Theorem, if $\lambda < 2^{\aleph_0}$, then the existence of a separable p-group of cardinality λ in which every subgroup of $\widehat{G}_0$ of cardinality λ is embedded, implies the existence of a universal separable p-group in λ (every separable p-group of cardinality λ is embedded in $\widehat{G}_\lambda$; $\widehat{G}_\lambda$ is embedded in $\widehat{G}_0$; so the composition of both embeddings embedds G in $\widehat{G}_0$ as a subgroups of cardinality λ).

c. The Universality Spectrum of Torsion-Free Group

We may restrict ourselves in this Section to the class of reduce torsion-free groups. We shall look at some subclasses of this class. We proceed to show now that in regular λ which satisfy $\lambda = \lambda^{\aleph_0}$ there is a universal reduced torsion-free group. This Theorem is an isolated point in the paper with respect to the technique, because its proof employs model theoretic notions of first order theory, elementary embedding and saturated model. These are available in every standard reference on model theory, like [CK].

3.8 Theorem: if $\lambda = \lambda^{\aleph_0} \geq 2^{\aleph_0}$, then there is a universal reduced torsion-free group in λ.

Proof: Let T be a complete first order theory of torsion free-groups. It is enough to find a reduced group G_T of cardinality λ such that $G_T \models T$ and for every $H \models T$, H is embedded in G_T; for if we have such a G for every T, the group $\bigoplus_T G_T$ is of cardinality λ (there are only $2^{\aleph_0}$ complete first order theories), and is evidently universal.

Let, then, G'_T be a saturated model of T of cardinality λ. Let D be its maximal division subgroup, and let $G_T \stackrel{\text{def}}{=} G'_T/D$. G_T is isomorphic to the direct summand of D, and is therefore torsion-free and reduced. Suppose that $H \models T$ is reduced (and, clearly, torsion-free). There is an elementary embedding $f : H \to G'_T$.

3.9 Claim: $\text{Im} f \cap D = 0$

Proof: Suppose $0 \neq a \in H$ and $f(a) \in D$. As f is elementary, a is divisible in H by every integer n. As H is torsion-free, the set of all divisors of a generates a divisible subgroup of H, contrary to H being reduced.

We conclude, therfore, that $\hat{f}$ defined by $\hat{f}(a) = f(a) + D$ is an embedding of H into G_T. 3.8

Next we show that below the continuum there is no purely-universal reduced torsion-free group. The reason for this is trivial: there are $2^{\aleph_0}$ types (over the empty set) in this class. Therefore we do not need the club guessing machinery, and gain an extra case – the case where $\lambda = \aleph_1$ — in comparison to Theorem 3.3.

3.10 Theorem: If $\lambda < 2^{\aleph_0}$ then there is no purely-universal reduced torsion-free group in cardinality λ. In fact, for every reduced torsion-free group G of cardinality λ there is a rank-1 group R which is not purely embeddable in G.

Proof: As $\lambda < 2^{\aleph_0}$, there is a characteristic $(k_1, k_2 \cdots)$, with all k_i finite, which is not equal to $\chi_G(g)$ for every $g \in G$ (The definitions of **characteristic** and **type** are from [Fu], II, 85). Let R be a rank-1 group such that $\chi_R(1) = (k_1, k_2 \cdots)$. As pure embeddings preserve the characteristic, R is not purely embeddable in G. 3.10

We may restrict ourselves to a subclasse of torsion-free groups which does not have many types above the empty set. This is the class of homogeneous groups (a group is

homogeneous if all non zero elements in the group have the same type. See [Fu] II p.109). Here we are able to prove with the aid of club guessing, that a cardinal $\lambda < 2^{\aleph_0}$ is not in the pure universality spectrum of the class of homogeneous groups with p-type $\mathbf{t}$, **if** $\lambda > \aleph_1$.

3.11 Theorem: If λ is a regular cardinal, $\aleph_1 < \lambda < 2^{\aleph_0}$, and $\mathbf{t}$ is a given p-type, then there is no purely universal group in λ for the class of homogeneous groups whose type is $\mathbf{t}$.

Proof: Let G be any homogeneous group with type $\mathbf{t}$, and fix some λ-filtration $\overline{G}$. Let $\overline{C}$ be a club guessing sequence, and for every $\delta \in S$ let A_δ be such that $A_\delta \notin P_\delta(\overline{G}, \overline{C})$. Such an A_δ exists, as $|P_\delta(\overline{G}, \overline{C})| \leq \lambda < 2^{\aleph_0}$, while there are $2^{\aleph_0}$ subsets of c_δ. By Theorem 2.5, there is a homogeneous group H with type $\mathbf{t}$ such that $A_\delta \in P_\delta(\overline{H}, \overline{C})$ for every $\delta \in S$. If there were a pure embedding $\varphi : H \to G$, then by Theorem 1.12, for almost every $\delta \in S$, A_δ would be in $P_\delta(\overline{G}, \overline{C})$. But by the choice of A_δ this is impossible. 3.11

We now examine cardinals λ for which $\lambda < \lambda^{\aleph_0}$. We shall prove for two classes of torsion-free groups, the class of slender groups and the class of homogeneous groups, that in certain regular λ, not only is there no universal group in the class, but also there is no torsion-free group (even outside the class) which is universal for the class.

3.12 Theorem: $\lambda > 2^{\aleph_0}$ is a regular cardinal, and there is some μ such that $\mu^+ < \lambda < \mu^{\aleph_0}$ then there is no universal homogeneous group of p-type $\mathbf{t}$ in λ. In fact, there is no reduced torsion-free group G such that every homogeneous group of type $\mathbf{t}$ of cardinality λ is embeddable in it.

Proof: We use the same argument as in previous theorems. We exploit Theorem 2.7 to obtain a homogeneous group of type $\mathbf{t}$ with a prescribed "non-embeddable" INV. 3.12

3.13 Theorem: $\lambda > 2^{\aleph_0}$ and there is some μ such that $\mu^+ < \lambda < \mu^{\aleph_0}$ then there is no universal reduced slender group in λ. In fact, there is no reduced torsion-free group G

such that every slender group of cardinality λ is embeddable in it.

Proof: As before, using this time theorem 2.8. 3.13

3.14 Corollary: Under the assumptions of this theorem, there is no universal reduced torsion-free group in cardinality λ.

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