Part II: Induced Ramsey Theory

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Review of Part 1

- $\{E_{\max}\}$ is a basis for all finite equivalnce relation over $[\omega]^d$.
- $\{D_D : D \subseteq d\}$ is a basis for all equivalence relations over $[\omega]^d$.
- Hales-Jewett: for large enough n a partition of d^n to c parts yields a combinatorial line in one of the parts.
- Nešetřil-Rödl Induces Ramsey theorem (for edge cologings): for every finite graph H and finite c there is a finite graph G such that

$$G \rightarrowtail (H)_c^{K_2}.$$

Reminder: for every k, c there is n such that

$$n \longrightarrow (k)_c^2.$$

Continue with Nešetřil-Rödl

Reminder:

Theorem (Nešetřil-Rödl)

for every finite graph H there exists a finite graph G such that

$$G \rightarrowtail (H)^{K_2}_2.$$

Sterp 1: for every finite bipartite graph $G = (L, R, E)$ and finite c there is n such that

$$G^{(n)} \rightarrowtail (G)^{K_2}_c.$$

bipartite means that $L \cap R = \emptyset$, $E \subseteq [L \cup R]^2$ and $|e \cap L| = |e \cap R| = 1$ for al $e \in E$. $G^{(n)} = (L^n, R^n, E^{(n)})$, $\{\bar{v}, \bar{u}\} \in E^{(n)} \iff \bigwedge_i \{v(i), u(i)\} \in E$.

The full Nešetřil-Rödl

The isomorphism types of ordered d-graphs form a basis in finite induceed Ramsey theory.

Theorem (Nešetřil and Rödl)

For any two ordered graphs $\bar{A}, \bar{B}$ and finite c there is a finite ordered graph $\bar{C}$ such that

$$\bar{C} \rightarrowtail (\bar{B})^{\bar{A}}_{c}.$$

That is: for every coloring of the copies of $\bar{A}$ in $\bar{C}$ by c colors there is an induced copy of $\bar{B}$ all of whose $\bar{A}$ are colored in on color.

Moral, again...

Order matters. Beyond the distinction of d-tuples to their isomorphism types, we also have to adopt an ordering!

The infintie universal Random graph

A graph $G = (\omega, E)$ satisfies the Randomness axiom if for every two finite disjoint $A, B \subseteq \omega$ there is $x \in \omega \setminus A \cup B$ such that $A \subseteq G[x]$ and $B \cap G[x[= \emptyset$.

Facts:

- any two random graphs on ω are isomorphic to each other.
- Lebesgue measure 1 of graphs on ω satisfy randomness.
- every countable graph is embeddable into the random graph $\Gamma = (\omega, E^\Gamma)$.
- every ordered countable graph is order embedded into the random graph.
- Exercise: $\Gamma \rightarrowtail (\Gamma)^1_2$.

The Erdős-Poša coloring

Γ is really the ω of countabl graphs. It should be true that $\Gamma \rightarrowtail (\Gamma)_2^{K_2}$— But it isn't ...

Theorem (Erdős-Poša)

There exists a coloring of the edges of Γ by 2 colors so that in every copy of Γ edges of both colors appear.

Proof.

Let $\langle r_n : n \in \omega \rangle$ enumerate all eventually 0 sequences in 2^ω. Put $\{x_n, x_m\} \in E$ for $n < m$ if $x_m(n) = 1$. This is the random graph, ordered in a dense order type by the lexicographic ordering of 2^ω. Let an edge be increasing if both orderings coincide on it and decreasing otherwise. Suppose $K_{\omega,\omega}$ is monochromatically embedded into Γ. □

Without loss of generality, $L \cup R = \{x_n : n \in \omega\}$, $x_{2n} \in L$ and $x_{2n+1} \in R$ for all n, Also, $x_{2_n} \restriction n_0$ is fixed for all $n > 0$.
Now:

$$
\begin{array}{llll}
 & \vdots & & \\
A \ni x_{n_4} = \ldots & & 1 & 0\ldots \\
B \ni x_{n_3} = \ldots & & 0 & 1\ldots \\
A \ni x_{n_2} = \ldots & & 1 & 0\ldots \\
B \ni x_{n_1} = \ldots & & & 1\ldots \\
A \ni x_{n_0} = \ldots & & & \ldots \\
 & & & n_0
\end{array}
$$

Either x_{n_4}, x_{n_2} are both lexicographically below or both lexicographically above x_{n_3}.

Puzet-Sauer and Sauer Basis theorem

Theorem

For every finite graph H there is a finite basis $\{E^H\}$ for all finite partitions of $\binom{\Gamma}{H}$ with respect to all isomorphic copies of Γ in Γ. In particular, there is a finite number $t(H)$ such that

$$\Gamma \rightarrowtail (\Gamma)^H_{<\omega|t}.$$

E.g.,

$$\Gamma \rightarrowtail (\Gamma)^H_{<\omega|4}.$$

Tomorrow's title is:

Ramsey's theorem and its friends go Borel.

