Almost isometric embeddings of metric spaces in the set theoretic lense

Menachem Kojman

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The geometry of metric spaces.

Consider the metric spaces $\mathbb{Q}$ and $\pi\mathbb{Q}$. Is there any geometric difference between them?

They are not isomorphic, and not embeddable isometrically into each other. However, for every constant $K > 1$ there is $f : \mathbb{Q} \to \pi\mathbb{Q}$, a homeomorphism such that f and f^{-1} satisfy the Lipschitz condition with constant K.

Definition 1 *1. Two metric spaces* (X_1, d_1) *and* (X_2, d_2) *are almost isometric if for every* $K > 1$ *there is a* K*-bi-Lipschitz homeomorphism between them.*

2. (X_1, d_1) *is almost isometrically embedded into* (X_2, d_2) *if for every* $K > 1$ *there is a* K*-bi-Lipschitz embedding of* (X_1, d_2) *into* (X_2, d_2).

Uniqueness and universality

Theorem 1 (Hrusak, Zamora-Avilles) *Any two countable dense sets are almost isometric in every* $\mathbb{R}^d$ *and in the separable Hilbert space.*

$\mathbb{R}^d$ is almost isometry unique: every space that is almost isometric to (almost isometrically embedded into) $\mathbb{R}^d$ isometric to (isometrically embedded into) $\mathbb{R}^d$.

So $\mathbb{R}^d$ is isometry-universal for all metric spaces that can be almost isometrically embedded in it.

Definition 2 *A metric space* X *has the almost extension property if for every finite metric space* $\{x_0, \dots, x_{n-1}, x_n\}$ *every* K*-bi-Lipschitz map* $f' : \{x_0, \dots, x_{n-1}\} \to X$ *can be extended to a* K*-bi-Lipschitz* $f : \{x_0, \dots, x_n\} \to X$*.*

There is a unique countable metric space up to almost isometry with the property of almost extension property. Its completion is unique up to isometry among all separable spaces that satisfy the extension property and is called the Uryson space. $\mathbb{U}$ is almost isometry unique and isometry universal for separable metric spaces.

Set theoretic problems

If two $\aleph_1$ dense subsets of $\mathbb{R}$ are Lipschitz homeomorphic, they are either order-isomorphic or order anti-isomorphic. There are $\aleph_1$-dense subsets of $\mathbb{R}$ which are order isomorphic to their inverse. Thus

- "all $\aleph_1$-dense subsets of $\mathbb{R}$ are almost isometric"

or even just

- "all $\aleph_1$-dense subsets of $\mathbb{R}$ are bi-Lipschitz homeomorphic"

imply

- "all $\aleph_1$-dense subsets of $\mathbb{R}$ are order-isomorphic" which is consistent by Baumgartners's theorem, and follows from PFA and from Woodin's axiom $(*)$.

Both strengthenings are false in all reasonable separable space.

Universality

$(*)_{\kappa,\lambda}$ There is a metric space of size λ into which every metric space of size κ can be almost isometrically embedded.

$(**)_{\kappa,\lambda}$ There are λ metric spaces, each of size κ, so that every metric space of size κ is almost isometrically embeddable into one of them.

$(**)\Rightarrow(*)$. If $\mathrm{Cov}(\lambda,\kappa)=\lambda$ the converse implications also holds.

$(*)_{\kappa,\lambda} \wedge \mathrm{Cov}(\kappa,\omega) \leq \lambda < 2^{\aleph_0}$ imply that $(*)^{sep}_{\kappa,\lambda}$, for separable metric spaces, holds. Namely, there is $A \in [\mathbb{U}]^{\lambda}$ so that every $B \in [\mathbb{U}]^{\kappa}$ (=every separable metric space of size κ) almost isometrically embeds into A.

For every regular $\lambda > \aleph_0$ it is consistent that $\lambda^+ < 2^{\aleph_0}$ and that $(*)^{sep}_{\lambda,\lambda^+}$ holds.

The return of the invariants in nonseparable spaces.

Theorem 2 *If $\lambda = \mathrm{cf}\lambda > \aleph_1$ and $2^{\aleph_0} > \lambda$, fewer than $2^{\aleph_0}$ metric spaces on λ do not suffice to bi-Lipschitz embed all metric spaces on λ.*

Club guessing method.

Theorem 3 *Suppose μ is singular and smaller than the first second-order fixed point of the $\aleph$-function. Then if $\kappa = \mathrm{cf}\kappa < \mu > 2^\kappa$ (namely, when μ is is not a strong limit) there is no metric space on μ into which all metric spaces on κ^{++} are almost isometrically embedded.*

Because the invariants are robust, and for such μ it holds that $\mathrm{Cov}(\mu, \kappa^{++}, \kappa^{++}) = \mu$ for arbitrarily large $\kappa^{++} < \mu$.

Discussion

Gödel's verifiability: did he mean prediction? $V = L$ is "restrictive" and has, in Gdel's opinion, poor verifiability.

What was seen to follow from $V = L$ by Jensen's work, echoes in V: weakenings of diamonds and of square are true in ZFC. Club guessing holds and is useful also when GCH fails.

Two Problems

Problem 1 *Is it consistent to have full universality for almost isometric embeddings in* $\aleph_1 < 2^{\aleph_0}$ *in* $\mathbb{R}$, $\mathbb{R}^d$ *or* $\mathbb{U}$*?*

Problem 2 (Erdős) *Does* $\mathbb{R}^2$ *have a dense rational subset?*