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Exact upper bounds and their uses in set theory[1]

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Abstract

The existence of exact upper bounds for increasing sequences of ordinal functions modulo an ideal is discussed. The main theorem (Theorem 18 below) gives a necessary and sufficient condition for the existence of an exact upper bound f for a $<_I$-increasing sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \mathrm{On}^A$ where $\lambda > |A|^+$ is regular: an eub f with $\liminf_I \mathrm{cf} f(a) = \mu$ exists if and only if for every regular $\kappa \in (|A|, \mu)$ the set of flat points in $\overline{f}$ of cofinality κ is stationary.

Two applications of the main Theorem to set theory are presented. A theorem of Magidor's on covering between models of ZFC is proved using the main theorem (Theorem 22): If $V \subseteq W$ are transitive models of set theory with ω-covering and GCH holds in V, then κ-covering holds between V and W for all cardinals κ. A new proof of a Theorem by Cummings on collapsing successors of singulars is also given (Theorem 24). The appendix to the paper contains a short proof of Shelah's trichotomy theorem, for the reader's convenience.

1. Introduction

Shelah's work on Cardinal Arithmetic (see [7, 1, 4]) introduced the theory of possible true cofinalities of products of sets of regular cardinals modulo an ideal – pcf theory. The relevance of pcf theory to set theory and other branches of mathematics was demonstrated by a series of applications.

In this paper the dual problem is addressed: suppose a set of ordinal functions on an infinite set A has true cofinality modulo an ideal I; is it equivalent to a product of regular cardinals modulo the same ideal? If so, to which product? Since a set of functions with true cofinality modulo I is equivalent modulo I to a product if and only if it has an exact upper bound (see below), the reformulation of the problem is: under

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which condition does a $<_I$-increasing sequence of ordinal functions on a set A have an exact upper bound.

In Section 3 below we derive the existence of an exact upper bound for a $<_I$-increasing $\overline{f}$, and partially determine the shape of the exact upper bound when it exists, from information which is stored in the sequence $\overline{f}$ itself, or, rather, in the collection of *flat points* in the sequence. A flat point in $\overline{f}$ is an initial segment of $\overline{f}$ which is equivalent modulo I to a product of sets of ordinals of *constant* regular order type.

A necessary and sufficient condition for the existence of an exact upper bound f such that $\liminf_I \operatorname{cf} f(a) = \mu$ is given below for a sufficiently long $<_I$-increasing sequence of functions $\overline{f} \subseteq \mathrm{On}^A$ where I is an ideal over an infinite set A (Theorem 18). The condition is: for every regular κ between $|A|$ and μ the indices in $\overline{f}$ of flat points of cofinality κ form a stationary subset of λ.

Theorem 18 is useful for set theory for two reasons. First, it enables a reconstruction of the exact upper bound – or of the product to which a given sequence is equivalent – from the collection of flat points of the sequence. Second, the flatness of a point is preserved in extensions in which the cofinality of the point remains greater than $|A|$. Two applications of the main theorem in Section 4 illustrate this.

The same theorem serves also as a convenient tool in the presenting pcf theory (see the new version of [4]).

1.1. Structure of the paper

The paper is organized as follows. In Section 2 the main definitions are introduced, notation is established and a few easy facts concerning true cofinality and exact upper bounds are collected. Section 3 is devoted to the proof of the main theorem. Section 4 presents two applications: an unpublished theorem of Magidor and a theorem by Cummings are proved using Theorem 18. A common feature of both proofs is the use of the fact that flat points are upwards hereditary between transitive models of ZFC which agree on the relevant cardinals. In the first proof the assumption that a transitive model V_2 agrees on cofinalities with a transitive model $V_1 \subseteq V_2$ that satisfies the GCH is utilized to show, via Theorem 18, that certain eubs remain eubs in V_2. This fact leads, in turn, to κ-covering between V_1 and V_2 for all κ – provided that ω covering holds. In the second proof the fact that the cofinality of a singular μ is coded in the eub of a μ^+-scale is utilized to show that, without severely damaging the structure of the scale one cannot collapse μ^+ to become a successor of a cardinal of cofinality different than $\operatorname{cf}\mu$ (Lemma 3.1 in [2]). This has several corollaries, as Cummings shows in [2].

A short proof of shelah's trichotomy theorem is found in the appendix to the paper.

2. Exact upper bounds modulo an ideal

The basic object we are examining is the following: let A be some infinite set and I be an ideal over A. Let $\overline{f} = \langle f_\alpha : \alpha < \delta \rangle$ be a sequence of functions from A to the ordinals which is increasing modulo I, where δ is some limit ordinal. The question we address is the existence and structure of an exact upper bound modulo I for this sequence. In this Section we establish notation and prove a few basic facts about exact upper bounds modulo an ideal which are needed to facilitate the rest of the discussion.

2.1. Basics

Let A be a fixed infinite set. By On^{A} we denote the class of all functions from A to the ordinal numbers. Given an ideal I over A, we quasi order On^{A} by defining $f \leqslant_I g$ for $f, g \in \mathrm{On}^{\mathrm{A}}$ iff $\{a \in A : f(a) > g(a)\} \in I$. Similarly, $=_I$ and $<_I$ are defined.

In the special case that $I = \{\emptyset\}$, the relation $<_I$ is the relation of domination everywhere and is denoted by $<$.

For subsets $F_1, F_2 \subseteq \mathrm{On}^{\mathrm{A}}$ write $F_1 \sim_I F_2$ if for every $f \in F_1$ there is $g \in F_2$ such that $f \leqslant_I g$ and for every $g \in F_2$ there is $f \in F_1$ such that $g \leqslant_I F$. The relation $\sim_I$ is an equivalence relation.

We shall investigate the relation between the following two properties of subsets of On^{A}:

Definition 1. 1. $F \subseteq \mathrm{On}^{\mathrm{A}}$ *has an exact upper bound* iff there exists $g \in \mathrm{On}^{\mathrm{A}}$ such that:
(a) $(\forall f \in F)(f \leqslant_I g)$,
(b) $g' <_I g \Rightarrow (\exists f \in F)(g' <_I f)$.
A function g which satisfies (a) and (b) is *an exact upper bound* (an eub) of F.
2. $F \subseteq \mathrm{On}^{\mathrm{A}}$ *has true cofinality* iff there exists some $F' \subseteq \mathrm{On}^{\mathrm{A}}$ such that $F' \sim_I F$ and F' is linearly ordered by $<_I$ and has no last element.

If F has true cofinality then the *true cofinality* of F is denoted by $\mathrm{tcf}\, F$ and is the cofinality of the order type of some (of every) linearly ordered F' that is equivalent to F.

The following points should be noticed: first, each of the properties defined below is invariant under $\sim_I$. Second, neither property the other. Third, eubs are also least upper bounds, except in the trivial case where F has an upper bound which assumes the value 0 on a positive set, and is therefore an eub vacuously.

Fact 2. (1) *A set $F \subseteq \mathrm{On}^A$ without a maximum with respect to $\leqslant_I$ has an eub f iff F is equivalent to a copy of a product of regular cardinals, namely there exists sets $S(a) \subseteq f(a)$ for $a \in A$ such that* otp $S(a) = \mathrm{cf}\, f(a)$ *is and* $F \sim_I \prod_{a \in A} S(a)$.

(2) *A set $F \subseteq \mathrm{On}^A$ has true cofinality λ iff it is equivalent to a $<_I$ increasing sequence $\langle f_\alpha : \alpha < \lambda \rangle$.*

Both properties above are preserved when the ideal I is extended. We shall be using the following fact freely:

Fact 3. *Suppose $I_1 \subseteq I_2$ are ideals over an infinite set A and $F \subseteq \mathrm{On}^{\mathrm{A}}$. Then:*
1. *If g is an eub of F modulo I_1 and $0 <_{I_1} g$ then g is also an eub of F modulo I_2.*
2. *If F has true cofinality λ modulo I_1 than F has true cofinality λ modulo I_2.*

A particular instance of this fact is when $I_2 = I_1 \restriction B$ for some $B \in I_1^+$.

