CANTOR BENDIXON DEGREES AND CONVEXITY IN $\mathbb{R}^2$

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Abstract. We present an ordinal rank, $\delta^3$, which refines the standard classification of non-convexity among closed planar sets. The class of closed planar sets falls into a hierarchy of order type $\omega_1 + 1$ when ordered by $\delta$-rank.

The rank $\delta^3(S)$ of a set $S$ is defined by means of topological complexity of 3-cliques in the set. A 3-clique in a set $S$ is a subset of $S$ all of whose unordered 3-tuples fail to have their convex hull in $S$. Similarly, $\delta^n(S)$ is defined for all $n > 1$.

The classification cannot be done using $\delta^2$, which considers only 2-cliques (known in the literature also as “visually independent subsets”), and in dimension 3 or higher, the analogous classification is not valid.

1. Introduction

Let $S$ be a set in a linear space, and suppose that $S$ is not convex. One would like to measure how far $S$ is from being convex. The most natural number for measuring non-convexity of a set $S$ is the least number of convex subsets of $S$ needed to cover $S$. Let us, then, define $\gamma(S)$ as the least cardinality of a collection of convex sets whose union equals $S$. The function $\gamma$ is adopted as the basic measurement of non convexity. Classification by $\gamma$ gives countably many different classes of sets with finite $\gamma$ and (potentially) only two classes with infinite $\gamma$: sets with countable $\gamma$ and sets with uncountable $\gamma$.

In this paper we define for each $n > 1$ a degree functions $\delta^n$, and show that $\delta^3$ refines the $\gamma$-classification for closed, planar sets. The class $\{S : S \subseteq \mathbb{R}^2$ is closed and $\gamma(S) \leq \aleph_0\}$ is divided by $\delta^3$ to $\aleph_1$ sub-classes, while $\{S : S \subseteq \mathbb{R}^2$ is closed and $\gamma(S) > \aleph_0\}$ is a single $\delta^3$-class.

The first step in understanding the structure of a set $S$ with $\gamma(S) = \lambda$ is to understand why $S$ fails to decompose into a union of fewer than $\lambda$ convex sets.

There is an easy sufficient condition for $S$ not to be a union of fewer than $\lambda$ convex sets: the existence of a subset $P \subseteq S$ of cardinality $\lambda$, with the property that for any two points in $P$, the line segment connecting them is not contained in $S$. No two of those points can sit in the same convex subset of $S$, hence $S$ is not a union of $n$ convex sets. Call a subset of $S$ with this property “visually independent”. Let $\alpha(S)$ be the supremum of cardinalities of all visually independent subsets in $S$.

Does $\alpha$ measure non-convexity adequately? This can be rephrased as whether there exists a “reasonable” function $f$ so that $\gamma(S) \leq f(\alpha(S))$.

For general sets this is badly false (see [3], Section 5), and also in “nice” sets in dimension 3 or higher the connection between $\alpha$ and $\gamma$ is not well behaved. Nevertheless, closed sets in $\mathbb{R}^2$ show some tight connections between $\alpha$ and $\gamma$. A long sequence of results [4, 9, 2, 1] culminated in the discovery [3] that $\gamma(S) \leq f(\alpha(S))$ for some function $f$, for closed planar sets. Later it was shown that $f$ is at most $n^6$. 


in [8]. Recently, $n^6$ was lowered to $18n^3$ by Matousek and Valtr in [6], where also a lower bound of $O(n^2)$ was set.

In sets which are not a finite union of convex sets, the connection between $\alpha$ and $\gamma$ is not as tight. There exist compact, planar sets with countable $\alpha$ and uncountable $\gamma$ ([5], Example 2.1). Put differently, the class of closed planar sets with countable $\alpha$ contains also sets with uncountable $\gamma$. This means that the notion of visual independence does not capture all the information as to why a closed $S \subseteq R^2$ cannot be covered by countably many convex subsets.

However, a generalization of visual independence does. Call a subsets $P \subseteq S$ a 3-clique if every 3-element subset $X \subseteq P$ satisfies that its convex closure is not contained in $S$. Theorem 2.2 in [5] says that a closed set in the plane is not a countable union of convex sets if, and only if, it contains an uncountable (actually perfect) 3-clique. Namely, the only reason for such $S$ not to be a countable union of convex sets is that it contains an uncountable 3-clique.

