Extensions of Ramsey's theorem

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- For a set A and $d \leq \omega$, denote by $[A]^d$ all subsets of A of cardinality d.
- An equivalence relation E on $[A]^d$, a partition of $[A]^d$ and a coloring of $[A]^d$ are freely interchanged.
- id is the identity equivalence relation which corresponds to the partition to singletons and to an injective coloring.
- $E_{\max}$ is the maximal equivalence relation which corresponds to the partition to a single cell and to a constant coloring.

Ramsey's Theorem

Theorem (Ramsey)

For every finite d and c and for every function $f : [\omega]^d \to c$ there exists an infinite set $A \subseteq \omega$ such that $f \restriction [A]^d$ is constant.

Proof.

For a given d, a simple induction on $c \geq 2$ gives the theorem for all finite c. Fix $c = 2$ and proceed by induction on d. For $d = 1$, a partition of ω to two must have an infinite part.

Suppose $d + 1 > 1$ and suppose $f : [\omega]^{d+1} \to 2$ is given. Define by induction $A_n, a_n = \min A_n$ so that $A_0 = \omega$, $A_{n+1} \subseteq A_n \setminus a_n$ is infinite and $f_n : [A_{n+1}]^d \to 2$, where $f_n(X) = f(\{a_n\} \cup X)$, is constant with color $c_n \in 2$. After thinning out we assume $c_n = c$ for all n. Now $f(X) = c$ for all $X \in [\{a_n : n \in \omega\}]^{d+1}$. □

In Hungarian partition notation Ramsey's theorem is written like that:

$$\omega \longrightarrow (\omega)^d_c$$

Ramsey's theorem reformulated

- Let $\mathcal{E}$ be a class of equivalence relations on $[\omega]^d$.
- Let $\mathcal{A}$ be a collection of admisible target subsets of ω
- A set $B \subseteq \mathcal{E}$ is called a basis for $\mathcal{E}$ with respect to $\mathcal{A}$ if for every $E \in \mathcal{E}$ there is $E^* \in B$ and $A \in \mathcal{A}$ such that

$$E \restriction [A]^d = E^* \restriction [A]^d$$

Now, Ramsey's theorem is: $\{E_{\max}\}$ is a basis for all finite equivalence relations on $[\omega]^d$ with respect to $[\omega]^\omega$.
In this talk and the next one, we shall mainly be looking for finite bases for various classes $\mathcal{E}$.

The Erdős-Rado Canonical Ramsey theorem

Theorem (Erdős-Rado)

for every $d \geq 1$ there is a basis of size 2^d, $\{E_D : D \subseteq d\}$ for all equivalence relations on ω with respect to $[\omega]^\omega$.

Proof.

For $X \in [\omega]^d$ let $\bar{X}$ denote the increasing enumeration of X. For $D \subseteq d$ and $X, Y \in [\omega]^d$ let $X\ E_D\, Y \iff \bar{X} \restriction D = \bar{Y} \restriction D$.
If $D = \emptyset$, then $E_D = E_{\max}$; If $D = d$ the $E_d = \text{id}$;
$D \subseteq D' \implies E_{D'} \subseteq E_D$; If $D \neq \emptyset$, in every infinite subset of ω there are infinitely many pairwise E_D-inequivalent d-tuples.

The proof goes by induction on $d \geq 1$. For $d = 1$ it holds that $\{E_\emptyset, E_{\{1\}} = \{E_{\max}, \text{id}\}$ is indeed a basis for all partitions of $[\omega]^1$.

□

Induction hypothesis: for every E on $[\omega]^{d-1}$ there is $A \in [\omega]^\omega$ and $D \subseteq d-1$ such that for all $X, Y \in [A]^{d-1}$, $X \; E \; Y \iff X \; E_D Y$.
Let E be an equivalence relation on $[\omega]^{d+1}$. For every $Z \in [\omega]^{2d}$, let E_Z be the restriction of E to $[Z]^{2d}$. Let $F(Z)$ be the unique equivalence relation on $[2d]^d$ onto which E_z is mapped to via the order isomorphism between Z and $2d = \{0, 1, \ldots, 2d-1\}$.
F is a finite coloring of $[\omega]^{2d}$. By Ramsey's theorem, fix an infinite subset if ω which is homogeneous for F, namely, for every $Z \in [\omega]^{2d}$, E_Z is the same equivalence relation up to increasing enumeration. Call this set ω again.