The following is a simple, yet important example of a set of functions which has both an eub and true cofinality regardless to which ideal I over A is involved:

Fact 4. *Suppose λ is regular and $\lambda > |A|$. Then* $\mathrm{tcf}(\lambda^A, <) = \lambda$. *Consequently,* $\mathrm{tcf}(\lambda^A, <_I) = \lambda$ *for every ideal I over A, by Fact* 3.

Proof. Let $g_\gamma(a) = \gamma$ for $\gamma < \lambda$ and $a \in A$. The sequence $\bar{g} = \langle g_\gamma : \gamma < \lambda \rangle$ is $<$-increasing. It is also cofinal in $(\lambda^A, <)$ by the following "rectangle argument": Let $g \in \lambda^A$ be arbitrary. Since $\lambda > |A|$ is regular, $\gamma := \sup\{g(a) : a \in A\} < \lambda$ and therefore $g \leqslant g_\gamma$. □

Claim 5. *Suppose that $\lambda > |A|$ is regular and $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle \subseteq \mathrm{On}^A$ is $<_I$-increasing. The following are equivalent:*
1. *There is an eub f of $\bar{f}$ such that $\mathrm{cf}\, f(a) = \lambda$ for all $a \in A$.*
2. *There are sets $S(a)$ for $a \in A$ with $\mathrm{otp}\, S(a) = \lambda$ such that $\bar{f} \sim_I \prod S(a)$.*
3. *There is some $<$-increasing $\bar{g} = \langle g_\gamma : \gamma < \lambda \rangle$ and some increasing, continuous and cofinal subsequence $\langle \alpha(\gamma) : \gamma < \lambda \rangle \subseteq \lambda$ such that $f_{\alpha(\gamma)} <_I g_{\gamma+1} <_I f_{\alpha(\gamma+1)}$.*

Definition 6. (1) A $<_I$-increasing $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle \subseteq \mathrm{On}^A$ is *flat* (of cofinality λ) $\mathrm{mod}\, I$ if and only if one of the equivalent conditions in 5 holds for $\bar{f}$.

(2) $\alpha < \delta$ is a *flat point* in a $<_I$-increasing $\bar{f} = \langle f_\beta : \beta < \delta \rangle$ if and only if $\bar{f} \restriction \alpha$ is flat.

The third of the three equivalent definitions for flatness makes the following remark obvious:

Remark 7. If $\bar{f} \subseteq \mathrm{On}^A$ is flat of cofinality $\lambda > |A|$ in some transitive universe V of set theory, then $\bar{f}$ is flat of cofinality λ in every transitive extension V' of V in which λ is regular $> |A|$.

Let us state now an easy property of flat sequences:

Fact 8. *Suppose $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle \subseteq \mathrm{On}^A$ is flat of cofinality λ. Then there is a closed unbounded set $E \subseteq \delta$ such that every point in E of cofinality $> |A|$ is a flat point for $\bar{f}$.*

Proof. Suppose that $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$ is flat of cofinality λ and that $\langle g_\gamma : \gamma < \lambda \rangle$ and $\langle \alpha(\gamma) : \gamma < \lambda \rangle$ are as in clause 3. of Claim 5. Every limit point $\delta < \lambda$ of $\langle \alpha(\gamma) : \gamma < \lambda \rangle$ of cofinality greater than $|A|$ is flat. □

In the next section we shall see that this property characterizes flat sequences except in the case that λ is a successor of a singular.

2.2. The structure of eubs

The question we are addressing is the following: given a set of functions $F \subseteq \mathrm{On}^A$ which has true cofinality modulo I, determine whether F has an eub, and, in case an eub exists, determine to which product of regular order types F is equivalent.

An example to this is the equivalence between conditions (3) and (2) in Definition 6 above: If $F = \langle f_\alpha : \alpha < \lambda \rangle$, is a $<$-increasing sequence of regular length $\lambda > |A|$ then F is flat, that is equivalent to a product of a constant order type.

But this case of a flat sequence is hardly the interesting case, of course. However, the structure of the set of *flat points* in a given $<_I$-increasing $\overline{f}$ is quite revealing about the existence and shape of an eub of $\overline{f}$.

We defer the existence problem for later and start with a preliminary simple classification of eubs. Assume for the moment that $F \subseteq \mathrm{On}^A$ with no maximum is given and has both true cofinality λ and an eub g. Since F itself matters only up to $\sim_I$, we assume wlog that F is a $<_I$-increasing sequence of functions $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$. We assume that $\lambda > |A|$. There are interesting questions involving true cofinality $\lambda < |A|$, but we do not address those here. To avoid trivialities, we assume also that $g(a) > 0$ and is limit for all $a \in A$.

Let λ_a be the cofinality of $g(a)$. We examine now the constraints on $a \mapsto \lambda_a$ which follow from $\mathrm{tcf}(\prod_{a\in A} \lambda_a, <_I) = \lambda$. We are interested, of course, in $a \mapsto \lambda_a$ only mod I.

Lemma 9. *Suppose $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \mathrm{On}^A$ is increasing modulo I, $\lambda > |A|$ is regular and g is an eub of f. Then for every regular θ satisfying $|A| < \theta < \lambda$, the set $\{a \in A : \lambda_a = \theta\}$ is measure zero.*

Proof. Suppose to the contrary that $|A| < \theta < \lambda$ and that $B = \{a \in A : \lambda_a = \theta\}$ is positive. Replacing I by $I \restriction B$, g remains an eub, and λ remains the true cofinality. But modulo $I \restriction B$, $\overline{f}$ is flat of cofinality θ — a contradiction, as $\theta < \lambda$. □

Since the set of $a \in A$ for which $\lambda_a > \lambda$ is clearly measure zero, we may assume it is empty by changing the eub g on a measure zero set. Now partition A as follows: $A = A_0 \cup A_1 \cup A_2$ where

- $A_0 = \{a \in A : \lambda_a \leqslant |A|\}$
- $A_1 = \{a \in A : |A| < \lambda_a < \lambda\}$,
- $A_2 = \{a \in A : \lambda_a = \lambda\}$.

Corollary 10. *If $\mathrm{tcf}(\prod \lambda_a, <_I) = \lambda$ and $A_1 \in I^+$, then $\lambda \geqslant |A|^{+\mathrm{add}(I)}$, where $\mathrm{add}(I)$ is the additivity of I.*

Proof. Suppose $\lambda \in (|A|, |A|^{+\mathrm{add}(I)})$. Then $A = \bigcup_{\kappa \in (|A|,\lambda)} \{a \in A : \lambda_a = \kappa\}$. Therefore one of the cardinals $\kappa \in (|A|, \lambda)$ satisfies $\{a \in A : \lambda_a = \kappa\} \in I^+$, contrary to Claim 9. □

The following example shows that each of the A_i'-s can indeed be positive:

Example 11. Let V be a model of set theory obtained from a ground model which satisfies GCH by ccc forcing, such that $V \models$ "$MA + 2^{\aleph_0} = \aleph_{\omega+1}$". Let $A_i = \{i\} \times \omega$ for $i \in \{0,1,2\}$ and let $A = A_0 \cup A_1 \cup A_2$". Let $\lambda_{(i,n)}$ be ω if $i = 0$; $\aleph_n$, if $i = 1$ and $\aleph_{\omega+1}$, if $i = 2$. The ideal I over A is the ideal of finite sets. The product of the $\lambda_{(i,n)}$ has true cofinality $\aleph_\omega + 1$.