Since, by this theorem, the full information about non-convexity of a closed planar set is stored in the collection of its 3-cliques, it is natural to try and classify non-convexity of such sets by classifying their 3-cliques. It turns out that the standard topological classification of closed countable sets — the Cantor-Bendixson degree — indeed works: for every closed planar set which is a countable union of convex sets there exists a countable ordinal which bounds the Cantor-Bendixson degrees of all 3-cliques in $S$. This ordinal is, then, the degree of non-convexity of $S$.

1.1. Statement of the results. Let $S$ be a subset of a linear space. Call $P \subseteq S$ an $n$-clique in $S$ if $n \geq 2$ and for every $n$-element subset $X$ of $P$ the convex closure of $X$ is not contained in $S$. Let $\delta^n(S)$ be the supremum of Cantor-Bendixson degrees of all $n$-cliques in $S$. Since every $n$-clique is also an $n + 1$-clique, $\delta^n(S) \leq \delta^{n+1}(S)$ for all $n \geq 2$. The rank $\delta(S)$ is the supremum over Cantor-Bendixson degrees of $n$-cliques for all $n$, that is, $\delta(S) = \sup\{\delta^n(S) : n < \omega\}$.

It is proved that for every closed set in a polish linear space, an uncountable $\delta$ implies an uncountable $\gamma$. Surely, if $\delta(S)$ is uncountable then $\delta^n(S)$ is uncountable for some $n$. In the case of closed planar sets this $n$ has to be $\leq 3$, by [5], theorem 2.2. Thus, (Corollary 6 below) for a closed planar $S$, the rank $\delta^3(S)$ is countable if and only if $S$ is a countable union of convex sets. In other words, $\delta^3$ refines $\gamma$ on closed planar sets by breaking the class of closed $S \subseteq R^2$ to $\aleph_1 + 1$ classes, so that the top class is that of sets with uncountable $\gamma$ and all smaller classes stratify the class of sets with countable $\gamma$. Since for countable sets the rank $\delta^n$ clearly coincides with the usual Cantor-Bendixson degree, any closed set of Cantor-Bendixson degree $\alpha$ is an example of a set with $\delta^3(S) = \alpha$; is is easy to construct uncountable sets of degree $\alpha$ as well.

Perles’ Example 2.1 in [5], in which $\gamma$ is uncountable yet $\delta^2$ equals 1, shows that one cannot get similar classification with $\delta^2$ instead of $\delta^3$. In dimension $d > 2$ a compact set may have $\delta$-rank 1 but still have an uncountable $\gamma$ (see [5], Example 4.2). Thus, Corollary 6 is sharp in two senses: first, $\delta^3$ cannot be weakened to $\delta^2$ and $R^2$ cannot be replaced by $R^d$.

The last remark suggests that classification of closed, non-convex sets in $R^d$ requires other methods. A more complicated rank function classifies non convexity of closed sets in all finite dimensions. Such rank function exists and will be presented in [7].
1.2. History. Infinite unions of convex sets were studied in [5]. We refer the reader to that paper for basic facts and examples concerning such sets.

The following problem is still open:

**Problem 1.** Is it true that a closed planar set in which the closure of every visually independent subset is countable, is a countable union of convex sets?

This problem was asked by G. Kalai and only very minor step towards solving it direction was made ([5], Theorem 4.2).

1.3. Notation. Our notation is standard, except, maybe, for denoting the set of natural numbers by $\omega$. A topological space $X$ is polish if it is complete, metric and separable. A sequence is a function whose domain is an initial segment of $\omega$. By $n^\omega$ we denote the space of all infinite sequences over $n$ symbols and by $n^k$ the set of sequences of length $k$ over $n$ symbols. The space $n^\omega$ is topologized by declaring the set of all infinite sequences that extend a given finite sequence as a basic open set.

This topology is polish, by the metrics which assigns to two sequences $\eta, \nu \in n^\omega$ the distance $1/k$ where $k$ is the first coordinate in which $\eta$ and $\nu$ are different. We write $\eta \prec \nu$ to denote that the sequence $\eta$ is an initial segment of the sequence $\nu$ and by $\eta \cdot \nu$ the concatenation of $\eta$ with $\nu$ is denoted.

