Extensions of Ramsey's theorem

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Ramsey's Theorem

Theorem (Ramsey)

For every finite d and c and for every function $f : [\omega]^d \to c$ there exists an infinite set $A \subseteq \omega$ such that $f \restriction [A]^d$ is constant.

In the partition calculus notation:

$$\omega \to (\omega)^d_c \quad \text{for all } d, c \in \omega.$$

The finife Ramsey theorem is

$$\forall d, c, k \in \omega \, \exists n \;\; n \to (k)^d_c.$$

Ramsey's theorem reformulated

- Let $\mathcal{E}$ be a class of equivalence relations on $[\omega]^d$.
- Let $\mathcal{A}$ be a collection of admisible target subsets of ω
- A set $B \subseteq \mathcal{E}$ is called a basis for $\mathcal{E}$ with respect to $\mathcal{A}$ if for every $E \in \mathcal{E}$ there is $E^* \in B$ and $A \in \mathcal{A}$ such that

$$E \restriction [A]^d = E^* \restriction [A]^d$$

Now, Ramsey's theorem is: $\{E_{\max}\}$ is a basis, or $E_{\max}$ is basic for all finite equivalence relations on $[\omega]^d$ with respect to $[\omega]^\omega$.

The Erdős-Rado Canonical Ramsey theorem

Theorem (Erdős-Rado)

for every $d \geq 1$ *there is a basis of size* 2^d, $\{E_D : D \subseteq d\}$ *for all equivalence relations on* ω *with respect to* $[\omega]^\omega$.

$$X\ E_D Y \iff X \restriction D = Y \restriction D$$

when X, Y are enumerated increasingly.

The induced Ramsey theorem

Theorem (Nešetril-Rödl, Erdős-Hajnal-Poša, Deuber,...)

for every finite graph H there exists a finite graph G such that

$$G \rightarrowtail (H)_2^{K_2}.$$

Reformulation: target sets are those that span a prescribed isomorphism type.

Theorem (Nešetřil and Rödl)

For any two ordered graphs $\bar{A}, \bar{B}$ and finite c there is a finite ordered graph $\bar{C}$ such that

$$\bar{C} \rightarrowtail (\bar{B})_c^{\bar{A}}.$$

That is: for every coloring of the copies of $\bar{A}$ in $\bar{C}$ by c colors there is an induced copy of $\bar{B}$ all of whose $\bar{A}$ are colored in on color.

The necessity of order

Theorem (Nešetril-Rödl)

For every ordered graph $\bar{H}$ there is an graph B such that for every ordering of B there is a copy of H in B which is ordered correctly. Thus, unless H is a complete or an empty finite graph, there is no Ramsey theorem for coloring unordered copies of H.

This makes order necessary. (But not law . . .)

The infinite induced case

Let $\Gamma = (\omega, E^\Gamma)$ be the random graph, that is, the countable saturated model of graph theory. If any infinite induced Ramsey theorem holds, than Γ should work as a source. So: can we get:

$$\Gamma \rightarrowtail (\Gamma)_2^{K_2}?$$

The thing is, you give the infinite random graph an ω-type order (which you have to, by finite reasons) and it cooks itself another ordering of type $\mathbb{Q}$!

Theorem (Erdős-Poša)

$$\Gamma \not\rightarrowtail (\Gamma)_2^{K_2}.$$

Puzet-Sauer and Sauer Basis theorem

Theorem

For every finite graph H there is a basic E^H for all finite partitions of $\binom{\Gamma}{H}$ with respect to all isomorphic copies of Γ in Γ.
In particular, there is a finite number $t(H)$, usually strictly larger than the number of orderings of H, such that

$$\Gamma \rightarrowtail (\Gamma)^H_{<\omega|t}.$$

E.g.,

$$\Gamma \rightarrowtail (\Gamma)^H_{<\omega|4}.$$

The universal pro-finite graph

Let $G_{\max}$ denote the inverse limit of all finite ordered graphs. Then:

Theorem (S. Frick-K.)

For every finite ordered graph H there is a basic continuous f_H for all finite Borel colorings of $\binom{G_{\max}}{H}$.
A particular instance of this is:

$$G_{\max} \rightarrowtail_{Borel} (G_{\max})^{K_2}_2.$$

The range of f_H is all normal sequences of H. Every simple graphs works instead of K_2 above (most finite graphs are simple).

Blass' theorem

Theorem (Blass)

For every $d > 0$ there is a continuous f_d which is basic for all finite Borel coloring of $[2^\omega]^d$ with respect to infinite perfect subsets of 2^ω. The range of f_d has size $(d-1)!$.