The subset A_2 in the decomposition above is not particularly interesting – if we augment I to $I \lceil A_2$ then the sequence becomes flat, and is then well understood. The subset A_0 is slightly more interesting, but there is little one can say about it. It is A_1 which is more interesting in the context of pcf theory, of course.

In what follows, we shall see that it is possible to find out from the structure of flat points in a $<_I$-increasing sequence whether indeed A_0 and A_2 are null and some facts about which cardinals appear as λ_a on A_1.

Definition 12. For a function $f \in \mathrm{On}^{\mathrm{A}}$ and an ideal I over A,
- α is an accumulation point of $f \bmod I$ iff for all $\beta < \alpha$, $f^{-1}[(\beta,\alpha]] \in I^+$.
- $\liminf_I f := \min\{\alpha \in \mathrm{On} : \mathrm{f}^{-1}[\alpha + 1] \in \mathrm{I}^+\}$ is the smallest accumulation point $f \bmod I$.
- $\limsup_I f := \sup\{\alpha \in \mathrm{On} : \mathrm{f}^{-1}[\alpha + 1] \in \mathrm{I}^*\}$ is the largest accumulation point of $f \bmod I$.
- If $\liminf_I f = \limsup_I f$ then $\lim_I f$ is defined and equals both.

Fact 13. *If* $\mathrm{tcf}(\prod \lambda_a, <_I) = \lambda$, $|A| < \lambda_a < \lambda$ *for all* $a \in A$ *and* μ *is an accumulation point of* λ_a *mod* I *then* $\mathrm{add}(I) \leqslant \mathrm{cf}\,\mu \leqslant |A|$.

Proof. Suppose that μ is an accumulation point of λ_a mod I. By 9 we know that $\{a \in A : \lambda_a = \mu\} \in I$, and therefore $\{a : \alpha < \lambda_a < \mu\} \in I^+$ for every $\alpha < \mu$. Therefore $\{\lambda_a : a \in A\} \cap \mu$ is cofinal in μ. The inequality $\mathrm{add}(I) \leqslant \mathrm{cf}\,\mu$ is obvious. □

What about the possibility $A =_I A_0$, or $\lambda_a \leqslant |A|$ for all $a \in A$? In the simplest case of $A = \omega$, and $1 \leqslant \lambda_n \leqslant \omega$ everything is possible: let λ be your favorite uncountable regular cardinal, and in a model of MA with $2^{\aleph_0} = \lambda$, the true cofinality of $(\prod \lambda_n, <_I)$ is λ with I being the ideal of bounded sets, and hence with I being any non-principal ideal.

A similar case is that of $|A| = \kappa$ inaccessible. Here too there is no bound to what can occur as $\mathrm{tcf}(\prod(\mathrm{Reg} \cap \kappa), <_{\mathrm{I}})$ where I is the ideal of bounded subsets of κ.

Neither of those cases is dealt here.

A fundamental theorem concerning eubs of sufficiently long $<_I$ increasing sequences $\overline{f} \subseteq \mathrm{On}^{\mathrm{A}}$ is Shelah's trichotomy theorem, quoted below. Shelah's trichotomy is Claim 1.2 in Ch. II of [7]. A proof of this theorem is found also in the appendix to the present paper. The proof in the appendix is a little shorter that the one in [7].

Theorem 14 (Shelah's Trichotomy). *Suppose $\lambda > |A|^+$ is regular, I is an ideal over A and $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is an $<_I$-increasing sequence of ordinal functions on A. Then $\overline{f}$ satisfies one of the following conditions:*

- *(Good) $\overline{f}$ has an eub f with $\operatorname{cf} f(a) > |A|$ for all $a \in A$;*
- *(Bad) there are sets $S(a)$ for $a \in A$ satisfying $|S(a)| \leqslant |A|$ and an ultra filter D over A extending the dual of I so that for all $\alpha < \lambda$ there exists $h_\alpha \in \prod S(a)$ and $\beta < \lambda$ such that $f_\alpha <_D h_\alpha < D f_\beta$.*
- *(Ugly) there is a function $g : A \to \mathrm{On}$ such that the sequence $\bar{t} = \langle t_\alpha : \alpha < \lambda \rangle$ does not stabilize modulo I, where $t_\alpha = \{a \in A : f_\alpha(a) > g(a)\}$ (notice that $\bar{t}$ is $\subseteq_I$-increasing, because $\overline{f}$ is $<_I$-increasing).*

3. The main theorem

Theorem 18 below is the main theorem of this paper. It extends Shelah's trichotomy and provides a necessary and sufficient condition for the existence of an eub for a sequence $\overline{f}$ that satisfies the hypothesis of the trichotomy theorem. It also determines the shape of such an eub when it exists.

We need three preparatory Lemmas. The first Lemma guarantees existence of an eub from the existence of a stationary set of flat points. The second Lemma is needed to guarantee that flat points of cofinality larger than the lim inf of the cofinalities of values of the eub are not stationary. the last Lemma shows that for every cofinality between $|A|$ and the lim inf the set of flat points of that cofinality is stationary.

Lemma 15. *Suppose $\overline{f} \subseteq \mathrm{On}^A$ is $<_I$ increasing of length $\lambda = \operatorname{cf} \lambda > |A|^+$. If there is a stationary set of flat points of cofinality κ in $\overline{f}$ for some $|A| < \kappa < \lambda$, then $\overline{f}$ has an eub f with $\operatorname{cf} f(a) > |A|$ for all $a \in A$.*

Proof. Since $\lambda > |A|^+$, the trichotomy theorem applies to $\overline{f}$. The existence of an eub as required can be derived from the trichotomy theorem once we show that Bad and Ugly fail.

We are assuming that $\lambda > \kappa > |A|$ and the collection of flat points of cofinality κ for $\overline{f}$ is stationary in λ.

Assume first, to the contrary, that Ugly holds. Fix $g \in \mathrm{On}^A$ such that letting $t_\alpha = \{a \in A : f_\alpha(a) > g(a)\}$ the sequence $\bar{t} = \langle t_\alpha : \alpha < \lambda \rangle$ does not stabilize mod I. Let $E \subseteq \lambda$ be a club of λ such that $\alpha < \beta$ in E implies $t_\alpha \subsetneqq_I t_\beta$. Choose a flat point $\alpha < \lambda$, $\operatorname{cf} \alpha = \kappa$ and α is a limit of E.

Fix a $<$-increasing sequence $\langle g_i : i < \kappa \rangle$ such that $\langle g_i : i < \kappa \rangle \sim_I \overline{f} \restriction \alpha$. Since α is a limit point of E we may assume, by passing to a subsequence of $\bar{g}$, that for every $i < \kappa$ there are $\beta < \gamma$ in E for which $g_i <_I f_\beta <_I f_\gamma <_I g_{i+1}$.

Let $s_i = \{a \in A : g_i(a) > g(a)\}$ for $i < \kappa$. By the above, $s_i \subseteq_I t_\alpha \subsetneqq_I t_\gamma \subseteq_I s_{i+1}$. In particular we have for $i < j < \kappa$ that $s_i \subsetneqq_I s_j$. Since $\langle g_i : i < \kappa \rangle$ is $<$-increasing, $\langle s_i : i < \kappa \rangle$ is also $\subseteq$-increasing. But now this is absurd, because to increase in both

$\subseteq$ and $\subsetneq_I$ means to increase in $\subsetneq$; and there is no $\subsetneq$-increasing sequence of subsets of A of length $\kappa > |A|$.

Assume now that Bad holds with respect to sets $S(a)$ with $|S(a)| \leqslant |A|$ and an ultra filter D. Find a club $E \subseteq \lambda$ such that for $\alpha < \beta$ in E it holds that $f_\alpha <_D h_\alpha <_D f_\beta$. Choose a flat limit point α of E of cofinality κ and choose an eub f of $\overline{f} \restriction \alpha$ with $\mathrm{cf} f(a) = \kappa$ for $a \in A$. Since $S(a) \cap f(a)$ is bounded below $f(a)$ for all $a \in A$ no subset of $\prod S(a)$ can be cofinal with $\overline{f} \restriction \alpha$, contrary to α being a limit of E.

