Bounds on coloring numbers

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Basic definitions

- The *chromatic number* of a graph $G = (V, E)$ is the least cardinal κ for which there exists a proper vertex coloring $c : V \to \kappa$ of G.
- The *list-chromatic* or *choice* number $\chi_\ell(G)$ is a variation on the chromatic number in which each vertex $v \in V$ is assigned its own list of colors $L(v)$ and the proper coloring chooses $c(v) \in L(v)$.
- $\chi_\ell(G)$ is the least cardinal κ such that for every assignment of lists $L(v)$ of size $|L(v)| = \kappa$ for each $v \in V$ there exists a choice function c which is a proper coloring of the graph.

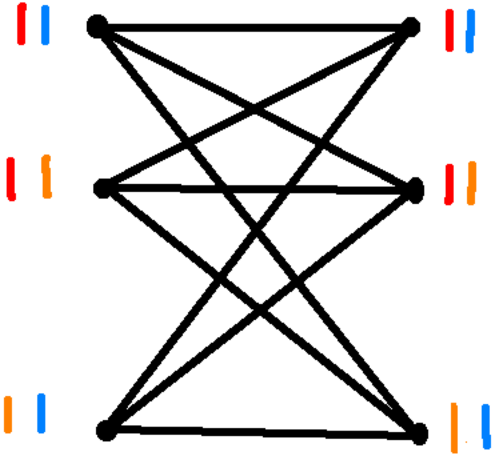
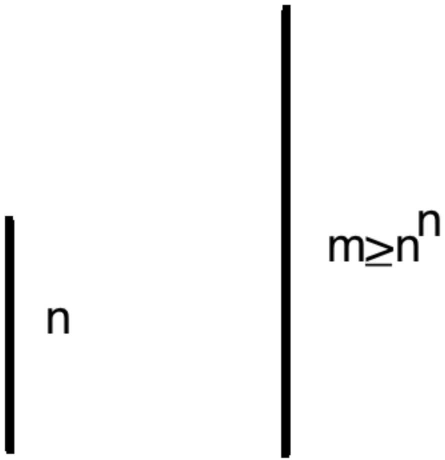


Figure: $\chi(K(3,3)) = 2, \chi_\ell(K(3,3)) = 3$

- For a bipartite graph $K(\kappa, \lambda)$ the list-chromatic number is at most $(\min\{\kappa, \lambda\})^+$.
- If $m \geq n^n$ then $\chi_\ell(K(n, m) = n + 1$



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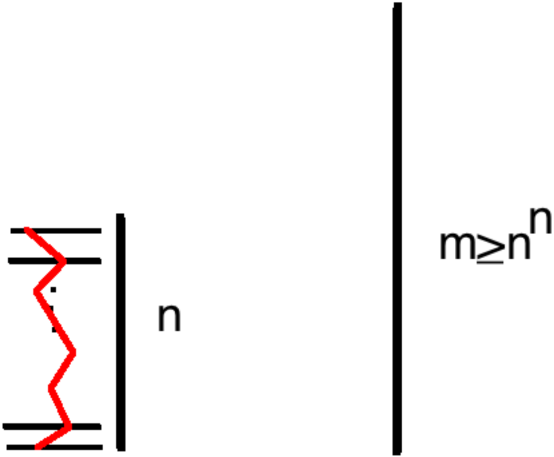


Figure: $\chi(K(n, m)) = n + 1$

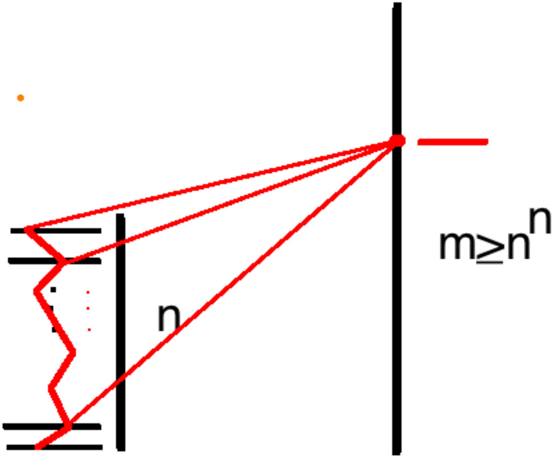


Figure: $\chi(K(3,3)) = 2, \chi_\ell(K(3,3)) = 3$

- The *coloring number* $\mathrm{Col}(G)$ is the least cardinal κ for which there exists a well ordering $\prec$ of V such that $|G_{\prec}[v]| = |\{u : u \prec v \wedge \{u, v\} \in E\}| < \kappa$ for every $v \in V$.
- For every graph G, the coloring number is at most $1 + \max\{\deg(v) : v \in V\} \leq |V|$. The coloring number of $K(3, 3)$ is 4.