Case 1: the constant E_Z is id.
Case 2:
$x_0 < x_1 < \ldots, x_{i-1}, x_i \qquad < \ldots$
$y_0 < x_1 < \ldots, y_{i-1}, \qquad y_i < \ldots$
Thinning out to 2ω we define a coloring on d-tuples $\langle x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d \rangle$ by inserting an extra member between x_{i-1} and x_i. Now by the induction hypothesis there is $D \subseteq \{0, 1, \ldots, i-1, i+1, \ldots, d-1\}$ and D works when viewed as $D \subseteq d$.

Moral

Order matters in Ramsey theory, even when coloring unordered sets.

$\mathbb{N}$

Let's begin with ω with the arithmetic structure, aka $\mathbb{N}$.
An arithmetic progression is a sequence of the form $\langle a_0 + id : i \leq i(*)\rangle$ for some $i(*) \leq \omega$.

Theorem (van der Waerden)

For ever finite partition of $\mathbb{N}$ one of the cells contains arbitrarily long finite arithmetic progressions.

Definition

The n-dimentional combinaotrial cube over d letters, denotet d^n is the set of all sequences ("words") of length n over the set $d = \{0, 1, \ldots, d-1\}$. Let X be a variable (that is, $X \notin d$) and let an n-dimentional 1-pattern $w(X)$ be a word of length n over $d \cup \{X\}$ in which X occurs at least once.

Example

$13X3X$ is a 5-dimentional 1-pattern over 4.

A combinatorial line in n^d is the set of all substitutions $\{w(0), w(1), \dots w(d-1)\}$ in a pattern $w(X)$.

Example

$13030, 13131, 13232, 13333$ is a combinatorial line in 4^5. It is also an arithmetic progression of numbers written in base 4 with $d = 101$ in base 4, that is $4^2 + 1 = 17$.

Hales-Jewett Theorem

Theorem (Hales-Jewett)

For every d and c there is a sufficently large n so that for every partition of d^n to c parts there exists a combinatorial line contained in one of the parts.

Corollary

van der Waerden's theorem.

Proof.

By induction on d, for fixed c. Let $n = HJ(d, c)$. We define a number N and prove that $N \geq HJ(d+1, c)$.

Let $N_0 = c^{(d+1)^n} + 1$. Let $N_{i+1} = c^{(d+1)^{((\sum_{j \leq i} N_j)+n)}} + 1$. Let $N = \sum_{i<n} N_i$.

$$\overbrace{\underbrace{11\ldots10\ldots0}_{N_0}\underbrace{11\ldots10\ldots0}_{N_1}\ldots\underbrace{11\ldots10\ldots0}_{N_{n-1}}}^{N}$$

□

Nešetřil-Rödl Theorem

Ramsey's theorem for $d = 2$ can be described in terms of colorings of the edges of the complete graph on ω. All infinite subsets correspond to complete, induced subgraphs.
Can we require a monochromatic induced subgraph which is not complete? We have to:

- start from a non-complete graph;
- distinguish between edges and non-edges.

Theorem (Nešetril-Rödl)

for every finite graph H there exists a finite graph G such that

$$G \rightarrowtail (H)_2^{K_2}.$$

Proof

First, use the Hales-Jewett theorem to get the theorem for bipartitie graphs. $G^{(n)}$ is the symmetric power of $G = (L, R, E)$:

$$(L^n, R^n, E^{(n)}).$$

Let n be the Hales-Jewett number. Now $G^{(n)}$ works.
From the bipartite version, the full version follows by the amalgamation argument.