Theorem (Sheu)

Let $I_{\min}$ be the σ-ideal generated over 2^ω by all sets which are either 0-homogeneous or 1-homogeneous, where $S \subseteq 2^\omega$ is i-homogeneous if $\Delta(\rho,\nu) \equiv i \mod 2$ for all $\{\rho,\nu\} \in [S]^2$. Then there is a continuous f_d which is basic for all finite Borel colorings of $[2^\omega]^d$ with respect to $I_{\min}$-positive subsets of 2^ω. The range of f_d has size $2^{d-1}(d-1)!$.

Sheu's theorem is also an induced one: let $G_{\min} = (2^\omega, E)$ with $\{\rho,\nu\} \in E \iff \Delta(\rho,\nu) \equiv 0 \mod 2$. Sheu's theorem is $G_{\min} \rightarrowtail_{\text{Borel}} (G_{\min})^d_{<\omega|2^d d!}$.

Symmetrized Substructures

A structure N is a symmetrized substructure of a structure M if $N \subseteq M$ and every automorphism of N extends to an automorphism of M. All subsets of $(\omega, =)$ are symmetrized substructures, so Ramsey's theorem can be written as

$$\omega \rightarrowtail^{S} (\omega)^{d}_{c}.$$

Which of the induced theorems has a symmetrized version?

All countable graph embedd symmetrically into Γ

Not all copies of K_ω in Γ are $\subseteq^S \Gamma$. There are too many types. Are there any symmetrized subgraphs of Γ of a given isomorphism type G?

Fact

For every countable G, $G \subseteq^S \Gamma$.

Proof.

Let $G = G_0$ and
$G_{n+1} = G_n \cup \{v_D : D \subseteq G_n \text{ finite and } G_{n+1}[v_D] = D\}$. *Now* $\bigcup_n G_n = \Gamma$ *and* $G_0 \subseteq^S \Gamma$. □

Wreath products

Wreath products

Let $\Gamma \otimes \Gamma = (\omega^2, E^{\oplus})$ where $\{(m,n),(s,t)\} \in E^{\oplus}$ iff $\{m,s\} \in E^{\Gamma}$ or $m = s$ and $\{n,t\} \in E^{\Gamma}$.

Fact

$$\Gamma \oplus \Gamma \rightarrowtail^{S} (\Gamma)^1_{\omega|1-1,1}.$$

Theorem (Geschke-K.)

$$\Gamma \rightarrowtail^{S} (\Gamma)^1_{\omega|1-1,1}.$$

The ordered $\mathbb{Q}$

Theorem (Devlin)

For every $d > 0$ there exists a number t_k (the 2k-1 "tangent number") such that

$$\mathbb{Q} \longrightarrow (\mathbb{Q})^{d}_{<\omega|t_k}.$$

Theorem (Hasson-K.)

$$\mathbb{Q} \rightarrowtail^{S} (\mathbb{Q})^{1}_{\omega|1-1,1}.$$

Metric spaces

Theorem

For every finite metric space X there exists a fintie metric space Y and $r > 0$ such that

$$Y \rightarrowtail^{S} (X, rX)^1_2.$$

Theorem (Solecki)

For every fintie metric space X there exists a finite metric space Y containing X as a subspace so that every partial isometry between subspaces of X extends to a total isometry of Y.

Solecki's proof uses a theorem by Herwig and Laskar. Herwig and Laskar extend, of course, Hrushovski's original extension lemma.

Finite graphs

Theorem (Geschke-K)

For every finite graph H there is a finite graph G such that

$$G \rightarrowtail^{P} (H)^{K_2}_{2}.$$

What about infinite edge colorings? Still open. . .

Countable metric spaces

Uryson's universal separable space $\mathbb{U}$ is the unique metric space which is:

- separable and complete;
- satisfies the isometric extension property: every isometry of n points from a metric space X can be extended to an isometry including a prescribed $n+1$-th point.

All countable dense subsets of $\mathbb{U}$ are almost isometric to each other, and characterized up to almost-isometry by

- countability;
- almost extension property.

This almost-isometry type is sometimes called the rational Uryson space, and is denoted $\mathbb{A}$.

It is known that $\mathbb{A} \not\to (\mathbb{A})^1_2$.
Let $\mathbb{A}_1$ be the bounded rational Uryson space.

Problem

$$\mathbb{A}_1 \to^S (\mathbb{A}_1)^1_2?$$

Without the ^{S} this is a conjecture/problem by Greg Hjorth.

The End

御静聴有難うございました。