Thus, Bad and Ugly fail, and hence Good holds by the trichotomy theorem, namely there exist and eub f for $\overline{f}$ with $\mathrm{cf} f(a) > |A|$ for all $a \in A$. $\square$

Lemma 16. *Suppose $\overline{f}$ is as in the previous lemma and f is an eub of $\overline{f}$. If for regular $\kappa \in (|A|, \lambda)$ the set of flat points in $\overline{f}$ of cofinality κ is stationary in λ then $\{a \in A : \mathrm{cf} f(a) \leqslant \kappa\} \in I$.*

Proof. First we observe that if $\{a \in A : \mathrm{cf}\, f(a) = \kappa\} \in I$ by 9. Suppose that $B = \{a \in A : \mathrm{cf}\, f(a) < \kappa\}$ is positive, and consider $I \restriction B$. Every flat point is flat with respect to this ideal as well. Modulo $I \restriction B$, we may assume that $\mathrm{cf} f(a) < \kappa$ for all $a \in A$.

Fix a set $S(a) \subseteq f(a)$ which is cofinal in $f(a)$ and of order type $\mathrm{cf} f(a)$. Since $\overline{f} \sim_I \prod_{a \in A} S(a)$, there is a closed unbounded $E \subseteq \lambda$ such that for every $\alpha \in E$, there is some sequence $\overline{h} = \langle h_i : i < \mathrm{cf}\alpha \rangle \subseteq \prod_{a \in A} S(a)$ with $\overline{f} \restriction \alpha \sim_I \overline{h}$.

Since there are stationarily many flat points of cofinality κ, we can choose some $\alpha \in E$ of cofinality κ so that $\overline{f} \restriction \alpha$ is flat, and let g be an eub of $\overline{f} \restriction \alpha$ with $\mathrm{cf} g(a) = \kappa$ for all $a \in A$. Let $\overline{h} = \langle h_i : i < \kappa \rangle \subseteq \prod_{a \in A} S(a)$ be chosen so that $\overline{h} \sim_I \overline{f} \restriction \alpha$. Since for every $i < \kappa$ we have $h_i <_I g$, we may assume that for all $a \in A$, $h_i(a) < g(a)$.

Let $h(a) = \sup[S(a) \cap g(a)]$. Since $\mathrm{otp}\, S(a) = \mathrm{cf} f(a) < \kappa$, $h < g$. Therefore there is some $\gamma < \alpha$ such that $h <_I f_\gamma$. But $f_\gamma <_I h_\gamma \in \prod_{a \in A} S(a) \cap g(a)$, hence $h_\gamma \leqslant h$ — a contradiction. $\square$

Lemma 17. *Suppose that $\overline{f}$ is as above, f is an eub of $\overline{f}$ and $\mu = \liminf_I \mathrm{cf}\, f(a)$. Then for every regular $\kappa \in (|A|, \mu)$ the set of flat points of cofinality κ in $\overline{f}$ is stationary in λ.*

Proof. It is obvious that $\mu \leqslant \lambda$. Let κ be any regular in $(|A|, \mu)$.

Correcting f on a null set we may assume that $\mathrm{cf}\, f(a) > \kappa$ for all $a \in A$. To establish stationarity of the set of flat points of cofinality κ, let $\overline{M} = \bigcup_{\zeta < \kappa} M_\zeta$ be an elementary chain of sub-models of $H(\chi)$ for a sufficiently large regular χ such that for every $\zeta < \kappa$, $\langle M_\xi : \xi < \zeta \rangle \in M_{\zeta+1}$, the cardinality of each M_ζ is κ and $\overline{f}, f \in M_0$. Let $M = \bigcup_{\zeta < \kappa}$. We show that $\sup M \cap \lambda$ is a flat point (of cofinality κ, of course). This guarantees stationarity, as the heights $\sup M \cap \lambda$ of such models form a stationary subset of λ.

For every $a \in A$ and $\zeta < \kappa$ define $\chi_\zeta(a) = \sup M_\zeta \cap f(a)$. Since $\mathrm{cf} f(a) > \kappa$ and $|M| \leqslant \kappa$ it follows that $\chi_\zeta < f$.

$\chi_\alpha \in M_{\zeta+1}$ and $M_{\zeta+1} \models$ "f is an eub of $\overline{f}$" so there is some $\alpha_\zeta \in M_{\zeta+1} \cap \lambda$ for which $\chi_\zeta <_I f_{\alpha_\zeta}$. Since $\overline{f} \in M_{\zeta+1}$ also $f_{\zeta_\zeta} \in M_{\zeta+1}$ and hence $f_{\alpha_\zeta} < \chi_{\zeta+1}$.

Conversely, if $\beta < \alpha$ then by increasing β we may assume that $\beta \in M$ and therefore there is some $\zeta < \kappa^+$ for which $\beta \in M_\zeta$. So also $f_\beta \in M_\zeta$ by elementarity and $f_\beta < \chi_{\zeta+1}$.

The sequence $\langle \chi_\zeta : \zeta < \kappa \rangle$ is thus $<$-increasing and equivalent mod I to $\overline{f} \restriction \alpha \bmod I$, proving that α is a flat point in $\overline{f}$. □

Theorem 18. *Suppose $\lambda > |A|^+$ is regular, I an ideal over A and $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is $<_I$-increasing. Let S be the set of regular cardinals κ for which the set of flat points of cofinality κ in $\overline{f}$ is stationary in λ.*

The following conditions are equivalent for a cardinal $\mu \in (|A|^+, \lambda]$:
1. *there exists an eub f of $\overline{f}$ and* $\liminf_I \operatorname{cf} f(a) = \mu$,
2. $\mu = \sup\{\kappa^+ : \kappa \in S\}$,
3. $S = \mathrm{Reg} \cap (|A|, \mu)$.

Proof. 1 $\Rightarrow$ 3: Suppose 1. holds. Lemma 16 gives $S \subseteq Reg \cap (|A|, \mu)$. Lemma 17 provides the converse inclusion.

3 $\Rightarrow$ 2: trivial.

2 $\Rightarrow$ 1: The existence of *some* regular κ in S guarantees the existence of an eub by Lemma 15. Fix an eub f for $\overline{f}$. The conjunction of Lemma 16 with Lemma 17 implies $\liminf_I \operatorname{cf} f(a) = \mu$. □

The next example may assist in understanding the previous theorem better. Work in a model of set theory in which $\operatorname{pcf}\{\aleph_n : n < \omega\}$ contains $\aleph_{\omega+\omega+1}$, namely, in which there is some ideal I over ω such that $\operatorname{tcf}(\prod_n \aleph_n, <_{I_1}) = \aleph_{\omega+\omega+1}$. Such a model is available in [6] or in [3]. By pcf theory, there is some ideal over $(\omega, \omega+\omega)$ such that $\operatorname{tcf}(\prod_n(\aleph_{\omega+n}, <_{I_2})) = \aleph_{\omega+\omega+1}$. Let I be the ideal over $\omega+\omega$ obtained by joining I_1 and I_2 as follows: $X \in I \Leftrightarrow X \cap \omega \in I_1 \wedge X \cap (\omega, \omega+\omega) \in I_2$. Let $\overline{f} \subseteq \prod_{\alpha \in (\omega+\omega)} \aleph_\alpha$ be $<_I$-increasing and $<_I$ cofinal in the product. The eub of $\overline{f}$ is the function $f(\alpha) = \aleph_\alpha$ for $\alpha \in (\omega+\omega)$. There are two accumulation points of $\{\operatorname{cf} f(\alpha) : \alpha \in (\omega+\omega)\}$ modulo I: $\aleph_\omega$ and $\aleph_{\omega+\omega}$. The set S of cofinalities for which the flat points in $\overline{f}$ are stationary is $\{\aleph_n : n < \omega\}$.

