Part III: Ramsey's Theorem and its friends go Borel

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Blass' theorem

Theorem (Blass)

For every $0 < d \in \omega$ there exists a basis for all finite Borel partitions of $[2^\omega]^d$ with respect to (nonempty) perfect subsets of 2^ω. The basis for d-partitions consists of a single clopen partition P_d which has $(d-1)!$ cells.

Some clarifications:

- The topology on unordered d-tuples is inherited from $(2^\omega)^d$.
- $2^\omega \to (2^\omega)^d_{<\omega|(d-1)!}$ for all $d > 0$.
- Galvin proved the cases $d = 1, 2, 3$.
- The Galvin-Prikry theorem about Borel colorings of infinite subsets of ω came before this!

This is the "Ramsey theorem" of perfect sets.

The universal profinite graph

Fix a random graph $\Gamma = (\omega, E^\Gamma)$.
Let $G_{\max} = (V, E)$ where $V = \prod_n (n+1)$ and
$E = \{\{\bar{x}, \bar{y}\} : \{x(\Delta(\bar{x}, \bar{y}), y(\Delta(\bar{x}, \bar{y})\} \in E^\Gamma\}$.
Properties of $G_{\max}$:

- $G_{\max}$ is homeomorphic to the Cantor space and its edge relation is clopen.
- $G_{\max}$ can be chosen so that for every vertex $\bar{x} \in G_{\max}$, every finite graph H with a distinguished vertex v and every open neighborhood U of $\bar{x}$ there is an embedding of H into U which takes v to $\bar{x}$. Let us call this the local randomness property of $G_{\max}$.
- $G_{\max}$ does not contain an infinite induced path — hence it is not universal for countable graphs.
- As local randomness together with ω-homogeneity implies randomness (and hence universality), $G_{\max}$ cannot satisfy ω-homogeneity.

The induced Borel Ramsey theorem

Theorem (K. -S. Frick)

For every $d > 0$ there is a basis for all Borel partitions of $[V]^d$ with respect to closed, ordered copies of $G_{\max}$. The basis for d-partitions consists of a single clopen partition P_d whose number of cells is smaller than 2^{d^2} and is equal to the number of linear normal series of ordered d-graphs. For a simple ordered graph H we have

$$G_{\max} \rightarrowtail_{Borel} (G_{\max})^H_2,$$

in particular, $G_{\max} \rightarrowtail_{Borel} (G_{\max})^{K_2}_2$.

The proof

Notation:

- Let the norm of a finite ordered graph G be the maximal n for which there is an ordered graph embedding of $\Gamma \restriction n$ into G.
- Let $T_{\max}$ be the tree $\prod_n (n+1)$. For $T \subseteq T_{max}$ the norm of $t \in T$ is the graph norm of $\mathrm{Succ}_T(t')$ where t' is the highest splitting node below t.
- $T \subseteq T_{\max}$ is a tree of infinite norm if for every $t \in T$ there is $t' > t$ with larger norm.

Let $\pi_n(\bar{x}) = \bar{x} \restriction n$ and π^m_n is defined identically. These projections are ordered graph homomorphisms.
The Bad Guy is required to respect the Borel structure when handing us a d-partition; but he may consult whatever other structure he wants. Still, for every Borel partition, we shall find a tree of infinite norm on whose branches the cell of a graph depends only on the normal series induced on the graph.

Steps of the proof

- Reduce to a tree on which the given Borel partition is clopen.
- Reduce to a skew tree, that is, one with no more than 1 splitting node in each level.
- For each normal series separately:
  - color depends only on highest split node and $\pi^n_{n-1}(H)$.
  - Halpern-Läuchli.

  continue fusion.

Comments

- Blass' theorem was proved similarly, with $(d-1)!$ cells.
- Sheu proved a similar result using ideals: the target set in Blass' theorem is required to be positive modulo the G_{min} ideal. This is also an induced Ramsey theorem.
- The names "max" and "min" come from the classification of continuous graph structure in Geschke,Goldstern and Kojman, *Continuous Ramsey theory on Polish spaces and covering the plane by functions* , J. Math. Log. 4 (2004), No.2, 109145. in which it was proved that G_{min} and G_{max} are a complete set of representatives for uncountalble co-chromatic numbers of continuous graphs on Polish spaces.