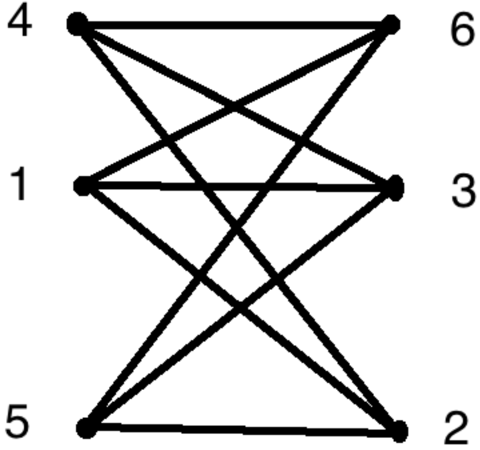


Figure: $\mathrm{Col}(K(3,3)) = 4$

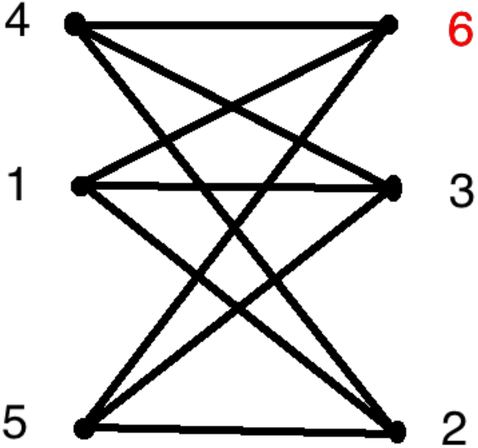


Figure: $\mathrm{Col}(K(3,3)) = 4$

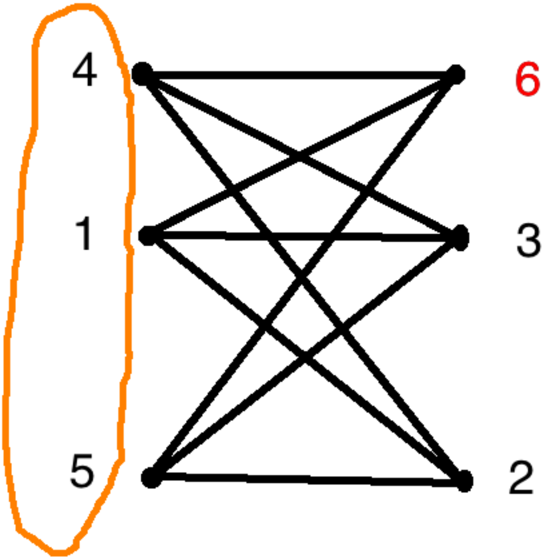


Figure: $\text{Col}(K(3,3)) = 4$

Summary of definitions

For every graph G,

$$\chi(G) \leq \chi_\ell(G) \leq \mathrm{Col}(G).$$

and the inequalities may be strict.

Some History

- The list-chromatic number was introduced independently by Vizing in 1976 and Erdős, Rubin and Taylor in 1979 and then lay dormant for a long time. Since the 1990s this number attracts a lot of interest in the graph theory community.
- The coloring number was introduced by Erdős and Hajnal in their work on graphs of uncountable chromatic number in 1966 (or earlier?). They observed that some of their results remained valid in the broader class of graphs with uncountable coloring number.
- Recently some interest has been given to list-chromatic numbers of relatives of the unit distance graph on $\mathbb{R}^2$.

Alon's result and question

- Let $d = d(G)$ denote the minimum degree of a vertex in G.
- In a finite graph, $\mathrm{Col}(G) \geq d(G)$, because, as mentioned earlier, some vertex has to be the last in every ordering of the graph. The λ-branching tree of height ω has uniform degree λ but has colorability 2.
- In 2000 Alon proved that $d(G) \leq (4+\epsilon)^{\chi_\ell(G)}$ for every *finite* graph G, using the probabilistic method.Given a finite G, find a vertex v with $\deg(V) \leq (4+\epsilon)^{\chi_\ell(G)}$ and mark it as the last vertex. Then eliminate this vertex from the graph and continue inductively. Thus:

$$\mathrm{Col}(G) \leq (4+\epsilon)^{\chi_\ell(G)}.$$

- Question (Alon): is there a similar bound on $\mathrm{Col}(G)$ for infinite graphs?