Now increase the ideal I be throwing ω into I. Modulo the revised ideal there is only one accumulation point of $\{\operatorname{cf} f(\alpha) : \alpha \in (\omega+\omega)\}$ and ther are stationarily many flat points in $\overline{f}$ also for cofinality $\aleph_{\omega+n}$ for all n.

Thus although true cofinality and the eub have not changed, the expansion of the ideal turned many points to flat points.

Theorem 19. *Suppose that $\overline{f} \subseteq \mathrm{On}^A$ is $<_I$ increasing of length $\lambda > |A|^+$ and has an eub f with $\operatorname{cf} f(a) > |A|$ for all $a \in A$. If λ is not a successor of singular whose cofinality is $\leqslant |A|$, then $\overline{f}$ has a closed unbounded set of flat points in every cofinality $\kappa < \lambda$ iff $\overline{f}$ is flat.*

Proof. If $\overline{f}$ is flat then indeed almost all points of cofinality κ in $\overline{f}$ are flat for every regular $\kappa \in (|A|, \lambda)$, by Fact 8.

Suppose now that $\overline{f}$ is not flat, and that for every regular $\kappa \in (|A|, \lambda)$ almost all points of cofinality κ in $\overline{f}$ are flat. In the notation of Theorem 18, $S = \mathrm{Reg} \cap (|A|, \lambda)$. Since $\overline{f}$ is not flat, $A_1 = \{a \in A : \mathrm{cf} f(a) < \lambda\} \in I^+$. Let $\mu = \liminf_I \mathrm{cf} f(a)$. By Fact 13 applied to $I \restriction A_1$, $\mathrm{cf}\mu \leqslant |A|$, and by Theorem 18, $\mathrm{Reg} \cap \mu = \mathrm{Reg} \cap \lambda$. So necessarily $\lambda = \mu^+$. □

We remark that it is possible to have $\liminf_I = \mu$, $\lambda = \mu^+$ and that for every $\kappa \in S$ the set of flat points in $\overline{f}$ of cofinality κ is not only stationary but almost all points of cofinality κ.

The next lemma describes a condition under which a non-flat sequences have club many flat points.

Lemma 20. *Suppose that $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \mathrm{On}^A$ is $<_I$ increasing, f an eub of $\overline{f}$ and $\liminf_I \mathrm{cf}\, f(a) = \mu$. If $\kappa = \mathrm{cf}\kappa < \mu$ and $\theta^{|A|} < \kappa$ for all $\theta < \kappa$, then every point of cofinality κ in $\overline{f}$ is flat.*

Proof. Suppose that $\alpha < \lambda$ has cofinality κ. Since $2^{|A|} < \kappa$ we have that $\kappa > |A|^+$. By the Trichotomy Theorem applied to $\overline{f} \restriction \alpha$ and as $2^{|A|} < \kappa$ excludes Bad and Ugly, we conclude that $\overline{f} \restriction \alpha$ has an eub, say h. Denote $\mu = \liminf_I \mathrm{cf} h(a)$. We know that $\mu \leqslant \kappa$; we easily see that $\mu \geqslant \kappa$ as well, since if $B = \{a \in A : \mathrm{cf} h(a) < \kappa\} \in I^+$, by regularity of $\kappa > |A|$, $\theta = \sup\{\mathrm{cf} f(a) : a \in A\} < \kappa$. Since h is an eub mod $I \restriction B$ we conclude that $\overline{f} \restriction \alpha \sim_{I \restriction B} \prod_{a \in B} S(a)$, where $S(a) \subseteq h(a)$ is cofinal of order type $\mathrm{cf} h(a)$. This is absurd, because $|\prod_{a \in B} S(a)| \leqslant \theta^{|A|} < \kappa$. □

4. Applications

In this section we apply Theorem 18 to two set theoretic problems.

First, we present an unpublished theorem of Magidor's about covering properties between models of ZFC. Then we re-prove a Lemma by Cummings concerning successors of singulars. In both proofs, some $\overline{f}$ with an eub f is fixed in an inner model V_1, and Theorem 18 is used to argue that f remains an eub of $\overline{f}$ in some extension V_2 which preserve sufficiently many cofinalities. One direction of the Theorem is used to encode f by the set of flat points in V_1 and the other direction is used in V_2 to reconstruct f.

4.1. Magidor's Theorem

Definition 21. Let $\mathrm{On} \subseteq \mathrm{V}_1 \subseteq \mathrm{V}_2$ be transitive models of ZFC. We say that κ-covering holds between V_1 and V_2 iff for all $X \in V_2$, if $X \subseteq \mathrm{On}$ and $V_2 \models |X| \leqslant \kappa$ then there is $Y \in V_1$ such that $X \subseteq Y$ and $V_2 \models |Y| \leqslant \kappa$.

If k-covering holds between V_1, V_2 for all κ, then V_1 and V_2 are "close" to each other in several senses: V_1, V_2 agree on cofinalities and hence on cardinalities; V_1, V_2

agree on $\mathrm{cf}([\lambda]^\kappa, \subseteq)$ for all $\lambda > \kappa$; every singular μ in V_2 is singular in V_1 and many other useful properties. Such is the situation between L and V in the absence of $0^\#$, for all $\kappa \geqslant \aleph_1$, by Jensen's Covering Theorem.

It is interesting to reverse the question, and ask: Suppose that two universes $V_1 \subseteq V_2$ are "close" to each other in the sense that the agree on cofinalities and cardinalities which are $\geqslant \kappa_0$ for some κ_0; do they necessarily satisfy κ-covering for all $\kappa \geqslant \kappa_0$?.

An example in which ω-covering fails between $V_1 \subseteq V_2$ which agree on cofinalities and cardinal arithmetic is the following: let κ_n be a Prikry sequence in a measurable κ. Let $V_1 := V[\langle \kappa_{2n} : n < \omega \rangle$ and let $V_2 := V[\kappa_n : n < \omega\rangle]$. The countable set $\{\kappa_{2n+1} : n < \omega\}$ is not covered by any set of cardinality $< \kappa$ from V_1, although both models agree on all cofinalities.

We wish now to obtain a violation of ω_1-covering without violating ω-covering between a pair $V_1 \subseteq V_2$ of universes. For that we use a model of Segal [5] which is constructed by starting from a ground model that satisfies GCH and collapsing some large cardinal λ by adding simultaneously $> \lambda^+$ ω_1-Prikry sequences to it. Thus, in the generic extension V^{P}, λ is a singular of cofinality $\aleph_1$ which violates the singular cardinal hypothesis.

Note that by Silver's theorem, SCH is violated by many singulars of countable cofinality below μ, so necessarily many new ω sequences are added below λ in V^{P}.

The property we invoke from Segal's forcing extension is that every subset of size $\leqslant \lambda$ belongs to an intermediate forcing extension obtained by a sub-forcing notion whose cardinality is λ.

Choose now *two* intermediate models $V \subseteq V_1 \subseteq V_2 \subseteq V^{\mathrm{P}}$, between the ground model and the forcing extension as follows: let V_1 be a model which contains all bounded subsets of λ and one cofinal ω_1 sequence. Thus, V_1 thinks that λ is a singular of cofinality $\aleph_1$ (below which SCH is violated in many singulars of countable cofinality), but that λ satisfies SCH. This is because V_1 knows at most λ^+ many new subsets of λ.

V_2 is obtained over V_1 similarly, by adding one new ω_1 cofinal sequence of λ which is not covered by any subset of λ of cardinality $\aleph_1$ from V_1. This is possible, since there are λ^{++} many ω_1 cofinal sequences of λ in $V^{\mathrm{P}} - V_1$. Now V_1 and V_2 agree on cardinalities, cofinalities and even on the power set function, but do not satisfy ω_1 covering.