We shall quote the "Borel Ramsey Theorem" due to Blass, and prove a Puzet-Sauer Basis theorem for it. But, we need two more ingredients. One is finite, the other infinite. The finite ingredient is something like the group theoretic Jordan-Hölder theorem — for graphs. The other is a very useful countable partition theorem: the Halpern-Lüchli theorem.

Halpern-Läuchli theorem

Let T be a finitely branching ω-tree. A subset $A \subseteq T$ is uniformly dominating if for every n there is m so that $A \cap T(m)$ dominates $T(n)$. Let $L(A) = \{n : A \cap T(n) \neq \emptyset\}$. If A is uniformly dominating it is in particular dense in T, and its restriction to any infinite subsets of its levels is also uniformly dominating.
Let T_1, T_2 be finitely branching ω-trees, and suppose $A \in [\omega]^\omega$. By $T_1 \otimes_A T_2$ we denote the level product for heights in A: all ordered pairs $\langle t_1, t_2 \rangle$ such that $t_i \in T_i$ and $h_1(t_1) = h_2(t_2) \in A$.

Theorem (Halpern-Läuchli)

For every finite sequence of finitely branching ω-trees $\langle T_1, \dots T_d \rangle$, $W \in [\omega]^\omega$ and uniformly dense $A_i \subseteq T_i$ such that $L(A_i) = W$, for every finite partition of $\bigoplus_i A_i$ there exists $t_i \in A_i$, a set $V \in [W]^\omega$ and $B_i \subseteq A_i$ with $L(B_i) = V$ such that B_i is uniformly dense above t_i and $\bigoplus_i B_i$ is monochromatic.

Sketch of proof for $d = 2$

First, coloring points from a uniformly dominating $A \subseteq T$ gives a monochromatic uniformly dominating set in T_t for some $t \in T$.
Suppose $A_i \subseteq T_i$, $i \in \{1, 2\}$ are uniformly dense and $L(A_i) = W$, and f is a coloring of $A_1 \otimes_W A_2$ by 2 colors.
Consider the statement

$P(t_1, t_2, A_1, A_2, i)$:

for every $s \in T_1$ and $A_2' \subseteq A_2 \cap T$ there is $A_2'' \subseteq A_2'$ and an increasing sequence $\langle s = s_0, s_1, \ldots \rangle$ such that $L(\{s_j\}) = L(A_2'')$ and $\{s_j\} \times A_2''(h(s_j))$ is monochromatic of color i.
Either $P(A_1, A_2, 0)$ holds, or there are $s \in T_1$, $t \in T_2$, $A_1' \subseteq A_1$ and $A_2' \subseteq A_2$ such that $P(A_1' \cap T_{t_1}, A_2' \cap T_{t_2}, 1)$ holds.
From that the theorem follows.

Substitution decompositions of graphs

- A homomorphism $f : G_1 \to G_2$ is a function which satisfies: for every $u, v \in G_1$, if $f(u) \neq f(v)$ then $\{u, v\} \in E \iff \{f(u), f(v)\} \in E$.
- A normal partition of a graph G is a partition of $V(G)$ such that for all $X, Y \in P$ and $u, u' \in X, v, v; \in Y$, $\{u, v\} \in E \iff \{u', v'\} \in E$.
- If $f : G_1 \to G_2$ is a surjective homomorphism then $P = \{f^{-1}(u) : u \in V(G_2)\}$ is a normal partition, and the quotient graph G_1/P is isomorphic to G_2 (in fact, every selector is a subgraph of G_1 which is isomorphic to G_2).

- A graph G is simple if $V(G) > 1$ and the only normal partitions of G are id and the single cell (alternatively, the only hom. images are itself and $\{u\}$). A homomorphism (or, equivalently, quotient) $f : G_1 \to G_3$ is simple if it is not injective and there is no nontrivial factorization G_2 and $G_1 \xrightarrow{g} G_2 \xrightarrow{h} G_3$.
- A surjective homomorphism $f : G_1 \to G_2$ is simple iff there is a unique $u \in V(G_2)$ for which $f^{-1}(u)$ is a simple subgraph of G and $f^{-1}(v)$ is a singleton for all other $v \in V(G_2)$.
- A composition series of a graph G is $G = G_0 \xrightarrow{g_1} G_1 \xrightarrow{g_2} G_2 \ldots \xrightarrow{g_n} G_n = \{u\}$ in which all maps are simple. The simple graph $F_i \subseteq G_i$ which is collapsed by g_{i+1} is called the i-th factor in the series.

Jordan-Hölder therem for graphs

Theorem (Möhring?)

For every finite graph G any two composition series of G have the same length and, up to a permutation, the same factors.

Proof.

If $F, E \subseteq G$ are two simple subgraphs their collapses commute □