1. The bipartite graph $K_{\kappa,\kappa}$ has chromatic number 2, coloring number $1+\kappa$ and in the finite case $\chi_\ell(K(n,n))$ grows to infinity with n.
2. $\chi_\ell(K(\aleph_0, 2^{\aleph_0})) = \aleph_1$ and more generally, $\chi_\ell(K(\kappa, 2^\kappa)) = \kappa^+$.
3. If $\kappa < 2^{\aleph_0}$ then $\chi_\ell(K(\aleph_0, \kappa)) = \aleph_0$.

Komjath's consistency results

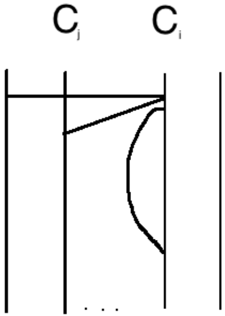
- MA implies that for every graph $|G| < 2^{\aleph_0}$ with countable chromatic number, $\chi_\ell(G) = \aleph_0$. By this result, there is no upper bound on coloring numbers in terms of list-chromatic numbers using only the $\aleph$ function; some exponentiation is needed for a ZFC bound.
- It is consistent that $\aleph_1 < 2^{\aleph_0}$ and that there exists a graph $G = (\omega_1, E)$ with countable chromatic number and $\chi_\ell(G) = \aleph_1$.
- It is consistent with the GCH that

$$\chi_\ell(G) = \aleph_0 \implies \mathrm{Col}(G) = \aleph_0$$

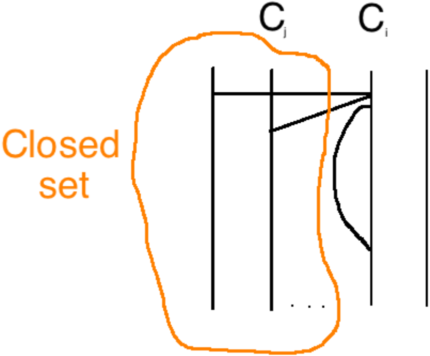
  for every graph G.
- It is consistent with the GCH to have a graph with $\chi_\ell(G) = \aleph_0 < \mathrm{Col}(G) = \aleph_1$.

Bounding coloring numbers inductively

Erdős and Hajnal introduced in 1966 a natural scheme for bounding colorability inductively: partition $V = \{C_i : i < \theta\}$ with $|C_i| < |V|$ and use the induction hypothesis to well-order each C_i separately. If the partition can be found so that $G[v] \cap \bigcup_{j<i} C_j$ is bounded for $v \in C_i$, then we are done.



Their idea was to use the assumption that G is $K(n,\omega_1)$-free to find sets C_i which are "closed" under common neighbors of n-element sets.



Closure operation

- For every $G = (V, E)$ let $F : \mathcal{P}(V) \to \mathcal{P}(V)$ be the function $F(X) = \bigcap_{v \in X} G[v]$ associating to X the set of common neighbors of all vertices in X. For a cardinal κ let $F_\kappa = F \restriction [V]^\kappa$.
- F and each F_κ are *anti-monotone*: $X \subseteq Y \implies F(Y) \subseteq F(X)$.
- $A \subseteq V$ is *κ-closed* if $F(X) \subseteq A$ for every $X \in [A]^\kappa$.
- If A is κ-closed then $|G[v] \cap A| < \kappa$ for all $v \in V \setminus A$.

The finitary case: Erdős-Hajnal

Theorem (Erdős-Hajnal 1966)

Suppose G is $K(n, \omega_1)$-free for some n. Then $\mathrm{Col}(G) \leq \aleph_0$.

Corollary

For every graph G, if $\chi_\ell(G) < \aleph_0$ then $\mathrm{Col}(G) \leq \aleph_0$.

Proof.