An interesting fact is, that V_1 violates SCH in many singulars of countable cofinality. Is this coincidental?

The following theorem by Magidor sheds some light on this phenomenon, by showing that if ω-covering holds between V_1, V_2 that agree on cofinalities and $V_1 \models GCH$ then k-covering holds for all κ. In other words, a violating ω_1 covering but still maintaining ω-covering between "close" models of set theory must occur between models that violate GCH.

The relation of this to eubs is the following: if $\overline{f} = \langle f_\alpha : \alpha < \aleph_{\omega_1} \rangle$ is $<_I$-increasing and cofinal in $\prod \lambda_i^+$ in V_1 for some normal sequence $\langle \lambda_i : i < \omega_1 \rangle$ of cardinals in μ and ω_1-covering holds between V_1 and V_2, then $\overline{f}$ is increasing cofinal in V_2 as well, because every $f \in \prod \lambda_i^+$ from V_2 is dominated by some such function from V_1. The main point in the proof is that the converse is also true if GCH holds in V_1, namely this consequence of covering for eubs actually implies ω_1-covering.

Theorem 22 (Magidor). *Suppose that* $\mathrm{On} \subseteq V_1 \subseteq V_2$ *are universes of ZFC, and* V_1, V_2 *agree on cofinalities. If* $V_1 \models GCH$ *and every countable set of ordinals in* V_2 *is covered by a countable set of ordinals from* V_1, *then every set of ordinals* $X \in V_2$ *is covered by some set of ordinals* $Y \in V_1$ *with* $|Y| = |X|$.

Proof. The proof goes by induction on $\mu := \sup X$.

If μ is not a cardinal, the claim follows easily from the induction hypothesis via a bijection of μ with its cardinality. So we assume that μ is a cardinal. If μ is regular, then $|X| = \mu$ and μ is the required set. The remaining case is that $\mu = \sup X$ is a singular cardinal and we divide it to two subcases.

If μ is singular of countable cofinality then fix in V_1 (by GCH in V_1) an enumeration e of length μ of all bounded subsets of μ which belong to V_1 and an ω-sequence $\langle \alpha_n : n < \omega \rangle$ with supremum μ. The set $\{e(X \cap \alpha_n) : n < \omega\} \subseteq \mu$ belongs to V_2 and since ω-covering holds between V_1 and V_2 it can be covered by a countable set of ordinals $Y \in V_1$. Now $X \subseteq \bigcup \{e^{-1}(\alpha) : \alpha \in Y \wedge |e^{-1}(\alpha)| \leqslant |X|\}$ belongs to V_1, covers X and has cardinality $\leqslant |X|$.

We are left with the interesting case: μ is singular of uncountable cofinality.

Let $\kappa := \mathrm{cf}\, \mu$. Fix in V_1 a sequence $\langle \lambda_i : i < \kappa \rangle$ increasing, continuous and cofinal in μ, and assume, without loss of generality, that $\lambda_0 > |X|$.

Since $V_1 \models GCH$ we can fix in V_1 a bijection $e_i : \mathscr{P}(\lambda_i) \to \lambda_i^+$ for every $i < \kappa$. Suppose that the following claim holds:

Claim 23. *For every* $g \in \prod_{i<\kappa} \lambda_i^+$ *in* V_2 *there is* $h \in \prod_{i<\kappa} \lambda_i^+$ *in* V_1 *such that* $g <_I h$, *where* I *is the non-stationary ideal on* κ *in* V_2; *in other words,* $(\prod_{i<\kappa} \lambda_i^+)^{V_1} \sim_I (\prod_{i<\kappa} \lambda_i^+)^{V_2}$.

We shall show that this suffices to cover X. For every $i < \kappa$ we can find, by the induction hypothesis, a set $Y_i \in \mathscr{P}(\lambda_i) \cap V_1$ such that $X \cap \lambda_i \subseteq Y_i$ and $|Y_i| \leqslant |X|$. Let $g(i)$ be the first index of such Y_i in the enumeration e_i we have fixed. Thus, $g \in \prod_{i<\kappa} \lambda_i^+$ and belongs to V_2. Find $h \in \prod_{i<\kappa} \cap V_1$ which bounds $g \bmod I$. For every $i < \kappa$ there is an injection π_i^h in V_1 from $h(i) \in \lambda_i^+$ into λ_i. Let $F(\lambda_i) \in \lambda_i$ be the image of $g(i)$ under this injection for each i such that $g(i) < h(i)$. Since $\{\lambda_i : g(i) < h(i)\}$ contains a club of κ, by Fodor's lemma there is some $\eta(*) < \mu$ and a stationary subset $S \subseteq \kappa$ such that $F(\lambda_i) < \eta(*)$ for all $i \in S$. Now let $Z = \{Y : (\exists i < \kappa)(\pi_i^h(e_i(Y)) < \eta \wedge |Y| \leqslant |X|)\}$. $Z \in V_1$ and by its choice, since $\{i < \kappa : h(i) > g(i)\}$ is stationary, $X \subseteq \bigcup Z$. Also, $|Z| < \mu$. So we have managed to cover X by a set in V_1 of cardinality smaller than μ. This is enough, because fixing a bijection between Z and $|Z|$ and using the induction hypothesis for $|Z|$, we can cover X.

We prove now Claim 23.

We have fixed $\langle \lambda_i : i < \kappa \rangle$, an increasing continuous sequence of cardinals with supremum μ. Using GCH in V_1 and standard diagonalization find a sequence $\overline{f} = \langle f_\alpha : \alpha < \mu^+ \rangle$, $<_I$-increasing and cofinal in $(\prod \lambda_i^+, <_I)$, where I is the non-stationary ideal over κ. So d, defined by $d(i) = \lambda_i^+$, is the eub of $\overline{f}$ mod I.

From GCH in V_1 it follows that $\theta^\kappa = \theta$ for all regular $\theta > \kappa$. Thus *every* point of cofinality θ^+ for regular $\theta > \kappa$ is flat in V_1 by 20. As V_1, V_2 agree on cofinalities and cardinals (by Remark 7), all points of cofinality θ^+ are flat in V_2 as well. By Theorem 18 there is in V_2 an eub f of $\overline{f}$ with $\liminf_I \operatorname{cf} f = \mu$. Since $\overline{f} \subseteq \prod \lambda_i^+$, without loss of generality, $f \leqslant d$.

We argue next that $f =_I d$. If $f \lneq_I d$ then the following set is stationary in V_2: $B = \{i < \kappa : f(a) < \lambda_i^+\}$. Let $G(i) := \operatorname{cf} f(i)$ and let $F(i) = \min\{j < \kappa : G(i) \leqslant \lambda_j\}$. Since without loss of generality every $i \in B$ is limit, $\operatorname{cf} f(i) < \lambda_i$ and hence F is regressive. By Fodor's lemma we assume that F is constant on B with constant value $j < \kappa$. We have shown that for all $i \in B$ the cofinality $\operatorname{cf} f(i)$ is bounded by λ_{j+1}. Since B is positive this contradicts $\liminf_I \operatorname{cf} f(i) = \mu$. $\square$

4.2. Cummings' Theorem

Next we apply Theorem 18 to give a new proof of the main Lemma in Cummings [2]. Cummings discusses the constraints which are enforced on a pair of universes of set theory $V \subseteq W$ with the property that some successor of singular in V has different cofinality in W: W cannot be a ccc extension of V, cannot violate the SCH at that singular, and more. We remark that the existence of such a pair of universes is not known. As with the proof of Magidor's theorem, here too the preservation of flat points plays a crucial role.