2. Cantor-Bendixon degrees and convexity

We begin by recalling the definition of the Cantor-Bendixon degree of a set $S$ in some topological space $X$. A point $x \in S$ is isolated in $S$, if there is an open neighborhood $u \ni x$ so that $S \cap u = \{x\}$. By induction on ordinals define the $\alpha$-th derived subset of $S$:

1. $S^{(0)} = S$
2. $S^{(\alpha+1)} = S^{(\alpha)} \setminus \{x : x \text{ is isolated in } S^{(\alpha)}\}$
3. If $\alpha$ is limit, then $S^{(\alpha)} = \bigcap_{\beta < \alpha} S^{(\beta)}$

Let $\text{rk}(S)$, the Cantor-Bendixon degree of $S$ be the least ordinal $\alpha$ for which $S^{(\alpha)} = S^{(\alpha+1)}$. Thus, for example, the Cantor-Bendixon degree of a set which is dense in itself is 0.

**Fact 2.** If $S$ is a subset of a polish space and $\text{rk}(S) = \alpha + 1$ then there is a closed subset $C \subseteq S$ with $\text{rk}(C) = \text{rk}(S)$.

Given a set $S$ and a point $x \in S$, the degree of $x$ in $S$, which we denote by $\text{rk}_S(x)$ is the last ordinal $\alpha$ for which $x \in S^{(\alpha)}$, if $x$ does not belong to $S^{(\alpha)}$ for all $\alpha$; if $x \in S^{(\alpha)}$ for all $\alpha$, we say that $\text{rk}_S(x) = \infty$. Clearly, for every set $S$ and $\beta < \text{rk}(S)$ there are points $x \in S$ with $\text{rk}_S(x) = \beta$ (but $S^{\text{rk}(S)}$ may be null).

We remark that a separable metric space is second countable, and therefore the Cantor-Bendixon degree of every set in such a space is always countable.

3. Proofs

**Definition 3.** Let $S$ be a set in a topological vector space. Let $\delta^n(S)$ be the supremum over all Cantor-Bendixon degrees of $n$-cliques in $S$. Let $\delta(S) := \sup \{\delta^n(S) : n < \omega\}$.

**Theorem 4.** Suppose that $S$ is a closed set in a polish linear space $E$ and $\gamma(S) \leq \aleph_0$. Then $\delta(S) \leq \omega_1$. 

Proof. Suppose that $S \subseteq E$, $E$ is a polish linear space and $\delta(S) = \omega_1$. Let $n \geq 2$ be the least so that $\delta^n(S) = \omega_1$. We may assume, then, that there are closed $n$-cliques of unbounded (countable) Cantor-Bendixon degrees in $S$.

Lemma 5. Suppose that $u$ is an open neighborhood in $E$ and that $u$ contains $n$-cliques in $S$ of unbounded Cantor-Bendixon degrees. Then there exist open neighborhoods $u_0, \ldots, u_{n-1}$ such that for every $i < n$, $\text{cl}u_i \subseteq u$, $u_i$ contains cliques of unbounded Cantor-Bendixon degrees and so that for every choice of $y_i \in \text{cl}u_i$, $\text{conv}(y_0, \ldots, y_{n-1}) \not\subseteq S$.

Proof of Lemma. Fix a countable base $B$ for the topology of $E$ (e.g. all balls of rational radius and a center in some countable dense set).

Define now a mapping from $\omega_1$ to $n$-tuples from $B$, $\beta \mapsto (u_0^\beta, \ldots, u_{n-1}^\beta)$, as follows. Let $\beta < \omega_1$ be given. Choose first an $n$-clique $P \subseteq u$ and $n$ points in $P$, $x_0^\beta, \ldots, x_{n-1}^\beta$ such that $\text{rk}_P(x_i^\beta) \geq \beta$. Since the complement of $S$ is open, there are open neighborhoods $u_i$ of $x_i^\beta$ for $i < n$ so that for every choice of $y_i \in u_i$ it holds that $\text{conv}(y_0, \ldots, y_{n-1}) \not\subseteq S$. By shrinking each $u_i$, we may assume that $u_i \in B$, $\text{cl}u_i \subseteq u$ and that $\text{conv}(y_0, \ldots, y_{n-1}) \not\subseteq S$ for every choice of $y_i \in \text{cl}u_i$. Let $(u_0^\beta, \ldots, u_{n-1}^\beta) := (u_0, \ldots, u_{n-1})$.