Suppose $\chi_\ell(G) = n$. Then G is $K(M, M)$-free for some finite M by Alon's theorem or by our direct counting. Now apply the inductive scheme for F_M-closed sets. □

Getting started

Assume now that $\chi_\ell(G) = \kappa$ is infinite. G is $K(\kappa, 2^\kappa)$-free, and we shall try to work with κ-closure. So here are additional definitions and properties of κ-closed sets for an infinite κ:

1. A cardinal θ is *κ-stable* for a graph G if every set $A \in [V]^\theta$ is contained in a κ-closed set of the same cardinality.
2. If $\theta^\kappa = \theta$ then θ is κ-stable for any G which is $K(\kappa, 2^\kappa)$-free.
3. If $\{B_i : i < \theta\}$ is $\subseteq$-increasing, each B_i is κ-closed and $\mathrm{cf}\,\theta \neq \mathrm{cf}\,\kappa$, then $\bigcup_{i<\theta} B_i$ is κ-closed.

Proof of (3).

Just the case $\theta < \mathrm{cf}\,\kappa$. For every $X \in [\bigcup B_i]^\kappa$ there is some $i < \theta$ such that $|X \cap B_i| = \kappa$. Now use anti-monotonicity. □

Set $\kappa = \aleph_0$ for the moment. Assume $|V| = \lambda = (2^{\aleph_0})^+$ and G is $K(\aleph_0, 2^{\aleph_0})$-free. Let $V = \bigcup_{i<\lambda} B_i$, an increasing union with $|B_i| = 2^{\aleph_0}$ and $\aleph_0$-closed. To make a set closed iterate ω_1 times the operation $A \mapsto A \cup \bigcup\{F(X) : X \in [A]^{\aleph_0}\}$. Now let $I = \{i < \lambda : B_i \setminus \bigcup_{j<i} B_j \neq \emptyset\}$ and put $C_i = B_i \setminus \bigcup_{j<i} B_j$ for $i \in I$. This is a partition of V; if $\text{cf } i \neq \omega$ then $\bigcup_{j<i} B_j$ is $\aleph_0$-closed, so a vertex $v \in C_j$ will have a finite set of neighbors in this union. If $\text{cf } i = \omega$ then $v \in C_i$ may have $\leq \aleph_0$ neighbors in this union. So we are proving inductively that $\text{Col}(G) \leq \aleph_1$.

Similar for $\lambda = (2^{\aleph_0})^{+n}$.Similar for the first limit above $2^{\aleph_0}$.

What about the successor of the first limit above $2^{\aleph_0}$?

Limit of countable cofinality

Pay more. Use the weaker $\aleph_1$-closure operation. Recall that:

Recall:

A countable union of $\aleph_1$-closed sets is $\aleph_1$-closed.

Thus we can get $\aleph_1$-closed sets, but then for $|V| = \aleph_{\omega+1}$ we only get $\mathrm{Col}(G) \leq \aleph_2$, because of limits of cofinality $\aleph_1$ in a filtration to closed sets.
So we can pass every limit cardinal of cofinality $\aleph_0$.
This gets us as far as the first limit of cofinality ω_1; what next?
Settle for $\aleph_2$? And then what?

With weak SCH

Assume that every limit $\mu > 2^{\aleph_0}$ of cofinality ω_1 is closed under $\aleph_0$-exponentiation, that is, $\theta < \mu \Rightarrow \theta^{\aleph_0} < \mu$.

Lemma

Every cardinal $\theta \geq 2^{\aleph_0}$ is $\aleph_1$-stable for all $K(\aleph_0, 2^{\aleph_0})$-free G.

Proof.

By induction on $\theta \geq 2^{\aleph_0}$. Every limit of cofinality ω_0 maintains the induction hypothesis and limits of cofinality ω_1 are limits of $\aleph_0$-stable cardinals, so are even $\aleph_0$-stable. □

Theorem

$\mathrm{Col}(G) \leq \max\{2^{\aleph_0}, \aleph_2\}$ *for all* G *with* $\chi_\ell(G) = \aleph_0$.

Proof.