Theorem 24. *Suppose that $V \subseteq W$ are inner models of set theory, and μ is singular in V. Suppose that $\langle \kappa_i : i < \operatorname{cf} \mu \rangle$ is increasing and cofinal in μ and that $\overline{f} = \langle f_\alpha : \alpha < \mu^+ \rangle \subseteq \prod_{i < \operatorname{cf}\mu} \kappa_i$ is $<^*$-increasing and $<^*$-cofinal in $\prod_{i < \operatorname{cf}\mu} \kappa_i$. Suppose that $\mu_V^+ = \nu_W^+$ and $W \models \operatorname{cf}\mu \neq \operatorname{cf}\nu$. Then there is some $\rho < \nu$ such that for all $\kappa \in \mathrm{Reg}^W \cap (\rho, \nu]$ the set of flat points in $\overline{f}$ of cofinality κ is not stationary in W.*

Proof. Let $\delta = \operatorname{cf}_V \mu$ and let $\theta = \operatorname{cf}_W \mu$. Clearly, $\theta = \operatorname{cf}_W \kappa$.

The relation $<^*$ is $<_I$ where I is the ideal of bounded sets on δ. If $\operatorname{cf}_W \delta < \operatorname{cf}_V \delta$, fix a cofinal set C in δ or order type θ, and work with $I \restriction C$. Thus, wlog, $\delta = \theta$.

Work in W from now on. The sequence $\overline{f} \subseteq \prod_{i<\theta} \kappa_i$ is $<^*$ increasing of length λ^+. Also, $\sup\{\kappa_i : i < \theta\} < \lambda^+$. If for unboundedly many regulars $\leqslant \lambda$ there are stationarily many flat points in $\overline{f}$, then, by Theorem 18, $\overline{f}$ has an eub f with $\liminf_I \operatorname{cf} f(i) = \lambda$. Since $\operatorname{ran} f \subseteq \sup\{\kappa_i : i < \theta\}$, the cofinality if each $f(i)$ is at most λ. The set $\{i < \theta : \operatorname{cf} f(i) = \lambda\}$ is clearly null (or else $\operatorname{tcf} \overline{f} = \lambda$); so $\operatorname{cf} f(i) < \lambda$ for all $i < \theta$.

By Fact 13, $\operatorname{add}(I) \leqslant \operatorname{cf} \lambda \leqslant \theta$. Since the additivity of the ideal of bounded sets of θ is θ, it follows that $\operatorname{cf} \lambda = \theta$ — contrary to $W \models \operatorname{cf} \kappa \neq \operatorname{cf} \lambda$. $\square$

Appendix

Theorem A.1 (Shelah's Trichotomy). *Suppose $\lambda > |A|^+$, I is an ideal over A and $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is an $<_I$-increasing sequence of ordinal functions on A. Then $\overline{f}$ satisfies one of the following conditions:*

- *(Good) $\overline{f}$ has an eub f with $\operatorname{cf} f(a) > |A|$ for all $a \in A$;*
- *(Bad) there are sets $S(a)$ for $a \in A$ satisfying $|S(a)| \leqslant |A|$ and an ultra filter D over A extending the dual of I so that for all $\alpha < \lambda$ there exists $h_\alpha \in \prod S(a)$ and $\beta < \lambda$ such that $f_\alpha <_D h_\alpha < D f_\beta$.*
- *(Ugly) there is a function $g : A \to \mathrm{On}$ such that the sequence $\overline{t} = \langle t_\alpha : \alpha < \lambda \rangle$ does not stabilize modulo I, where $t_\alpha = \{a \in A : f_\alpha(a) > g(a)\}$ (notice that $\overline{t}$ is $\subseteq_I$-increasing, because $\overline{f}$ $<_I$-increasing).*

Proof of the Trichotomy Theorem. We argue first that Good can be weakened in the Theorem to the existence of a lub f: By the next claim either Ugly holds or every lub of $\overline{f}$ is an eub. If f is an eub and $\liminf \operatorname{cf} f(a) \leqslant |A|$ then Bad is witnessed by any ultra filter D extending the dual of $I \restriction \{a \in A : \operatorname{cf} f(a) \leqslant |A|\}$. Thus if Ugly fails, either Bad holds or f is an eub with $\liminf_I \operatorname{cf} f > |A|$.

Claim A.2. *If $\overline{f}$ is not Ugly then every lub of $\overline{f}$ is an eub.*

Proof of Claim. Assume to the contrary that f is a lub to $\overline{f}$ which is not an eub. This means that there is some function $g : A \to \mathrm{On}$ with $g <_I f$ but such that for all $\alpha < \lambda$ it holds that $g \not<_I f_\alpha$. Let $\overline{t} = \langle t_\alpha : \alpha < \lambda \rangle$ be defined by $t_\alpha := \{a \in A : f_\alpha(a) > g(a)\}$. This sequence is increasing in $\subseteq_I$, because $\overline{f}$ is increasing in $\leqslant_I$. Since $\overline{f}$ is not Ugly, there is some $\alpha(*) < \lambda$ at which $\overline{t}$ stabilizes, namely for all $\alpha(*) \leqslant \alpha < \lambda$ it holds that $t_{\alpha(*)} =_I t_\alpha$. If $t_{\alpha(*)} =_I A$ then $g <_I f_{\alpha(*)}$, which we assume does not happen; thus $A \setminus t_{\alpha(*)} \in I^+$.

Let f' be defined as follows:

$$f'(a) = \begin{cases} f(a) & \text{if } a \in t_{\alpha(*)}, \\ g(a) & \text{if } a \in A \setminus t_{\alpha(*)}. \end{cases}$$

Since $A \setminus t_{\alpha(*)} \in I^+$, it holds that $f' \lneq_I f$. But f' is still an upper bound of $\overline{f}$ because for all $\alpha \geqslant \alpha(*)$ the definition of f' implies $f_\alpha \leqslant_I f'$. This contradicts f being lub. □

Now it is enough to prove the following weaker form of Shelah's Trichotomy:

Claim A.3. *For $\overline{f}$ as in Theorem A.1, either $\overline{f}$ has a lub, or $\overline{f}$ is Bad or $\overline{f}$ is Ugly.*

Proof. Assume Ugly fails, and we will either produce a lub or find sets $S(a)$ and ultra filter $D \cap I = \emptyset$ that witness Bad.

Define by induction on $\zeta < |A|^+$ an upper bound g_ζ to $\overline{f}$ and functions h^ζ_α for $\alpha < \lambda$ so that:

1. $\xi < \zeta \Rightarrow g_\zeta \not\geq_I g_\xi$,
2. $S_\zeta(a) := \{g_\xi(a) : \xi < \zeta\}$,
3. $h^\zeta_\alpha(a) := \min(S_\zeta(a) \setminus f_\alpha lpha(a))$.

Observe that by Definition 3. above the sequence $\langle h^\zeta_\alpha : \alpha < \lambda\rangle$ is $\leqslant_I$-increasing in $\prod S_\zeta(a)$.

Either g_ζ will be a lub of $\overline{f}$ for some $\alpha < |A|^+$ or else we will find an ultra filter D that witnesses Bad with the sets $S_\zeta(a)$ for some (limit) $\zeta < |A|^+$.

Let $g_0(a) := \sup\{f_\alpha lpha(a) + 1 : \alpha < \lambda\}$. For every $\alpha < \lambda$, $g_0 > f_\alpha lpha$ so g_0 is an upper bound of $\overline{f}$.

At a successor $\zeta + 1 < |A|^+$ choose, if possible, an upper bound $g_{\zeta+1}$ to $\overline{f}$ which satisfies $g_{\zeta+1} \not\geq_I g_\zeta$. If this is not possible, g_ζ is, by definition, a lub of $\overline{f}$, and the Theorem is proved. □

Suppose that $\zeta < \kappa^+$ is limit and that g_ξ is defined for all $\xi < \zeta$. For every $\alpha < \beta < \lambda$ let $t_{\alpha,\beta} := \{a \in A : h^\zeta_\alpha(a) < f_\beta(a)\}$. Since $\langle h^\zeta_\alpha : \alpha < \lambda\rangle$ is $\leqslant_I$-increasing, $t_{\alpha,\beta}$ is $\subseteq_I$-decreasing in α and since $\overline{f}$ is $<_I$ increasing, $t_{\alpha,\beta}$ is $\subseteq_I$-increasing in β.