Since there are only countably many $n$-tuples from $B$, there is a fixed $n$-tuple $(u_0, \ldots, u_{n-1})$ and an unbounded $I \subseteq \omega_1$ so that $(u_0, \ldots, u_{n-1}) = (u_0^\beta, \ldots, u_{n-1}^\beta)$ for every $\beta \in I$.

Therefore, for every $i < n$ and an ordinal $\beta < \omega_1$, there exists a closed clique $P$ and a point $x \in P \cap u$ with $\text{rk}_P(x) \geq \beta$. Since $u_i$ is open, $\text{rk}(P \cap u_i) \geq \beta$. Therefore each $u_i$ contains cliques of unbounded degrees.

Suppose now that $S$ is closed, $\gamma(S) \leq \aleph_0$ and $S$ contains $n$-cliques of unbounded degrees. By induction on $k$ define neighborhoods $u_\eta$ for $\eta \in n^k$ so that:

1. $d(u_\eta) < 1/k$ for all $\eta, \nu \in n^k$.
2. $\eta < \nu \Rightarrow \text{cl}u_\nu \subseteq u_\eta$.
3. $u_\eta$ contains closed cliques in $S$ of unbounded Cantor-Bendixon degrees.
4. if $\eta_0, \ldots, \eta_{n-1}$ are distinct and agree up to $k - 1$ then for every choice of $y_i$ from $\text{clf}(\eta_i)$ the convex closure of $\{y_0, \ldots, y_{n-1}\}$ is not contained in $S$.

At stage $k + 1$ use Lemma 5 to find, for each $\eta \in n^k$, sub-neighbohoods $\{u_\eta^\gamma : i < k\}$ of $u_\eta$, which satisfy conditions 1-4 above.

Suppose now that $u_\eta$ is defined for every finite sequence over $n$ and define $g : n^\omega \to S$ by $g(\eta) := \bigcap_k u_\eta^k$. Since $E$ is complete, $g$ is well defined. Since $S$ is closed, $g(\eta) \in S$ for every $\eta \in n^\omega$.

Suppose now that $S = \bigcup_n C_n$. The space $n^\omega$ of all infinite sequences over $n$ symbols is a complete separable metric space under the metrics $d(\eta, \nu) = 1/k$ for the least $k$ such that $\eta(k) \neq \nu(k)$. By the Baire category theorem, there is some index $m$ so that $f^{-1}(C_m)$ is somewhere dense. Choose some $k$ and a sequence $\nu \in n^k$ so that $f^{-1}(C_m)$ is dense in $\{\eta \in n^\omega : \nu \not< \eta\}$. For every $i < n$ there must be, then, a sequence $\eta_i$ so that $\eta_i[k + 1 = \eta_i^\gamma$, and $f(\eta_i) \in C_m$. But then $f(\eta_i) \in u_{\eta_i^\gamma}$ by the definition of $g$, and therefore $\text{conv}(g(\eta_0), \ldots, g(\eta_{n-1})) \not\subseteq S$ by condition 4. Therefore $C_m$ is not convex.

Corollary 6. A closed planar set is a countable union of convex sets if and only if $\delta^3(S) < \omega_1$. 
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Proof. One direction is proved above.

For the other direction suppose that $S \subseteq \mathbb{R}^3$ is closed and is not a countable union of convex sets. By Shelah's theorem there is a perfect 3-clique $P \subseteq S$. Every subset of $P$ is a 3-clique, and since $P$ is perfect, it contains countable sets of unbounded Cantor-Bendixson degrees.

We observe that Perles' Example 2.1 in [5], of a compact planar set $S$, satisfies that $\delta^2(S) = 1$ while $\delta^3(S) = \omega_1$. Hence, classification by $\delta^2$ does not refine the classification by $\gamma$.

It is natural to ask at this point whether $\delta^4(S)$ classifies non-convexity of closed sets in $\mathbb{R}^3$ analogously to the manner $\delta^3$ classifies closed sets in $\mathbb{R}^2$. This is false by Example 4.1 in [5]. This example is of a compact $S \subseteq \mathbb{R}^3$ with $\delta(S) = 1$ and $\gamma(S) > \aleph_0$.

References


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