Assume for simplicity $2^{\aleph_0} = \aleph_2$.
By induction on $|V| = \lambda \geq 2^{\aleph_0}$ prove that $\mathrm{Col}(G) \leq \aleph_2$ for every $K(\aleph_0, 2^{\aleph_0})$-free G.
Case 1. $\mathrm{cf}\,\lambda = \aleph_1$. Fix $\langle \theta_i : i < \omega_1 \rangle$ increasing with limit λ such that $\theta_i^{\aleph_0} = \theta_i$. Possible by the assumption. Present $V = \bigcup_{i<\omega_1} B_i$, increasing union, where $|B_i| = \theta_i$ and B_i is $\aleph_0$-closed. Let $I = \{i < \omega_1 : B_i \setminus \bigcup_{j<i} B_j \neq \emptyset\}$ and $C_i = B_i \setminus \bigcup_{j<i} B_j$ for $i \in I$.
Case 2. $\mathrm{cf}\,\lambda \neq \aleph_1$ — even easier. □

Silver, Prikry and Gitik

Recall that modulo large cardinals it is consistent to have $\mu^{\aleph_0}$ arbitrarily large for a strong limit μ of cofinality ω. Gitik proved this for the first fixed point. In particular, for every $\mu' \in (\mu, \mu^{\aleph_0})$ it holds that $\mu^{\mathrm{cf}\,\mu} \geq \mu^{\aleph_0}$, so arbitrary high cofinalities may show up.

Thus, a simple counting argument using standard exponentiation will not work in ZFC.

But we are not restricted to using only cardinal-arithmetic functions which were created in the limited world of natural number.

With no assumptions

Shelah's revised power function:

$\lambda^{[\kappa]} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\lambda]^\kappa \wedge (\forall X \in [\lambda]^\kappa)(\exists \mathcal{Y} \in [\mathcal{A}]^{<\kappa})(X \subseteq \bigcup \mathcal{Y})\}$

Lemma

If $\theta \geq 2^\kappa$, $\kappa = \operatorname{cf} \kappa$ and $\theta^{[\kappa]} = \theta$ then θ is κ-stable for every $K(\kappa, \theta^+)$-free G.

Proof.

Suppose $A \in [V]^\theta$. Fix $\mathcal{A} \subseteq [A]^\kappa$ witnessing $\theta^{[\kappa]} = \theta$.

$$\bigcup_{X \in [A]^\kappa} F(X) = \bigcup_{Z \in \mathcal{A}} \bigcup_{W \in [Z]^\kappa} F(W)$$

Iterate κ^+ times the operation $A \mapsto A \cup \bigcup \{\{F(X) : X \in [A]^\kappa\}$. □

Shelah's revised GCH in ZFC:

For every $\lambda \geq \beth_\omega(\nu)$ for all but a bounded set of $\kappa < \beth_\omega(\nu)$

$$\lambda^{[\kappa]} = \lambda.$$

Lemma

For every cardinal ν, every $\theta \geq \beth_\omega(\nu)$ is κ-stable for all but a bounded set of $\kappa < \beth_\omega(\nu)$ for every $K(\nu, 2^\nu)$-free G.

Proof.

Fix a regular $\kappa < \beth_\omega(\nu)$ for which $\theta^{[\kappa]}$ (there is one by Shelah's theorem). Clearly $2^\kappa < \theta$. κ-stability of θ follows by the previous lemma and persists upwards with κ. □

In ZFC

Theorem

$\mathrm{Col}(G) \leq \beth_\omega(\nu)$ *for every graph* G *with* $\chi_\ell(G) = \nu$.

Proof.

For every $\theta \geq \beth_\omega(\nu)$ fix $\kappa(\theta) \in \{(\beth_n(\nu))^+ : n \in \omega\}$ for such that θ is $\kappa(\theta)$-stable for every $K(\aleph_0, 2^{\aleph_0})$-free G.

Case 1: $\mathrm{cf}\,\lambda = \omega$. Let $V = \bigcup B_n$, increasing union B_n is $\kappa(|B_n|)$ closed.

Case 2: $\mathrm{cf}\,\lambda > \omega$. Fix a $\leq$-increasing sequence $\langle \theta_i : i < \mathrm{cf}\,\lambda \rangle$ unbounded below λ and assume, without loss of generality, that $\kappa(\theta_i)$ is fixed. Let $V = \bigcup_{i < \mathrm{cf}\,\lambda} B_i$, each B_i κ-closed. Now at limits of cofinality other than κ the union is closed, and at limits of cofinality exactly κ the trace of every vertex from outside is $\leq \kappa$. □

Last remark

The case $\nu \in \omega$ is also included in the previous theorem by the corollary to the theorem by Erdős and Hajnal we stated above, as $\beth_\omega(n) = \omega$.

And now

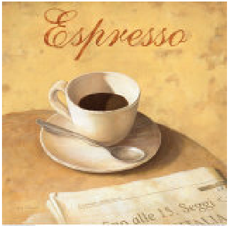


Figure: coffee break