If there exists $\alpha < \lambda$ for which $\langle t_{\alpha,\beta} : \beta < \lambda\rangle$ does not stabilize mod I then Ugly holds. Since Ugly fails, for every $\alpha < \lambda$ there exists $\beta(\alpha) < \lambda$ such that $t_{\alpha,\beta} =_I t_{\alpha,\beta'}$ for all $\beta < \beta' < \lambda$. By this find a club $E \subseteq \lambda$ such that for all $\alpha < \beta < \beta'$ in E we have $t_{\alpha,\beta} = t_{\alpha,\beta'}$. Let t_α, for $\alpha \in E$, be such that $t_\alpha =_I t_{\alpha,\beta}$ for all $\beta > \alpha$ in E. Thinning E, if necessary, we assume that either $t_\alpha \in I$ for all $\alpha \in E$ or that $t_\alpha \notin I$ for all $\alpha \in E$.

If $t_\alpha \in I$ for all $\alpha \in E$, let $\alpha(\zeta) = \min E$ and define

$$g_\zeta := h^\zeta_{\alpha(\zeta)}. \tag{A.1}$$

The assumption $t_{\alpha(\zeta)} \in I$ means that $f_\beta \leqslant_I h^\zeta_{\alpha(\zeta)}$ for all $\beta < \lambda$. Thus $h^\zeta_{\alpha(\zeta)} = g_\zeta$ is an upper bound to $\overline{f}$ and all we need for verifying the induction hypothesis for g_ζ is that $g_\zeta \not\geq_I g_\zeta$ for $\xi < \zeta$. Since ζ is limit and $\langle g_\xi : \xi < \zeta\rangle$ is $\not\geq_I$-decreasing by the induction hypothesis, it is enough to show that $g_\zeta \leqslant_I g_\xi$ for all $\xi < \zeta$. Now $g_\xi \in \prod S_\zeta(a)$ and $f_{\alpha(\zeta)} \leqslant h_{\alpha(\zeta)} = g_\zeta \in \prod S_\zeta(a)$. The definition of $h^\zeta_{\alpha(\zeta)}$ in 3. above implies that $h^\zeta_{\alpha(\zeta)} \leqslant_I h$ for every $h \in \prod S_\zeta(a)$ for which $f_\alpha \leqslant_I h$ — in particular $h_\alpha \leqslant_I g_\xi$.

If $t_\alpha \notin I$ for all $\alpha \in E$, observe first that if $\alpha < \beta$ are in E then $t_\beta \subseteq_I t_\alpha$ (because $t_\alpha =_I t_{\alpha,\gamma} \subseteq_I t_{\beta,\gamma} =_I t_\beta$ for any $\beta < \gamma \in E$). The sequence $\langle t_\alpha : \alpha \in E\rangle$ is a $\subseteq_I$-decreasing sequence of positive sets (a "tower" in $\mathscr{P}(A)/I$), so in particular $\{t_\alpha : \alpha \in E\} \cup I^*$ has the finite intersection property and can be extended to an ultra filter D. For every $\alpha < \beta$ in E it holds that $f_\alpha \leqslant h^\zeta_\alpha <_D f_\beta$ (first inequality by the definition in 3. above, the second because $t_\alpha \in D$). This is Bad.

Failure to find $g_{\zeta+1}$ when g_α is defined gives a lub, and failure to define g_α for limit $\zeta < |A|^+$ with g_ξ defined for $\xi < \zeta$ yields Bad. The Theorem follows then once we establish that failure to find γ_ζ must occur at *some* stage before $|A|^+$.

Claim A.4. *g_ζ cannot be defined as in A.1 above for all limit $\zeta < |A|^+$.*

Proof. Suppose to the contrary that the induction goes through all $\zeta < |A|^+$.

For every limit $\zeta < |A|^+$ we have, by 3 above, $g_\zeta = h^\zeta_{\alpha(\zeta)} =_I h^\zeta_\alpha$ for all $\alpha \geqslant \alpha_\zeta$. Since $\lambda > |A|^+$ we can find $\alpha(*) < \lambda$ such that $\alpha_\zeta \leqslant \alpha(*)$ and therefore $g_\zeta =_I h^\zeta_{\alpha(*)}$ for all limit $\zeta < |A|^+$.

The sequence $\langle h^\zeta_{\alpha(*)} : \zeta \in \mathrm{acc}(|A|^+)\rangle$ is $\leqslant$-decreasing because $\langle S_\zeta(a) : \zeta \in \mathrm{acc}|A|^+\rangle$ is $\subseteq$-increasing. Therefore $h^\zeta_{\alpha(*)}$ is fixed for an end segment of $\zeta \in \mathrm{acc}|A|^+$, starting, say, at $\zeta(*) \in \mathrm{acc}|A|^+$ (because there are $|A|$ many coordinates $a \in A$ and on each $a \in A$ the sequence $\langle g_\zeta(a) : \zeta \in \mathrm{acc}(|A|^+)$ can decrease at most finitely many times). So for all $\zeta(*) < \xi < \zeta$ limit ordinals in $|A|^+$ it holds that $g_\xi =_I h^\xi_{\alpha(*)} =_I h^\zeta_{\alpha(*)} =_I g_\zeta$. But by condition 3 of the induction, for every $\xi < \zeta$ limit points in $|A|^+$ it holds that $g_\zeta \lneqq_I g_\xi$ – contradiction. □

Let us make a few more remarks concerning the Trichotomy Theorem. The condition Ugly in the Theorem implies, in particular, that there are λ sets in I^+ whose pairwise intersections lie in I. Namely, I is not λ-saturated. If I is the dual of an ultra filter, than this is impossible (ultra filters are 2-saturated). Thus either Bad or Good must hold.

If I is the dual of an ultra filter, $<_I$ linearly orders On^A. The previous remark tells us in this case that every Dedekind cut of cofinality $> |A|^+$ is either determined by one element – if Good holds – or else belongs to a small product, if Bad holds. In the latter case there may or may not be an eub, namely the cut may or may not be realized.

Finally, the assumption $\lambda > |A|^+$ is necessary and cannot be replaced by $\lambda > |A|$. This, however, is not that important for pcf theory, because in a typical pcf situation $\lambda > |A|^{+\omega}$ anyway.

References

[1] M.R. Burke, M. Magidor, Shelah's pcf theory and its applications, Ann. Pure Appl. Logic 50 (3) (1990) 207–254.
[2] J. Cummings, Collapsing successors of singulars, Proc. Amer. Math. Soc. 125 (9) (1997) 2703–2709.
[3] M. Gitik, M. Magidor, The singular cardinal hypothesis revisited, in: Set Theory of the Continuum (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., Vol. 26, Springer, New York, 1992, pp. 243–279.
[4] M. Kojman, The abc of pcf., Logic Eprints.
[5] M. Segal, SCH at uncountable cofinality, Master's thesis, The Hebrew University of Jerusalem, 1993 (in Hebrew).
[6] S. Shelah, The singular cardinals problem: independence results, in: A.R.D. Mathais (Ed.), Surveys in set theory, London Math. Soc. Lecture Note Series, vol. 87, Cambridge Univ. Press, Cambridge, 1983, pp. 116–134. Proc. Symp. in Set Theory, Cambridge, August 1978.
[7] S. Shelah, Cardinal Arithmetics, Oxford Logic Guides, vol. 29, Oxford Science Publications, 1994.